On the Helly Number for Hyperplane Transversals to Unit Balls*

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Abstract. We prove two results about the Hadwiger problem of finding the Helly number for line transversals of disjoint unit disks in the plane, and about its higher-dimensional generalization to hyperplane transversals of unit balls in d -dimensional Euclidean space. These consist of (a) a proof of the fact that the Helly number remains 5 even for arbitrarily large sets of disjoint unit disks—thus correcting a 40-year-old error; and (b) a lower bound of $d + 3$ on the Helly number for hyperplane transversals to suitably separated families of unit balls in $\mathbb{R}^d$.

1. Introduction

In 1955 Hadwiger [12] posed the problem of determining the smallest number k with the property that if every collection of k members of a family of $n \geq k$ pairwise disjoint unit disks in the plane are met by a line, then all the disks are met by a line; i.e., the problem of determining the so-called *Helly number* (if one exists) for line transversals

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to disjoint unit disks in the plane. He pointed out, by means of an example consisting of disks centered at the vertices of a regular pentagon and almost touching, that $k = 4$ would not suffice. It is this problem, along with its generalization to higher dimensions, that we consider in this paper.

Hadwiger's problem was solved 2 years later in the planar case by Danzer [4], who gave an intricate case analysis to show that $k = 5$ works.

In 1958, in a paper in which he extended Danzer's result to parallelograms, Grünbaum [10] asserted the following strengthened version of Danzer's theorem: "For families of disjoint, congruent circles containing at least six members, $T(4)$ implies T ." (The notation $T(k)$, which has since come to be widely used, means that every k members of the family have a (line) transversal; similarly, " T " means that all do.) He provided only an outline of the proof.

It turns out, however, that this assertion is incorrect: even for arbitrarily large families of unit disks, $T(5)$ is necessary to get T . We give an example in Section 2 below. (This corrects an error that has become embedded in the literature during the past 40 years; see [5], [6], [8], and [13], for example.)

As Grünbaum points out [11], this means that the situation is now much clearer than before, in the following sense: In [10] he had conjectured that $T(5) \Rightarrow T$ for any family of disjoint translates of a compact convex set in the plane. This was finally proven in 1989 by Tverberg [16], thus generalizing Danzer's theorem on unit disks. It appeared, however, that the circular disk was exceptional among all planar convex sets, in that $T(4)$ was sufficient for T for its translates (in a family of sufficient size). One sees now that this is not the case after all.

In [4] Danzer also considered briefly two variants of the problem in higher dimensions: one for line transversals of pairwise disjoint unit balls (which we will not touch on here), the other for hyperplane transversals. In the latter case he observed that, in $\mathbb{R}^3$, no Helly number exists for pairwise disjoint unit balls.

In his example, however, the balls are permitted to have line transversals for more than just two balls; it is this that allows the result to fail, just as the existence of a Helly number for line transversals in the plane would fail if the disks were permitted to have points in common—see Section 4 of [4]. As demonstrated in a number of recent papers [2], [3], [7], [9], [14], [15], [17], the appropriate generalization of "pairwise disjoint," when considering k -dimensional transversals, is " $(k-1)$ -separated": A family of convex sets in $\mathbb{R}^d$ is $(k-1)$ -separated if no $k+1$ of them have a $(k-1)$ -transversal; thus "0-separated" means pairwise disjoint, and in the case where one is interested in the existence of a hyperplane transversal, " $(d-2)$ -separated" means that no d of the sets have a transversal of dimension less than $d-1$. If one assumes that the balls in question are $(d-2)$ -separated, Danzer's remark does not apply, and there is no a priori reason why a Helly number should not exist.

It is this problem, of finding the Helly number for separated unit balls (and, more generally, for separated translates of a single compact convex set), that we set out initially to investigate. While we were unable to find an *upper bound* for the Helly number, or even to prove that one exists, in dimension greater than 2,¹ we did discover a lower

¹ We were, however, able to find an alternate proof [1] of Danzer's theorem that the Helly number in the plane is 5, which we believe is more transparent than the original one, and which we hope may generalize to higher dimensions.

bound. This is presented in Section 3. There, we first prove a theorem about the strict monotonicity (as a function of d) of the Helly number, if one exists, for hyperplane transversals of $(d-2)$ -separated unit balls in $\mathbb{R}^d$. Starting with the example in Section 2, this then allows us to construct, for each $d \geq 2$, an example of $d+3$ unit balls in $\mathbb{R}^d$, no d having a $(d-2)$ -transversal, for which every $d+2$ have a hyperplane transversal, yet all $d+3$ do not. This shows, in particular, that the Helly number for hyperplane transversals of $(d-2)$ -separated unit balls in $\mathbb{R}^d$, if one exists, must be at least $d+3$.

Finally, we conjecture that such a Helly number does indeed exist:

Conjecture 1. *For each $d > 2$, there is a $k(d)$ such that, for hyperplane transversals to families of $(d-2)$ -separated unit balls in $\mathbb{R}^d$, $T(k(d)) \Rightarrow T$.*

For background and additional information on Helly's theorem and geometric transversals, we refer the reader to the excellent survey [5] and to several surveys containing more recent results [6], [8], [18], [19].

2. $T(4) \not\Rightarrow T$

The following example shows that $T(4)$ does not suffice for T , even in the case of six or more disjoint unit disks.

Theorem 1. *For each $n \geq 6$, there exist n pairwise disjoint unit disks in $\mathbb{R}^2$ such that any four have a common transversal but some five do not.*

Proof. We first describe the construction for $n = 6$, and then indicate how to extend it to $n > 6$.

Begin with six unit disks centered at (say) $(0, 0)$, $(3, 0)$, $(10, \pm 1)$, $(12, \pm 1)$. Then translate the fifth and sixth up and down (resp.) a distance ε , for ε suitably small (0.001 will do), and translate the third and fourth up and down (resp.) a distance ε^2 . Figure 1 shows the new disks (with the size of ε exaggerated). Notice that the disks are now pairwise disjoint.

One checks easily (by calculating the distances from various centers to the lines tangent to appropriate pairs of disks) that every four disks are met by a line; disks $\Delta_3, \Delta_4, \Delta_5, \Delta_6$, for example, are met by a vertical line. On the other hand, since a vertical line is the *only* common transversal to $\Delta_3, \Delta_4, \Delta_5, \Delta_6$, disks $\Delta_1, \Delta_3, \Delta_4, \Delta_5, \Delta_6$ have no common transversal.

Fig. 1. Six unit disks.

By making ε successively smaller, we may add an arbitrarily large (finite) number of additional disks with centers at (say) $(-3, 0)$, $(-6, 0)$, $\dots$, while maintaining $T(4)$ but avoiding T . $\square$

3. A Lower Bound for the Hyperplane Transversal Helly Number in $\mathbb{R}^d$

Danzer remarks in [4] that if k_d is the smallest k for which $T(k) \Rightarrow T$ for *line* transversals of disjoint unit balls in $\mathbb{R}^d$, then trivially $k_{d+1} \geq k_d$ for $d \geq 2$.

For *hyperplane* transversals to suitably separated unit balls, it turns out that the Helly number is, in fact, *strictly* monotone:

Theorem 2. *If there is a collection of n $(d-2)$ -separated unit balls in $\mathbb{R}^d$ for which $T(k)$ holds for hyperplanes, but not $T(k+1)$, then there is a collection of $n+1$ $(d-1)$ -separated unit balls in $\mathbb{R}^{d+1}$ for which $T(k+1)$ holds for hyperplanes, but not $T(k+2)$.*

Proof. Since the hypothesis remains valid if the given balls are enlarged very slightly, we may assume without loss of generality that for any k of them there is a hyperplane cutting the *interior* of each.

Suppose the given balls, $\Gamma_1, \dots, \Gamma_n$, have centers at $(x_{11}, \dots, x_{1d}), \dots, (x_{n1}, \dots, x_{nd})$, respectively. Embed $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ as the hyperplane of the first d coordinates, and call it H_0 . Let $\Delta_1, \dots, \Delta_{n+1}$ be the unit balls in $\mathbb{R}^{d+1}$ centered at $(x_{11}, \dots, x_{1d}, M_1), \dots, (x_{n1}, \dots, x_{nd}, M_d), (x_{n1}, \dots, x_{nd}, M_{d+1})$, respectively, where the coordinates M_i are chosen in the order $M_1, \dots, M_{n+1}$, and *each is chosen sufficiently larger than its predecessor*, in a manner to be determined below; see Fig. 2. (Notice that the center of Γ_n is used twice.)

Fig. 2. The $n+1$ lifted balls.

We must show that these $n + 1$ balls satisfy all three conditions.

(i) *They are $(d-1)$ -separated.* Suppose $\Delta_{i_1}, \dots, \Delta_{i_{d+1}}$ are any $d + 1$ of the balls, with $i_1 < \dots < i_{d+1}$. Since $\Gamma_{i_1}, \dots, \Gamma_{i_d}$ have no $(d-2)$ -transversal, a $(d-1)$ -flat meeting $\Delta_{i_1}, \dots, \Delta_{i_d}$ cannot be vertical. Hence if $M_{i_{d+1}}$ is chosen sufficiently large, the ball $\Delta_{i_{d+1}}$ will lie above all such hyperplanes. (If, for $M_{i_{d+1}}$ arbitrarily large, there were a hyperplane through $\Delta_{i_1}, \dots, \Delta_{i_{d+1}}$, by compactness we could find a sequence of hyperplanes converging to a vertical one, each meeting $\Delta_{i_1}, \dots, \Delta_{i_d}$; hence the limiting one would meet $\Delta_{i_1}, \dots, \Delta_{i_d}$ as well, which is impossible.)

(ii) *Any $k + 1$ of the balls $\Delta_1, \dots, \Delta_{n+1}$ have a hyperplane transversal.* This is clear if both Δ_n and Δ_{n+1} are included among them: just take the vertical extension of a $(d-1)$ -transversal to the k distinct projections into the hyperplane H_0 . If not, suppose the balls are $\Delta_{i_1}, \dots, \Delta_{i_{k+1}}$ with $i_1 < \dots < i_{k+1}$. Let F be a $(d-1)$ -flat in the space H_0 cutting the interior of each of $\Gamma_{i_1}, \dots, \Gamma_{i_k}$, and let H be the hyperplane obtained by extending F vertically. If H is tilted by a sufficiently small amount, the resulting hyperplane H' will still meet all of the lifted balls $\Delta_{i_1}, \dots, \Delta_{i_k}$; if $M_{i_{k+1}}$ is large enough, such an H' can be chosen to meet $\Delta_{i_{k+1}}$ as well.

(iii) *Some $k + 2$ of the balls $\Delta_1, \dots, \Delta_{n+1}$ have no hyperplane transversal.* Suppose $\Gamma_{i_1}, \dots, \Gamma_{i_{k+1}}$ have no $(d-1)$ -transversal in H_0 . We claim that $\Delta_{i_1}, \dots, \Delta_{i_{k+1}}, \Delta_{n+1}$ will have no d -transversal in $\mathbb{R}^{d+1}$ if M_{n+1} is chosen large enough.

If, for each choice of M_{n+1} , no matter how large, we can find a hyperplane through Δ_{n+1} meeting all of $\Delta_{i_1}, \dots, \Delta_{i_{k+1}}$, then (by the compactness of the space of hyperplanes meeting a ball containing all of $\Gamma_1, \dots, \Gamma_n$) there is a sequence $H_1, H_2, \dots$ of hyperplanes converging to a vertical hyperplane H , such that each H_i meets all of $\Delta_{i_1}, \dots, \Delta_{i_{k+1}}$. It follows that H does as well, so that $H \cap H_0$ is a $(d-1)$ -flat in H_0 meeting all of $\Gamma_{i_1}, \dots, \Gamma_{i_{k+1}}$, contrary to hypothesis. Hence for M_{n+1} large enough, $\Delta_{i_1}, \dots, \Delta_{i_{k+1}}, \Delta_{n+1}$ will have no d -transversal. $\square$

Corollary 1. *The Helly number for hyperplane transversals to families of $d + 3$ or more $(d-2)$ -separated unit balls in $\mathbb{R}^d$ is at least $d + 3$.*

Proof. This is immediate from Theorems 1 and 2, plus Hadwiger's example in [12]. $\square$

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