Covering Squares with Squares*

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Abstract. For $0 < x < 1$ denote by $f(x)$ the length of the side of the largest open square Q that can be covered by the sequence $\{Q_n\}_{n=0}^{\infty}$ of closed squares, where Q_n has side of length x^n , and is placed so that its sides are parallel to those of Q . We obtain some information concerning $f(x)$ and discuss some related questions.

1. Introduction

In this paper we consider a special case of the following problem: Let $S: 1 = s_0 > s_1 > s_2 > \dots$ be a sequence of positive numbers such that $A = \sum s_i^2 < \infty$. Determine the supremum $f(S)$ of the set of numbers s for which a closed square Q of side s may be covered by the sequence of closed squares $Q_0, Q_1, Q_2, \dots$, where Q_i has side of length s_i and is placed so that its sides are parallel to those of Q . One may give examples of sequences S for which there exist coverings of the open square Q of side $f(S)$ in which no boundary point of Q is covered. We do not give such examples here, but their existence shows that in the definition of $f(S)$ one cannot, in general, replace “supremum” by “maximum.” We remark that while covering problems have a long history and an extensive literature (see [2], [3], or [4] for some of the general background and references to some of the many works in the area), the particular question considered here seems not to have been studied before.

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Unless one imposes some additional conditions on S there is nothing that one can say beyond the trivial inequality

$$1 \leq f(S) \leq \sqrt{A}. \quad (1)$$

In fact, it is easy to give examples for which these bounds are realized. An example of a sequence S for which $f(S) = \sqrt{A}$ may be constructed as follows: Tile a square of side 1 by finitely many squares of different sizes. It is well known that this may be done [1]. Now tile the smallest of these squares by a scaled-down replica of the original tiling and continue this process indefinitely. It is clear that we get a tiling of the unit square with infinitely many different squares. If the largest of these squares has side a , then by magnifying by a factor $1/a$ we get an example for which the upper bound in (1) is realized, with $A = 1/a^2$. It is also the case that $f(S) = 1$ whenever the sequence S satisfies

$$\sum_{i=2}^{\infty} s_i \leq 2(1 - s_1). \quad (2)$$

In order to see this, note that if we attempt to cover a square Q of side just slightly larger than 1 and if we place Q_0 in the upper left corner of Q and Q_1 in the lower right corner, there remain to be covered two congruent rectangles with longer side of length exceeding $1 - s_1$. Inequality (2) then implies that these rectangles cannot be covered. It is also clear that it does not help to consider any other placement of Q_0 and Q_1 . Thus $f(S) = 1$ for any S satisfying (2).

Questions which seem to be of some interest and depth arise if restrictions are placed on S . In this paper we suppose that $s_i = x^i$, $0 < x < 1$. We write $f(x)$ instead of $f(S)$. A covering which covers an open square of side $f(x)$ is called a covering for x and a covering which uses the fewest possible number of squares is called a minimal covering for x . We denote by $g(x)$ the number of squares in a minimal covering. It may happen that $g(x)$ is infinite and we give examples of this subsequently. It goes almost without saying that unless $f(x)$ is determined, it is not possible to evaluate $g(x)$.

We now formulate some of the results that we have obtained. The upper bound in (1) yields

$$f(x) \leq \sqrt{\frac{1}{1-x^2}}. \quad (3)$$

By working out the simple details involved in (2) one finds that

$$f(x) = 1 \quad \text{when} \quad 0 < x \leq 2 - \sqrt{2} := \alpha, \quad (4)$$

where α satisfies

$$\alpha^2 - 4\alpha + 2 = 0. \quad (5)$$

We shall show that (4) is best possible in the sense that $x = \alpha$ is the largest number for which $f(x) = 1$. More precisely, we have the following result:

Theorem 1.

- (a) *There exist sets J and K of positive integers such that $J \cap K = \{1\}$, $J \cup K$ is the set of all positive integers, and*

$$\sum_{j \in J} \alpha^j = \sum_{k \in K} \alpha^k = 1. \quad (6)$$

- (b) *Let $\alpha < x \leq \beta$, where β satisfies $\beta^5 + \beta = 1$, and let m be the least integer for which*

$$\sum_{\substack{j \in J \\ j \leq m-2}} x^j \geq 1 \quad \text{and} \quad \sum_{\substack{k \in K \\ k \leq m-2}} x^k \geq 1. \quad (7)$$

Then

$$f(x) \geq 1 + \frac{x^m}{1 - x^m}.$$

When x is close to 1, the upper bound given by (3) is reasonably accurate in that it provides an asymptotic estimate for $f(x)$ as $x \rightarrow 1^-$.

Theorem 2.

$$f(x) > \sqrt{\frac{2 - x^2}{1 - x^2}} - 2 > \sqrt{\frac{1}{1 - x^2}} - 2.$$

It seems unlikely that one will be able to evaluate $f(x)$ for all x in $(\alpha, 1)$. However, for values of x in some small intervals we have been able to determine $f(x)$ exactly. In such an interval $f(x)$ is given by a polynomial and the endpoints of the interval are zeros of other polynomials. It is also the case that $g(x)$ is finite in these intervals. The bulk of the remainder of the paper is devoted to proving two of the many results of this type that we have obtained.

Theorem 3.

$$f(x) = \begin{cases} x + x^2 + x^5 & \text{for } a \leq x \leq b, \\ 1 + x^{10} + x^{11} & \text{for } b \leq x \leq c, \end{cases}$$

where $a = 0.589373824 \dots$ is a zero of $p(x) = x^{13} + x^{12} + x^{10} - x^5 - x^2 - x + 1$, $b = 0.589445230 \dots$ is a zero of $q(x) = x^{11} + x^{10} - x^5 - x^2 - x + 1$, and $c = 0.589496534 \dots$ is a zero of $r(x) = x^{18} - x^{10} + x^9 + x^8 + x^7 + x^6 + x^4 + x^3 + x - 1$. Furthermore, $g(x) \leq 15$ for $a \leq x \leq b$.

Theorem 4. *Let $h(x) = x + x^3 + x^4 + x^6 + x^7 + x^8 + x^9$. Then*

$$f(x) = \begin{cases} h(x) + x^{11} & \text{for } u \leq x \leq v, \\ 1 + x^9 & \text{for } v \leq x \leq w, \\ h(x) + x^{16} & \text{for } w \leq x \leq z, \end{cases}$$

where $u = 0.589533014 \dots$ is a zero of $x^{19} - (1-x)(h(x) - 1 - x^{10} - x^{12} - x^{14} - x^{15} - x^{16})$, $v = 0.589656902 \dots$ is a zero of $x^{11} + h(x) - x^9 - 1$, $w = 0.590390100 \dots$ is a zero of $x^{16} + h(x) - x^9 - 1$, and $z = 0.590390946 \dots$ is a zero of $x^{24} - x^{16} - h(x) + x^9 + 1$. Furthermore, $g(x) \leq 29$ for $u \leq x \leq z$.

We remark that it follows from Theorem 3 that f is not convex on $[\alpha, 1)$. In fact, as one may readily verify, $f((a+c)/2) > \frac{1}{2}(f(a) + f(c))$.

We could make little progress in determining $f(x)$ for $c < x < u$. Matters are complicated, and perhaps made more interesting, by the fact that $g(x)$ is infinite at at least one x in (c, u) . We have the following result:

Theorem 5. $f(d) = 1 + d^{10} + d^{10}(1 + d^{10})/(1 - d^{10})$ where $d = 0.589515850 \dots$ is a zero of $x^{10} - h(x) + 1$. Furthermore, $g(d) = \infty$.

We do not know whether $g(x)$ is infinite for any other x in (c, u) . We can show that $g(x)$ is unbounded in (d, u) , but we do not give a proof of this result in this paper. The situation is also complicated when x is slightly larger than z . We were able to show that $f(x)$ is given by a polynomial in $[z, z + \delta)$ for an extremely small value of δ . However, we also found a value of x , just slightly larger than z , for which $g(x)$ is infinite.

Theorem 6. $f(t) = 1 + t^9/(1 - t^{15})$, where $t = 0.590391160$ is a zero of

$$x^{20} + x^{17} - x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} - x^9 + x^8 + x^7 + x^6 + x^4 + x^3 + x - 1.$$

Furthermore, $g(t) = \infty$.

We can show that $g(x) \rightarrow \infty$ as $x \rightarrow t^-$, but we do not know whether $g(x) = \infty$ for any x in (z, t) .

Apart from Theorems 1 and 2, we do not discuss in this paper results for the intervals (α, a) or $(t, 1)$. We have been able to obtain many results similar to those given in Theorems 3–6, but their proofs are along the lines of those we present and, for this reason, we do not give them here. We mention, however, that we have been able to show that in $(t, 1)$ there are infinitely many values of x at which $g(x)$ is infinite. The proof of this involves ideas not described in this paper. We hope to give an account in a future article.

The plan of this paper is as follows: In Section 2 we give the proofs of Theorems 1 and 2. In Section 3 we describe the general method which leads to results like Theorems 3 and 4. In Section 4 we give proofs of Theorem 3 and part of Theorem 4. In Section 5 we make some general remarks concerning $g(x) = \infty$ and give coverings showing that $f(d)$ and $f(t)$ are at least the values given in Theorems 5 and 6.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. We first establish (a). Let $j_1 = k_1 = 1$ and suppose that we have defined sets $J_r = \{j_1, j_2, \dots, j_r\}$ and $K_s = \{k_1, k_2, \dots, k_s\}$ such that $J_r \cap K_s = \{1\}$,

$J_r \cup K_s = \{1, 2, \dots, r + s - 1\}$, and

$$\sum_{j \in J_r} \alpha^j < 1 \quad \text{and} \quad \sum_{k \in K_s} \alpha^k < 1.$$

If $\sum_{j \in J_r} \alpha^j \leq \sum_{k \in K_s} \alpha^k$ we set $j_{r+1} = r + s$ and $J_{r+1} = J_r \cup \{j_{r+1}\}$. Otherwise, we set $k_{s+1} = r + s$ and $K_{s+1} = K_s \cup \{k_{s+1}\}$.

If $j_{r+1} = r + s$, it cannot occur that $\sum_{j \in J_{r+1}} \alpha^j \geq 1$, because, if this were the case, we would have

$$\begin{aligned} \sum_{k \in K_s} \alpha^k + \sum_{i=r+s+1} \alpha^i &= \sum_{k \in K_s} \alpha^k + \frac{\alpha^{r+s+1}}{1-\alpha} \\ &> \sum_{k \in K_s} \alpha^k + \alpha^{r+s} \geq \sum_{j \in J_r} \alpha^j + \alpha^{r+s} = \sum_{j \in J_{r+1}} \alpha^j \geq 1. \end{aligned}$$

However, then we would have

$$\sum_{j \in J_{r+1}} \alpha^j + \sum_{k \in K_s} \alpha^k + \sum_{i=r+s+1}^{\infty} \alpha^i > 2.$$

This inequality is easily seen to be equivalent to $\alpha^2 - 4\alpha + 2 < 0$ which, by (5), is false.

Similarly, it may be shown that if $k_{s+1} = r + s$ it cannot occur that $\sum_{k \in K_{s+1}} \alpha^k \geq 1$. Thus the inductive procedure may be repeated indefinitely to produce sets J and K with the first two of the desired properties. To finish the argument, just observe that

$$\sum_{j \in J} \alpha^j + \sum_{k \in K} \alpha^k = 2\alpha + \sum_{i=2}^{\infty} \alpha^i = 2\alpha + \frac{\alpha^2}{1-\alpha} = 2,$$

where, at the last step, we appealed to (5). It now follows that (6) holds. Thus (a) is established.

Now we prove (b). Note first that an m (and hence a least m) satisfying (7) exists, by part (a). It is also clear that $m \geq 5$. Consider a square Q of side $1 + x^m/(1 - x^m)$. Place Q_0 in the upper left corner and Q_1 in the lower right corner. The condition $m \geq 5$ and the equation defining β ensure that Q_0 and Q_1 intersect. There remain uncovered two congruent rectangles, one along the lower side of Q and one along the right side of Q . Place the squares Q_j , $j \in J$ and $j \leq m - 2$, and the square Q_{m-1} on the lower rectangle. From the first inequality in (7) and the fact that $x^{m-1} \geq x^m/(1 - x^m)$ (this inequality is strict except when $m = 5$ and $x = \beta$) it is easily seen that this may be done so that the lower rectangle is covered. From the second inequality in (7) it is seen that the squares Q_k , $k \in K$, $k \leq m - 2$, may be placed on the upper rectangle in such a way that the only uncovered part is a rectangle whose longer side has length $x^m/(1 - x^m)$. This rectangle will be a square if it happens that equality holds in the right member of (7). It now follows that, for each $i \geq 0$, the squares $Q_{im}, Q_{im+1}, \dots, Q_{im+m-1}$ may be used to cover the whole of a square of side $x^{im} + x^{(i+1)m}/(1 - x^m)$ apart from a rectangle whose longer side has length $x^{(i+1)m}/(1 - x^m)$. The argument just given shows that this is so for $i = 0$ and the general case follows by scaling by a factor x^{im} . It now follows that Q may be covered by $Q_0, Q_1, \dots$, so that (b) is established. $\square$

The lower bound for $f(x)$ given in Theorem 1 seems not to be very good, even when x is close to α . For any particular x one may take into account which of the two sums in (7) is closer to 1 and use this to advantage. If, for a specified x , the small rectangle left uncovered after placing $Q_0, Q_1, \dots, Q_{m-1}$ happens not to be a square, then one may exploit this fact. However, there seems to be little that one can say of a general nature. We remark, in this regard, that it would be useful to have more information about the structure of J and K . The initial terms are given as follows:

$$\begin{aligned} J &= \{1, 2, 6, 7, 10, 12, 15, 16, 18, 21, 24, 25, 28, 31, 32, \dots\}, \\ K &= \{1, 3, 4, 5, 8, 9, 11, 13, 14, 17, 19, 20, 22, 23, 26, 27, 29, 30, \dots\}. \end{aligned}$$

It can be shown that neither J nor K contains four consecutive integers. We do not know whether J contains three consecutive integers or whether K contains three consecutive integers, apart from 3, 4, 5.

Proof of Theorem 2. We deduce the theorem from the following proposition.

Proposition. *Let R be a rectangle with sides a and b , $a \leq b$, and let S be a collection of $n + 1$ squares with sides $x_0 > x_1 > \dots > x_n$. If*

$$\sum_{i=0}^n x_i^2 \geq ab + (a + b)x_0, \quad (8)$$

then the squares from S may be used to cover R .

Proof. We use induction on n . If $n = 0$, then (8) implies that $x_0 > b$ and the result clearly holds. Suppose then that $n > 0$ and that the proposition holds when there are fewer than n rectangles. Let ℓ be the least integer such that $\sum_{i=0}^{\ell} x_i \geq a$. Such an ℓ exists since, by (8), $ax_0 < ab + (a + b)x_0 \leq \sum_{i=0}^n x_i^2 < x_0 \sum_{i=0}^n x_i$ so that $\sum_{i=0}^n x_i \geq a$. It cannot occur that $\ell = n$ since, if this were the case, we would have $\sum_{i=0}^{n-1} x_i < a$ and thus $\sum_{i=0}^{n-1} x_i^2 \leq x_0 \sum_{i=0}^{n-1} x_i < ax_0$. It would then follow that $\sum_{i=0}^n x_i^2 < ax_0 + x_n^2 < ax_0 + x_{n-1}x_0 < 2ax_0$ while, from (8), one has $\sum_{i=1}^n x_i^2 \geq ab + (a + b)x_0 > (a + b)x_0 \geq 2ax_0$.

Partition R into two rectangles, an a by x_ℓ rectangle R_1 and an a by $(b - x_\ell)$ rectangle R_2 . It is clear, from $\sum_{i=0}^{\ell} x_i \geq a$, that R_1 may be covered by the squares with sides $x_0, x_1, \dots, x_\ell$. By the induction hypothesis, R_2 may be covered by the squares with sides $x_{\ell+1}, x_{\ell+2}, \dots, x_n$ provided

$$\sum_{i=\ell+1}^n x_i^2 \geq a(b - x_\ell) + (a + b - x_\ell)x_{\ell+1}.$$

This will be so if

$$\sum_{i=\ell+1}^n x_i^2 \geq ab + bx_\ell - x_\ell^2. \quad (9)$$

Now, from (8), we have

$$\begin{aligned}
 \sum_{i=\ell+1}^n x_i^2 &\geq ab + (a+b)x_0 - \sum_{i=0}^{\ell} x_i^2 \\
 &\geq ab + (a+b)x_0 - x_0 \sum_{i=0}^{\ell-1} x_i - x_{\ell}^2 \\
 &\geq ab + (a+b)x_0 - ax_0 - x_{\ell}^2 \\
 &= ab + bx_0 - x_{\ell}^2 \\
 &\geq ab + bx_{\ell} - x_{\ell}^2,
 \end{aligned}$$

and this is (9). This completes the proof of the proposition. $\square$

Now we complete the proof of Theorem 2. For $\alpha < x < 1$ let a be determined by $(a+1)^2 + 2(a+1) = \sum_{i=0}^{\infty} x^{2i}$, so that $a = \sqrt{(2-x^2)/(1-x^2)} - 2$. Let n be the least integer such that $\sum_{i=0}^n x^{2i} \geq a^2 + 2a$. This inequality is just (8) with $x_i = x^i$ and $a = b$. It follows from the proposition that a square of side a may be covered by the squares $Q_0, Q_1, \dots, Q_n$ and hence that $f(x) \geq a$. This completes the proof. $\square$

We had originally proved a weaker version of Theorem 2; namely, $f(x) = (1 + o(1))\sqrt{1/(1-x^2)}$ as $x \rightarrow 1^-$, but the error implied by the $o(1)$ term was unbounded. The proposition and its proof is due to our friend A. Meir. We are grateful to him for providing it and for permitting us to use it in this paper.

3. Some General Remarks

The proofs of Theorems 3 and 4 are technically complicated but they use simple ideas. In this section we give a brief description of the methods; we hope that this will help guide the reader through the details.

Suppose that we have found a covering of a square Q with squares $Q_0, Q_1, \dots, Q_m$ from which we extract the bound $f(x) \geq F(x)$ for some specified F and all x in some interval I . Suppose that the covering is such that we have reason to expect that equality should hold, at least on some subinterval of I . This expectation will usually be based on the fact that, at some places, the covering is tight in the sense that a slight shift of one of the squares will destroy the covering. Suppose also that we can show that in attempting to cover a square Q' of side slightly longer than that of Q we are forced to place $Q_0, Q_1, \dots, Q_k$ so that part of Q' (it will usually consist of one or two rectangles) remains uncovered. Suppose further that we can show that no matter how the squares $Q_{k+1}, Q_{k+2}, \dots$ are placed the uncovered part cannot be entirely covered. It will then follow that $f(x) = F(x)$.

In carrying out this analysis we make use of certain constraints. If, after being forced to place $Q_0, Q_1, \dots, Q_{\ell-1}$ in a certain way there is left uncovered a rectangle R with sides $a(x)$ and $b(x)$, then, if there is to be a covering, we must have the following area

constraint:

$$\frac{x^{2\ell}}{1-x^2} = \sum_{i=\ell}^{\infty} x^{2i} \geq a(x)b(x).$$

Of more important and more frequent use are linear constraints. When we say that a side of R cannot be covered we mean that it is not possible to cover a very thin strip along this side. If $a(x) < b(x)$ we have the linear constraint

$$\frac{x^\ell}{1-x} = \sum_{i=\ell}^{\infty} x^i \geq b(x)$$

since otherwise it is not possible to cover the longer side of R . If it happens that $x^\ell \leq a(x)$, then we must have

$$2x^\ell + \frac{x^{\ell+1}}{1-x} \geq a(x) + b(x)$$

since otherwise there are two intersecting sides of R that cannot both be covered. If we have $x^\ell < a(x)$, then we must have

$$\frac{x^\ell}{1-x} \geq 2b(x)$$

since otherwise at most one of the longer sides of R can be covered. Still with the assumption $x^\ell < a(x)$, if Q_ℓ is not placed at a corner of R , then we must have

$$2x^{\ell+1} + 2x^{\ell+2} + \frac{x^{\ell+3}}{1-x} \geq 2a(x) + b(x).$$

It will usually be the case that after placing $Q_0, Q_1, \dots, Q_\ell$ there will be a second rectangle R' which is not covered. In principle, there is no reason why there should not be more than two uncovered rectangles, but this does not arise in the work we describe here. If R' has sides of length $a'(x)$ and $b'(x)$, where $a'(x) < b'(x)$, then additional constraints arise. We mention a few of them. If it happens that it is not possible to place Q_ℓ on R' so that R' is covered, then we must have

$$\frac{x^{\ell+1}}{1-x} \geq a'(x) + b(x)$$

since otherwise the shorter side of R' and the longer side of R cannot both be covered. If, in addition, it happens that $x^{\ell+1} \leq a(x)$, then we must have

$$2x^{\ell+1} + \frac{x^{\ell+2}}{1-x} \geq a'(x) + a(x) + b(x)$$

and if $x^{\ell+1} < a(x)$ we must have

$$\frac{x^{\ell+1}}{1-x} \geq a'(x) + 2b(x).$$

It is clear that there are further variants, but we do not give them here.

Our procedure is then roughly as follows: Suppose that in attempting to cover a square of side slightly larger than $F(x)$ we are forced to place $Q_0, Q_1, \dots, Q_{\ell-1}$ in a certain way and that there are two rectangles R and R' left uncovered. Suppose also that it is not possible to place Q_ℓ so that it covers parts of each of the rectangles. If the placement of Q_ℓ on R yields an area or linear constraint on R' (or $R \cup R'$) that cannot be satisfied, then we are forced to place Q_ℓ on R' . If it now happens that a constraint on R (or $R \cup R'$) cannot be satisfied, then we have achieved $f(x) = F(x)$. Otherwise, we examine the various ways that Q_ℓ may be placed (usually, we will be able to show, without loss, that there is only one way) and for each of these we then proceed to $Q_{\ell+1}$ and continue until our goal is reached.

In carrying out the analysis in the proofs of the various theorems there are many inequalities that need to be verified. We do not elaborate on this. We merely state that an inequality is valid (or fails to hold, if this is what is needed) on the relevant interval given in the statement of the theorem. Any particular inequality will usually hold (or fail to hold) in a longer interval. In developing the work we determined all of these intervals and then, essentially, took their intersection. The computations were done using the system “Derive.”

A rectangle R with sides $a(x)$ and $b(x)$ is denoted by $\langle a(x), b(x) \rangle$. If it happens that $a(x) < b(x)$, then when we say that Q_ℓ is placed on R it will be understood, in the absence of further discussion, that we have verified that $a(x) \leq x^k < b(x)$ and that Q_ℓ is placed so that one of its corners coincides with a corner of R . The resulting uncovered rectangle is then $\langle a(x), b(x) - x^k \rangle$. Similar remarks apply in other situations. For example, if we write that the constraint $x^\ell/(1-x) \geq 2b(x)$ must hold, it is understood that we have verified that $x^\ell < a(x)$.

We conclude this section with a lemma concerning the placement of Q_0, Q_1, Q_2, Q_3 , and Q_4 .

Lemma. *For $\alpha < x \leq t$, in a covering for x there is no loss in supposing that Q_0, Q_1, Q_2, Q_3 , and Q_4 are placed as shown in Fig. 1.*

We omit the straightforward proof which uses only linear constraints of the simplest kind. The lemma remains valid in an interval much longer than $(\alpha, t]$. We can show that it remains valid in $(\alpha, \beta]$ where $\beta = 0.754877666 \dots$ satisfies $\beta^5 + \beta = 1$. It is the number occurring in Theorem 1.

In Fig. 1 there are two uncovered rectangles; the one in the upper right corner, which we denote by U and the one in the lower left corner, which we denote by L . In the subsequent arguments the shapes of U and L change as additional squares are placed, but we continue to denote them by U and L .

4. Proofs of Theorems 3 and 4

Throughout the remainder of this section a, b, c, d, t, u, v, w , and z have the meanings given in Section 1.

Fig. 1

Proof of Theorem 3. We show first that

$$f(x) = x + x^2 + x^5 := F_1(x) \quad \text{for } a \leq x \leq b.$$

That $f(x) \geq F_1(x)$ on $[a, b]$ may be seen as follows: Consider a square Q of side $F_1(x)$. Place Q_0, Q_1, Q_2, Q_3 , and Q_4 as shown in Fig. 1, place Q_5 on U , and Q_6, Q_7, Q_8 , and Q_9 on L . U is then covered and, with $h(x)$ as given in Theorem 4, $L = \langle F_1(x) - h(x), F_1(x) - 1 \rangle$. Now place Q_{10} on L . This gives $L = \langle F_1(x) - h(x), F_1(x) - 1 - x^{10} \rangle$. Next place Q_{11} on L to get $L = \langle F_1(x) - h(x) - x^{11}, F_1(x) - 1 - x^{10} \rangle$. L may then be covered by Q_{12}, Q_{13} , and Q_{14} .

We remark that one of the inequalities that needs to be checked in verifying that Q is covered is that $1 + x^{10} + x^{11} \geq F_1(x)$. This holds for all $x \leq b$. The construction actually shows that $f(x) \geq F_1(x)$ for $a \leq x \leq b$. The reader should also note the tight fit involving Q_1, Q_2 , and Q_5 and recall the second paragraph of Section 3.

It remains to be shown that $f(x) = F_1(x)$ on $[a, b]$. Suppose that $f(x) = F_1(x) + \varepsilon(x)$ where $\varepsilon(x) \geq 0$ for all x in $[a, b]$ and consider a covering of a square Q of side $f(x)$. We may suppose, by the lemma, that Q_0, Q_1, Q_2, Q_3 , and Q_4 are placed as in Fig. 1. Let $G(x) = f(x) - x - x^3 - x^4$, $H(x) = f(x) - x - x^2$, and $M(x) = f(x) - 1$. Then $L = \langle G(x), M(x) \rangle$ and $U = \langle H(x), M(x) \rangle$.

Suppose that Q_5 is placed on L . Then $L = \langle G(x) - x^5, M(x) \rangle$. If Q_6 is placed on L , then there results the constraint $x^7/(1 - x) \geq H(x)$ on U . However, this is false for all $x \in [a, b]$. Thus Q_6 must be placed on U and we get $U = \langle H(x) - x^6, M(x) \rangle$. If Q_7 is placed on L , then L is covered. After placing Q_8 and Q_9 on U we get $U = \langle H(x) - x^6 - x^8 - x^9, M(x) \rangle$. There now results the area constraint $x^{20}/(1 - x^2) \geq M(x)(H(x) - x^6 - x^8 - x^9)$. This is false for all x in $[a, b]$. It follows that Q_7 must

be placed on U and we get $U = \langle H(x) - x^6 - x^7, M(x) \rangle$. If Q_8 is placed on U there results on L the linear constraint $x^9/(1-x) \geq G(x) - x^5$. Since this fails to hold on $[a, b]$, we must place Q_8 on L . Then $L = \langle G(x) - x^5 - x^8, M(x) \rangle$. If Q_9 is placed on U , the area constraint on L is $x^{20}/(1-x^2) \geq M(x)(G(x) - x^5 - x^8)$. However, this is false for all x in $[a, b]$. Thus Q_9 must be placed on L and one finds that L is covered. After placing Q_{10} on U we get $U = \langle H(x) - x^6 - x^7, M(x) - x^{10} \rangle$. When Q_{11} is placed on U one of two things may occur. It may happen that no matter how Q_{11} is placed there are two intersecting sides of U not touched by Q_{11} , and this will occur in case $x = b$ and $\varepsilon(b) > 0$, or it may happen that it is possible to place Q_{11} so that it touches both of the longer sides of U . In the first instance there arises the linear constraint $2x^{12} + x^{13}/(1-x) \geq (H(x) - x^6 - x^7) + (M(x) - x^{10})$; but this is false for all x in $[a, b]$. In the second instance we get $U = \langle H(x) - x^6 - x^7 - x^{11}, M(x) - x^{10} \rangle$. After placing Q_{12} on U we get $U = \langle H(x) - x^6 - x^7 - x^{11}, M(x) - x^{10} - x^{12} \rangle$. It may be verified that $x^{13} \leq M(x) - x^{10} - x^{12}$ for all x in $[a, b]$. The inequality is strict for x in $(a, b]$ and also at $x = a$ if $\varepsilon(a) > 0$. If $\varepsilon(a) = 0$, then equality holds at $x = a$, but this is just the statement that $p(a) = 0$. We have, for $a < x \leq b$, the constraint $x^{13}/(1-x) \geq 2(H(x) - x^6 - x^7 - x^{11})$, but this is false for all x in $(a, b]$. It follows that Q_5 must be placed on U and hence that $U = \langle \varepsilon(x), M(x) \rangle$.

It is easy to verify that unless $Q_6, Q_7, Q_8, Q_9, Q_{10}$, and Q_{11} are placed on L , area constraints are violated. After placing Q_6, Q_7, Q_8 , and Q_9 we obtain $L = \langle G(x) - x^6 - x^7 - x^8 - x^9, M(x) \rangle$. After placing Q_{10} we get $L = \langle G(x) - x^6 - x^7 - x^8 - x^9, M(x) - x^{10} \rangle$. No matter how Q_{11} is placed on L one of the longer sides of length $M(x) - x^{10}$ will be uncovered. We may suppose $\varepsilon(x) > 0$ since otherwise we are done. Then, since U is not covered, we get the constraint $x^{12}/(1-x) \geq (M(x) - x^{10}) + M(x)$, but this is false on $[a, b]$, and the proof that $f(x) = F_1(x)$ on $[a, b]$ is complete. That $g(x) \leq 15$ on $[a, b]$ follows from the construction given in the first paragraph. With careful attention to detail one may show that $g(a) = 14$ and $g(x) = 15$ for $a < x \leq b$.

We now show that

$$f(x) = 1 + x^{10} + x^{11} := F_2(x) \quad \text{for } b \leq x \leq c.$$

The covering described in the first paragraph also shows that $f(x) \geq F_2(x)$ for all x in $[b, c]$. The construction actually shows that $f(x) \geq F_2(x)$ for all x in $[b, 1)$. We point out that there is no longer a tight fit involving Q_1, Q_2 , and Q_5 (except at $x = a$) but there is a tight fit involving Q_0, Q_{10} , and Q_{11} .

We now show that $f(x) = F_2(x)$ for $b \leq x \leq c$. Suppose that $f(x) = F_2(x) + \varepsilon(x)$ where $\varepsilon(x) \geq 0$ for all x in $[b, c]$ and consider a covering of a square Q of side $f(x)$. We may suppose that Q_0, Q_1, Q_2, Q_3 , and Q_4 are placed as in Fig. 1. Let $G(x) = f(x) - x - x^3 - x^4$, $H(x) = f(x) - x - x^2$, and $M(x) = f(x) - 1$. Then $L = \langle G(x), M(x) \rangle$ and $U = \langle H(x), M(x) \rangle$.

The proof that Q_5 must be placed on L now follows, word for word, that given in the first part of the proof up to the placement of Q_{10} . At this stage we have, as before, $U = \langle H(x) - x^6 - x^7, M(x) - x^{10} \rangle$. No matter how Q_{11} is placed on U it will fail to touch two intersecting sides. Thus we have the constraint $2x^{12} + x^{13}/(1-x) \geq (H(x) - x^6 - x^7) + M(x) - x^{10}$, but this is false for x in $[b, c]$. It follows that Q_5 must be placed on U . Nothing is lost in supposing that $\varepsilon(x)$ is such that U is now covered.

It remains to be shown that it is not possible to cover L . We find that after placing Q_6, Q_7, Q_8, Q_9 , and Q_{10} that $L = \langle G(x) - x^6 - x^7 - x^8 - x^9, M(x) - x^{10} \rangle$. Equivalently, $L = \langle f(x) - h(x), x^{11} + \varepsilon(x) \rangle$. If Q_{11} is not placed at a corner of L we get the constraint

$$2x^{12} + 2x^{13} + \frac{x^{14}}{1-x} \geq (f(x) - h(x)) + 2(x^{11} + \varepsilon(x)),$$

but this is false for all x in $[b, c]$. Thus Q_{11} is placed at a corner of L . Let P be the corner of L at which the sides not touched by Q_{11} intersect. If Q_{12} is not placed at P we get the constraint

$$x^{12} + 2x^{13} + \frac{x^{14}}{1-x} \geq (f(x) - h(x)) + (x^{11} + \varepsilon(x)),$$

but this is false for all x in $[b, c]$. We may therefore suppose that Q_{12} is placed on L so that one of its corners coincides with P . One may readily check that Q_{11} and Q_{12} intersect so that there now remain two uncovered rectangles: $L_1 = \langle f(x) - h(x) - x^{12}, \varepsilon(x) \rangle$ and $L_2 = \langle f(x) - h(x) - x^{11}, x^{11} - x^{12} + \varepsilon(x) \rangle$. For any x for which $\varepsilon(x) = 0$ the desired conclusion holds. Thus we may suppose $\varepsilon(x) > 0$ and L_1 is not degenerate. We note also that L_1 and L_2 are sufficiently separated that no unplaced square can cover parts of each.

Suppose that Q_{13} is placed on L_1 . We then have $L_1 = \langle f(x) - h(x) - x^{12} - x^{13}, \varepsilon(x) \rangle$. If Q_{14} is placed on L_2 we get $L_2 = \langle f(x) - h(x) - x^{11}, x^{11} - x^{12} - x^{14} + \varepsilon(x) \rangle$. However, then the area constraint

$$\frac{x^{30}}{1-x^2} \geq (f(x) - h(x) - x^{11})(x^{11} - x^{12} - x^{14} + \varepsilon(x))$$

fails at each x in $[b, c]$. Thus Q_{14} must be placed on L_1 and we get $L_1 = \langle f(x) - h(x) - x^{12} - x^{13} - x^{14}, \varepsilon(x) \rangle$. One now gets the following linear constraint on $L_1 \cup L_2$:

$$\frac{x^{15}}{1-x} \geq (f(x) - h(x) - x^{12} - x^{13} - x^{14}) + 2(x^{11} - x^{12} + \varepsilon(x)).$$

This is false for all x in $[b, c]$. It follows that Q_{13} must be placed on L_2 and one finds then that

$$L_2 = \langle f(x) - h(x) - x^{11}, x^{11} - x^{12} - x^{13} + \varepsilon(x) \rangle.$$

One may verify that unless Q_{14}, Q_{15} , and Q_{16} are placed on L_1 , linear constraints are violated. We therefore suppose that this is done and hence $L_1 = \langle f(x) - h(x) - x^{12} - x^{14} - x^{15} - x^{16}, \varepsilon(x) \rangle$. If Q_{17} is placed on L_1 , a linear constraint on L_2 is violated. Thus Q_{17} is placed on L_2 and we find that $L_2 = \langle f(x) - h(x) - x^{11}, x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x) \rangle$. If Q_{18} is placed on L_1 we obtain the linear constraint $x^{19}/(1-x) \geq 2(x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x))$ on L_2 . However, this is found to be false on $[b, c]$. Thus Q_{18} must be placed on L_2 . Now for all x in $[b, c]$, $x^{11} - x^{12} - x^{13} - x^{17} < x^{18} \leq f(x) - h(x) - x^{11}$. The inequality on the right is strict for all x in $[b, c]$ and it is strict at c unless $\varepsilon(c) = 0$, in which case equality holds. In fact, the assertion that equality holds at c when $\varepsilon(c) = 0$ is just the assertion that $r(c) = 0$, where $r(x)$ is the polynomial given in the statement of the theorem. That is, it is at this stage that the equation defining c emerges.

On placing Q_{18} on L_2 we obtain $L_2 = (f(x) - h(x) - x^{11} - x^{18}, x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x))$. The argument now splits into three cases, depending on the shape of L_2 . Consider the equations

$$f(x) - h(x) - x^{11} - x^{18} = x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x)$$

and

$$f(x) - h(x) - x^{11} - x^{18} - x^{19} - \varepsilon(x) = 0.$$

$z_1 = 0.589480093 \dots$ is a root of the first equation and $z_2 = 0.589485158 \dots$ is a root of the second. Note that $b < z_1 < z_2 < c$ and that, when $x = z_1$, L_2 is a square.

Case 1: $x \in [b, z_1)$. The following inequality holds:

$$x^{19} < x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x) < f(x) - h(x) - x^{11} - x^{18}.$$

Thus, no matter how Q_{19} is placed on L_2 it does not touch two intersecting sides of L_2 . It follows that we have the constraint (on $L_1 \cup L_2$)

$$\begin{aligned} \frac{x^{19}}{1-x} &\geq (x^{11} - x^{12} - x^{13} - x^{17}) + (f(x) - h(x) - x^{11} - x^{18}) \\ &\quad + (f(x) - h(x) - x^{12} - x^{14} - x^{15} - x^{16}). \end{aligned}$$

However, this is false for x in $[b, z_1)$.

Case 2: $x \in [z_1, z_2)$. We then have

$$x^{19} < f(x) - h(x) - x^{11} - x^{18} \leq x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x).$$

The second inequality is strict except when $x = z_1$; that is, except when L_2 is a square. We then get the following linear constraint on $L_1 \cup L_2$:

$$\frac{x^{19}}{1-x} \geq 2(x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x)) + (f(x) - h(x) - x^{12} - x^{14} - x^{15} - x^{16}).$$

However, this is false for all x in $[z_1, z_2)$.

Case 3: $x \in [z_2, c]$. The following inequality now holds:

$$f(x) - h(x) - x^{11} - x^{18} \leq x^{19} < x^{11} - x^{12} - x^{13} - x^{17} + \varepsilon(x).$$

The inequality on the left is strict except when $x = z_2$. We now obtain on $L_1 \cup L_2$ the constraint

$$\frac{x^{19}}{1-x} \geq 2(f(x) - h(x) - x^{11} - x^{18}) + (f(x) - h(x) - x^{12} - x^{14} - x^{15} - x^{16}).$$

However this is false for all x in $[z_2, c]$. It follows that $f(x) = F_2(x)$ on $[b, c]$.

The covering given at the beginning of the proof shows that $g(x) \leq 15$ on $[b, c]$. One may, with some attention to detail, show that equality holds. $\square$

Proof of (Part of) Theorem 4. We first describe coverings showing that $f(x)$ is at least as large as the values given in the theorem. For x in $[u, v]$ consider a square of side $h(x) + x^{11}$ and place $Q_0, Q_1, \dots, Q_8, Q_9$ as in the covering described in the opening paragraph of the proof of Theorem 3. Then U is covered and we have $L = \langle x^{11}, h(x) + x^{11} - 1 \rangle$. Place Q_{10} and Q_{11} on L to get $L = \langle x^{11}, h(x) - x^{10} - 1 \rangle$. Now L may be covered by Q_{12}, Q_{13} , and Q_{14} . For x in $[v, w]$ consider a square of side $1 + x^9$ and place $Q_0, Q_1, \dots, Q_{11}$ in the manner described above. U is then covered and we have $L = \langle 1 + x^9 - h(x), 1 + x^9 - x^{10} - x^{11} \rangle$. L may now be covered by Q_{12}, Q_{13} , and Q_{14} . There is, however, some waste in part of the interval. There are numbers w_1 and w_2 , $v < w_1 < w_2 < w$, such that for x in $[v, w_1)$ one needs all three squares, for $x \in [w_1, w_2)$ one can get by with Q_{12} and Q_{13} , and for x in $[w_2, w]$ one can get by with Q_{12} . For x in $[w, z]$ the situation is more complicated. Consider a square of side $h(x) + x^{16}$. Place $Q_0, Q_1, \dots, Q_8$ in the manner described above. Then U is covered and $L = \langle x^9 + x^{16}, h(x) + x^{16} - 1 \rangle$. Place Q_9 in a corner of L and Q_{10} in the opposite corner. Then there result two uncovered rectangles $L_1 = \langle x^{16}, h(x) + x^{16} - 1 - x^{10} \rangle$ and $L_2 = \langle x^9 + x^{16} - x^{18}, h(x) + x^{16} - 1 - x^9 \rangle$. L_1 may now be covered by Q_{11}, Q_{15} , and Q_{16} . Place $Q_{12}, Q_{13}, Q_{14}, Q_{17}, Q_{18}, Q_{19}, Q_{20}, Q_{21}, Q_{22}, Q_{23}$, and Q_{24} on L_2 . One may show that there are numbers z_1, z_2 , and z_3 , $w < z_1 < z_2 < z_3 < z$, such that for x in $(w, z_1]$, L_2 may be covered by Q_{25} , for x in $(z_1, z_2]$, L_2 may be covered by Q_{25} and Q_{26} , for x in $(z_2, z_3]$, L_2 may be covered by Q_{25}, Q_{26} , and Q_{27} , and for x in $(z_3, z]$, L_2 may be covered by Q_{25}, Q_{26}, Q_{27} , and Q_{28} .

We now outline a proof that $f(x) = 1 + x^9$ on $[v, w]$. Suppose that $f(x) = 1 + x^9 + \varepsilon(x)$ where $\varepsilon(x) \geq 0$ for $v \leq x \leq w$ and consider a covering of a square of side $f(x)$. We may suppose that Q_0, Q_1, Q_2, Q_3 , and Q_4 are placed as in Fig. 1. Let $G(x) = f(x) - x - x^3 - x^4$ and $H(x) = f(x) - x - x^2$. We then have $L = \langle G(x), x^9 + \varepsilon(x) \rangle$ and $U = \langle H(x), x^9 + \varepsilon(x) \rangle$.

Suppose that Q_5 is placed on L . The argument then follows, almost word for word, that given in the proof of Theorem 3 up to the placement of Q_8 . At this stage we have reached $L = \langle G(x) - x^5 - x^8, x^9 + \varepsilon(x) \rangle$ and $U = \langle H(x) - x^6 - x^7, x^9 + \varepsilon(x) \rangle$. There is no loss in supposing that we consider only those x for which $\varepsilon(x) > 0$. If Q_9 is placed on L we get $L = \langle G(x) - x^5 - x^8, \varepsilon(x) \rangle$. There is then the constraint $x^{10}/(1-x) \geq (G(x) - x^5 - x^8) + (x^9 + \varepsilon(x))$, on $L \cup U$, but this is false for all x in $[v, w]$. It follows that Q_9 must be placed on U . This gives $U = \langle H(x) - x^6 - x^7, \varepsilon(x) \rangle$. If Q_{10} is placed on U we obtain on $L \cup U$ the constraint $x^{11}/(1-x) \geq 2(x^9 + \varepsilon(x)) + (H(x) - x^6 - x^7 - x^{10})$, but this is false on $[v, w]$. Thus Q_{10} must be placed on L . Now if $x^{10} > G(x) - x^5 - x^8$ we have the constraint (on $L \cup U$) $x^{11}/(1-x) \geq 2(x^9 + \varepsilon(x) - x^{10}) + (H(x) - x^6 - x^7)$ while, if not, we have $x^{10}/(1-x) \geq 2(x^9 + \varepsilon(x)) + H(x) - x^6 - x^7$. Both inequalities fail at each x in $[v, w]$. It follows that Q_5 must be placed on U . U is now covered, or, at least, there is no loss in supposing that $\varepsilon(x)$ is such that this is so.

After placing Q_6, Q_7 , and Q_8 on L we obtain $L = \langle G(x) - x^6 - x^7 - x^8, x^9 + \varepsilon(x) \rangle$. Equivalently, $L = \langle f(x) - h(x) + x^9, x^9 + \varepsilon(x) \rangle$. If Q_9 is not placed at a corner of L we get the constraint

$$2x^{10} + 2x^{11} + \frac{x^{12}}{1-x} \geq 2(x^9 + \varepsilon(x)) + (f(x) - h(x) + x^9),$$

but this is false in $[v, w]$. Thus Q_9 must be placed at a corner of L . If Q_{10} is not placed at the opposite corner we get the constraint

$$x^{10} + 2x^{11} + \frac{x^{12}}{1-x} \geq (x^9 + \varepsilon(x)) + (f(x) - h(x) + x^9).$$

This is false on $[v, w]$. It follows that Q_{10} must be placed at the opposite corner. Q_9 and Q_{10} then intersect so that there result two uncovered rectangles L_1 and L_2 where $L_1 = \langle f(x) - h(x), x^9 - x^{10} + \varepsilon(x) \rangle$ and $L_2 = \langle f(x) - h(x) + x^9 - x^{10}, \varepsilon(x) \rangle$.

The equation

$$\frac{x^{11}}{1-x} = (f(x) - h(x) + x^9 - x^{10} - \varepsilon(x)) + (x^9 - x^{10})$$

has a root $x = v_1 = 0.590374895 \dots$. Note that $v < v_1 < w$. We have the following constraint on $L_1 \cup L_2$:

$$\frac{x^{11}}{1-x} \geq (f(x) - h(x) + x^9 - x^{10}) + (x^9 - x^{10} + \varepsilon(x)).$$

This is false for all x in $[v, v_1]$. Thus, in all of what follows we may suppose $v_1 \leq x \leq w$. For $x \in [v_1, w]$ we have $x^{15} > f(x) - h(x)$ and hence also $x^\ell > f(x) - h(x)$ for $\ell = 11, 12, 13, 14$. Thus, for these values of ℓ , Q_ℓ may be placed on L_1 so that it touches both of the longer sides. We now distinguish several cases.

Case 1: Q_{11} and Q_{12} are placed on L_2 . One may check that L_2 is then covered. After placing Q_{13} , Q_{14} , and Q_{15} on L_1 we get the area constraint

$$\frac{x^{32}}{1-x^2} \geq (f(x) - h(x))(x^9 - x^{10} - x^{13} - x^{14} - x^{15} + \varepsilon(x)),$$

but this is false for all x in $[v_1, w]$.

Case 2: Q_{11} and Q_{12} are placed on L_1 . One may check that L_1 is now covered. We then have the constraint $x^{13}/(1-x) \geq f(x) - h(x) + x^9 - x^{10}$ on L_2 , but this is false for all x in $[v_1, w]$.

Case 3: Q_{11} on L_1 and Q_{12} on L_2 . We then have $L_1 = \langle f(x) - h(x), x^9 - x^{10} - x^{11} + \varepsilon(x) \rangle$ and $L_2 = \langle f(x) - h(x) + x^9 - x^{10} - x^{12}, \varepsilon(x) \rangle$. It is now easy to show that unless Q_{13} and Q_{14} are placed on L_2 , linear constraints on L_1 are violated. Suppose then that this is done. This gives $L_2 = \langle f(x) - h(x) + x^9 - x^{10} - x^{12} - x^{13} - x^{14}, \varepsilon(x) \rangle$. If Q_{15} is placed on L_2 , then L_2 is covered. We find that for x in $[v_1, w]$ we have $x^{16} \leq f(x) - h(x)$. The inequality is strict on $[v_1, w)$ and it is strict at w unless $\varepsilon(w) = 0$. That equality holds at $x = w$ when $\varepsilon(w) = 0$ is just the defining relation for w . We therefore get the following constraint on L_1 : $x^{16}/(1-x) \geq 2(x^9 - x^{10} - x^{11} + \varepsilon(x))$, but this is false. It follows that Q_{15} is placed on L_1 and hence that $L_1 = \langle f(x) - h(x), x^9 - x^{10} - x^{11} - x^{15} + \varepsilon(x) \rangle$.

If Q_{16} is placed on L_2 there results the linear constraint $x^{17}/(1-x) \geq 2(x^9 - x^{10} - x^{11} - x^{15} + \varepsilon(x))$ on L_1 . This is false for all x in $[v_1, w]$. Thus Q_{16} must be placed on

L_1 . Since we may suppose $\varepsilon(w) > 0$ we then get the constraint

$$\frac{x^{17}}{1-x} \geq (x^9 - x^{10} - x^{11} - x^{15} + \varepsilon(x)) + (f(x) - h(x) + x^9 - x^{10} - x^{12} - x^{13} - x^{14})$$

on $L_1 \cup L_2$, but this fails to hold at each x in $[v_1, w]$.

Case 4: Q_{11} on L_2 and Q_{12} on L_1 . We then have $L_1 = \langle f(x) - h(x), x^9 - x^{10} - x^{12} + \varepsilon(x) \rangle$ and $L_2 = \langle f(x) - h(x) + x^9 - x^{10} - x^{11}, \varepsilon(x) \rangle$. If Q_{13} is placed on L_2 , then L_2 is covered. After placing Q_{14} and Q_{15} on L_1 we get the area constraint

$$\frac{x^{32}}{1-x^2} \geq (f(x) - h(x)) (x^9 - x^{10} - x^{12} - x^{14} - x^{15} + \varepsilon(x)),$$

but this is false on $[v_1, w]$. Thus Q_{13} must be placed on L_1 . We then get $L_1 = \langle f(x) - h(x), x^9 - x^{10} - x^{12} - x^{13} + \varepsilon(x) \rangle$.

Suppose that Q_{14} is placed on L_2 . We then get $L_2 = \langle f(x) - h(x) + x^9 - x^{10} - x^{11} - x^{14}, \varepsilon(x) \rangle$. If Q_{15} is placed on L_2 an area constraint on L_1 is violated. Thus Q_{15} must be placed on L_1 and we get $L_1 = \langle f(x) - h(x), x^9 - x^{10} - x^{12} - x^{13} - x^{15} + \varepsilon(x) \rangle$. If Q_{16} is placed on L_2 an area constraint on L_1 is violated. Thus Q_{16} must be placed on L_1 . The discussion in the first paragraph of Case 3 now implies the linear constraint

$$\frac{x^{16}}{1-x} \geq 2(x^9 - x^{10} - x^{12} - x^{13} - x^{15} + \varepsilon(x)) + (f(x) - h(x) + x^9 - x^{10} - x^{11} - x^{14}),$$

but this is false on $[v_1, w]$. It follows that Q_{14} must be placed on L_1 . Then $L_1 = \langle f(x) - h(x), x^9 - x^{10} - x^{12} - x^{13} - x^{14} + \varepsilon(x) \rangle$.

Suppose that Q_{15} is placed on L_2 . The discussion in the first paragraph of Case 3 then shows that on $L_1 \cup L_2$ we have the constraint

$$\frac{x^{16}}{1-x} \geq 2(x^9 - x^{10} - x^{12} - x^{13} - x^{14} + \varepsilon(x)) + (f(x) - h(x) + x^9 - x^{10} - x^{11} - x^{15}),$$

but this is false on $[v_1, w]$. It follows that Q_{15} must be placed on L_1 . L_1 is then covered. On L_2 we have the constraint

$$\frac{x^{16}}{1-x} \geq f(x) - h(x) + x^9 - x^{10} - x^{11},$$

but this is false on $[v_1, w]$.

It follows that $f(x) = 1 + x^9$ on $[v, w]$. We remark that the argument makes no explicit mention of v . However, the reader who checks the calculations involved in showing that the covering described in the first paragraph is indeed a covering will find that one of the inequalities requires that $x \geq v$.

We do not give the proofs of the first and third parts of Theorem 4, but we do point out that the proof of the third part is especially lengthy. We also remark that once all three parts have been established, the explicit coverings show that $g(x) \leq 29$ on $[v, z]$. In fact, we have shown that the bounds given for $g(x)$ on the various subintervals are the actual values. $\square$

5. Theorems 5 and 6

We first make some general remarks. Suppose that we have found, for a specified x , a partial covering of a square Q of side q with squares $Q_0, Q_1, \dots, Q_{\ell-1}, Q_\ell, Q_{\ell+1}, \dots, Q_{\ell+m-1}$ such that:

- (i) The part of Q left uncovered by $Q_0, Q_1, \dots, Q_{\ell-1}$ is a rectangle R_1 with sides a and b .
- (ii) The part of R_1 left uncovered by $Q_\ell, Q_{\ell+1}, \dots, Q_{\ell+m-1}$ is a rectangle R_2 with sides ax^m and bx^m .

It is then clear that we may use the squares $Q_{\ell+m}, Q_{\ell+m+1}, \dots, Q_{\ell+2m-1}$ to cover all of R_2 but a rectangle R_3 with sides ax^{2m} and bx^{2m} . The procedure may be repeated indefinitely in the obvious way to get a covering of Q . It will then follow that $f(x) \geq q$.

Suppose further that we can show, by arguments similar to those used in the proofs of Theorems 3 and 4, that every covering of Q has properties (i) and (ii). It will then follow that Q (and hence also any square larger than Q) has no covering with finitely many of the squares. It does not immediately follow that $f(x) = q$ but this has nearly been accomplished.

Suppose that $f(x) = q + \delta$, $\delta > 0$, and that there exists a number $\delta_1 \leq \delta$ such that for each ε , $0 < \varepsilon < \delta_1$, it can be shown that every covering of a square Q' of side $q + \varepsilon$ is such that the partial covering of Q' obtained from $Q_0, Q_1, \dots, Q_{\ell-1}, Q_\ell, Q_{\ell+1}, \dots, Q_{\ell+m}$ has the following properties:

- (i)' The part of Q left uncovered by $Q_0, Q_1, \dots, Q_{\ell-1}$ is a rectangle R'_1 with sides $a + \varepsilon$ and $b + \varepsilon$.
- (ii)' The part of R'_1 left uncovered by $Q_\ell, Q_{\ell+1}, \dots, Q_{\ell+m-1}$ is a rectangle R'_2 with sides $ax^m + \varepsilon$ and $bx^m + \varepsilon$.
- (iii)' Q_ℓ must be placed at a corner of R'_1 .

In any particular case the arguments used to establish (i)' and (ii)' will be almost word for word the argument used to prove (i) and (ii). Property (iii)' will emerge at the first step in proving (ii)' and will be a consequence of a simple linear constraint. Now let R''_2 be a rectangle with sides $ax^m + \varepsilon/2$ and $bx^m + \varepsilon/2$ whose boundary lies strictly inside the boundary of R'_2 , but which is otherwise arbitrary. If ε is chosen sufficiently small the constraint used in establishing (iii)' will show that R''_2 (and hence also R'_2) cannot be covered unless $Q_{\ell+m}$ is placed at a corner. Since R''_2 is arbitrary a contradiction results and we have $f(x) = q$.

Some Remarks on Theorem 5. Consider a square Q of side $1 + d^{10} + d^{10}(1 + d^{10})/(1 - d^{10})$ where d satisfies $d^{10} - h(d) + 1 = 0$. Place Q_0, Q_1, Q_2, Q_3 , and Q_4 as shown in Fig. 1 and let L and U have their usual meanings. Place Q_5 on U and Q_6, Q_7, Q_8 , and Q_9 on L . U is then covered and, keeping in mind that $d^{10} - h(d) + 1 = 0$, we find that

$$L = \left\langle \frac{d^{10}(1 + d^{10})}{1 - d^{10}}, d^{10} + \frac{d^{10}(1 + d^{10})}{(1 - d^{10})} \right\rangle.$$

If we now place Q_{10} on L we see that L becomes a square of side $d^{10}(1+d^{10})/(1-d^{10})$. Thus (i) in the general discussion is established with $\ell = 10$, $R_1 = L$, and $a = b = d^{10}(1+d^{10})/(1-d^{10})$.

Now observe that the ratio of the side of Q to the side a of R_1 is d^{-10} . It follows that by using the squares $Q_{10}, Q_{11}, \dots, Q_{20}$ we may cover all of R_1 but a square R_2 of side ad^{10} , so that (ii) is established, with $m = 10$, and hence that, by repeating this indefinitely, we get a covering of Q so that $f(d) \geq 1 + d^{10} + d^{10}(1+d^{10})/(1-d^{10})$.

The proofs that every covering of Q satisfies (i) and (ii) and that no larger square can be covered follows the now familiar arguments used in Section 4. No new devices are needed. We suppress the details.

Some Remarks on Theorem 6. We shall again be content with describing the construction of a covering of a square Q of side $1 + t^9/(1 - t^{15})$ satisfying (i) and (ii) for suitable a, b, ℓ , and m .

Consider such a square Q . Place Q_0, Q_1, Q_2, Q_3 , and Q_4 as in Fig. 1. Place Q_5 on U and Q_6, Q_7 , and Q_8 on L . Then U is covered and

$$L = \left\langle 1 + \frac{t^9}{1 - t^{15}} - h(t) + t^9, \frac{t^9}{1 - t^{15}} \right\rangle.$$

This we take as R_1 so that we have $\ell = 9$, $a = 1 + t^9/(1 - t^{15}) - h(t) + t^9$, and $b = t^9/(1 - t^{15})$. Place Q_9 in a corner of Q and Q_{10} in the opposite corner. (We remark that, in a full account of a proof that no square larger than Q can be covered, we would at this point have to show that Q_9 must be so placed, so as to establish (iii).) There now result two uncovered rectangles L_1 and L_2 where

$$L_1 = \left\langle 1 + \frac{t^9}{1 - t^{15}} - h(t), \frac{t^9}{1 - t^{15}} - t^{10} \right\rangle$$

and

$$L_2 = \left\langle 1 + \frac{t^9}{1 - t^{15}} - h(t) + t^9 - t^{10}, \frac{t^9}{1 - t^{15}} - t^9 \right\rangle.$$

Place Q_{11}, Q_{15} , and Q_{16} on L_1 . We find that L_1 is then covered. Place $Q_{12}, Q_{13}, Q_{14}, Q_{17}, Q_{18}, Q_{19}, Q_{20}, Q_{21}, Q_{22}$, and Q_{23} on L_2 . We then obtain

$$L_2 = \left\langle 1 + \frac{t^9}{1 - t^{15}} - h(t) + t^9 - t^{10} - t^{12} - t^{13} - t^{14} - t^{17} - t^{18} - t^{19} - t^{20} - t^{21} - t^{22} - t^{23}, \frac{t^9}{1 - t^{15}} - t^9 \right\rangle.$$

We claim that we may take this to be R_2 ; that is, this rectangle is similar to R_1 , and the scaling factor is t^{15} , so that $m = 15$. It is clear that

$$\frac{t^9/(1 - t^{15}) - t^9}{t^9/(1 - t^{15})} = t^{15}.$$

We also have

$$\frac{1 + t^9/(1 - t^{15}) - h(t) + t^9 - t^{10} - t^{12} - t^{13} - t^{14} - t^{17} - t^{18} - t^{19} - t^{20} - t^{21} - t^{22} - t^{23}}{1 + t^9/(1 - t^{15}) - h(t) + t^9} = t^{15}$$

since this is equivalent to

$$t^{20} + t^{17} - t^{16} + t^{15} + t^{14} + t^{13} + t^{12} + t^{10} - t^9 + t^8 + t^7 + t^6 + t^4 + t^3 + t - 1 = 0$$

and this is the defining relation for t . The covering therefore satisfies (ii).

We conclude by elaborating a little on some remarks made earlier. While we have been able to obtain a fairly large number of results like Theorems 3 and 4 giving values of $f(x)$ for x in certain intervals we have no such results for any intervals contained in $(\beta, 1)$ where β is the number given in the paragraph following the lemma. We cannot show that the lemma remains valid outside $(\alpha, \beta]$ and, without it, the analysis becomes unmanageable. In this regard it would be helpful to find a solution to the equation $f(x) = 1 + x$ and to show that a covering for x need not have Q_0 in a corner and Q_1 in the opposite corner.

We have been able to show that the set A of values of x for which $g(x) = \infty$ is infinite. There are infinitely many polynomials whose roots satisfy some rather remarkable equations (analogous to that satisfied by t). Each of these yields a number in A and, in fact, all of the numbers that we have shown to belong to A arise in this way. It would be of interest to know whether A contains any transcendental numbers. We know of only one well-known number in A ; namely, the golden ratio $\tau = (\sqrt{5} - 1)/2$, a number with countless geometric properties.

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