The Euler Characteristic Is the Unique Locally Determined Numerical Invariant of Finite Simplicial Complexes which Assigns the Same Number to Every Cone*

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Abstract. We show that a proof given by Levitt in [L] suffices to prove the stronger theorem stated in the title.

Let $\mathcal{S}$ denote the set of finite simplicial complexes, where we consider two complexes to be the same if they are combinatorially equivalent. Say a function

$$\rho: \mathcal{S} \rightarrow R$$

is *locally determined* if there is a function

$$d: \mathcal{S} \rightarrow R$$

such that, for all finite simplicial complexes M ,

$$\rho(M) = \sum_{\text{vertices } v \text{ of } M} d(\text{link}(v))$$

(where $\text{link}(v)$ denotes the link of v in M , and we sometimes write $\text{link}(v, M)$ if there is possible confusion). In this case, we write $\rho = \rho_d$. We are primarily interested in those locally determined functions with topological significance. For example, a natural problem is to find all locally defined functions which only depend on the homotopy type of the underlying complex. This problem was posed, and completely answered, by

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Levitt. In [L] it is proved that

Theorem A [L]. *The only locally determined real-valued homotopy invariants of finite simplicial complexes are multiples of the Euler characteristic.*

It is easy to see that the Euler characteristic is, in fact, locally determined. Namely, define

$$e: S \rightarrow R$$

by setting, for any finite simplicial complex N ,

$$e(N) = 1 + \sum_i \frac{(-1)^{i+1}}{i+2} (\text{number of } i - \text{simplices in } N).$$

Then, for any finite simplicial complex M ,

$$\rho_e(M) = \chi(M),$$

where

$$\chi(M) = \sum_i (-1)^i (\text{number of } i - \text{simplices in } M)$$

is the Euler characteristic of M .

Levitt also proves the following refinement of Theorem A.

Theorem B [L]. *Let ρ_d be any locally determined real-valued invariant of finite simplicial complexes which is a homotopy invariant, then there is a constant c so that $d = c \cdot e$.*

Our goal in this paper is simply to point out that an argument used in [L] suffices to give a simple direct proof of the stronger theorem stated in the title. We quickly review some standard definitions. For any simplicial complex N , $\text{cone}(N)$ is the join of N and a single vertex. We say a simplicial complex M is a cone if $M = \text{cone}(N)$ for some N . In the proof of Theorem B in [L], Levitt essentially proves the following theorem.

Theorem 1. *Let ρ_d be any locally determined real-valued function on the set of finite simplicial complexes. Suppose there is a constant c so that $\rho_d(M) = c$ whenever M is a cone. Then $d = c \cdot e$.*

Before proving this theorem we note that since all cones are homotopy equivalent, every locally determined homotopy invariant satisfies the hypotheses of Theorem 1. Thus Theorem 1 implies Theorem B (and hence Theorem A). It is interesting to observe that Theorem 1 implies that the converse is true. Namely, any locally determined invariant which gives the same answer for every cone is a homotopy invariant. Moreover, every cone is collapsible, and hence all cones are simple-homotopy equivalent (see [C] for definitions). Therefore, the above theorem implies the following strengthening of Levitt's Theorem A.

Corollary 2. *The only locally determined real-valued simple-homotopy invariants of finite simplicial complexes are multiples of the Euler characteristic.*

We actually prove something somewhat more general. Namely, ρ_d need not be defined on all finite simplicial complexes. Say

$$\mathcal{T} \subset \mathcal{S}$$

is *star-closed* if:

- (i) $\mathcal{T}$ contains the simplicial complex consisting of a single vertex.
- (ii) $\mathcal{T}$ is closed under taking stars. That is, if $M \in \mathcal{T}$ and v is a vertex in M , then $\text{star}(v, M) \in \mathcal{T}$.

We will prove the following generalization of Theorem 1.

Theorem 3. *Let $\mathcal{T} \subset \mathcal{S}$ be any star-closed set, and let ρ_d be any locally determined real-valued function on $\mathcal{T}$ such that there is a constant c with $\rho_d(M) = c$ for every cone $M \in \mathcal{T}$. Then $d = c \cdot e$.*

The set of combinatorial manifolds with (possibly empty) boundary is star-closed. Hence we have the corollary

Corollary 4. *Let ρ_d be any locally determined function on the set of combinatorial manifolds with (possibly empty) boundary such that there is a constant c such that $\rho_d(M) = c$ whenever M is a cone, then $d = c \cdot e$.*

This implies Theorem A'' of [L], where there is the stronger hypothesis that ρ_d be a PL homeomorphism invariant.

Proof of Theorem 3. (The reader should note that we are following the proof of Levitt's Theorem B in [L], and we include this proof merely for the sake of being self-contained.) Let ρ_d satisfy the hypotheses of the theorem. Without loss of generality, we may assume that, for every cone M , $\rho_d(M) = 1$. (Namely, if $c \neq 0$ we can consider $\rho_{d'}$ where $d' = (1/c)d$. If $c = 0$ we can consider $\rho_{d'}$ where $d' = d + e$.) Our goal is to show that, for each finite simplicial complex $M \in \mathcal{T}$ and every vertex $v \in M$, $d(\text{link}(v)) = e(\text{link}(v))$. The proof will be by induction on the number $\ell(v)$ (or $\ell(v, M)$) of simplices in the link of v in M .

Suppose $\ell(v) = 0$. Let $M' = \text{star}(v)$. Then M' consists only of the vertex v , and by hypothesis $M' \in \mathcal{T}$. Since $\text{star}(M')$ is a cone (on the empty set), we have

$$1 = \rho_d(M') = d(\text{link}(v, M')).$$

However, $\text{link}(v, M') = \emptyset = \text{link}(v, M)$, so $d(\text{link}(v, M)) = 1$. On the other hand, $1 = e(\text{link}(v))$. Thus, if $\ell(v) = 0$, $d(\text{link}(v)) = e(\text{link}(v))$.

Suppose $d(\text{link}(v)) = e(\text{link}(v))$ whenever $v \in M \in \mathcal{T}$ satisfies $\ell(v) \leq k - 1$. Let $M \in \mathcal{T}$ and let v be a vertex in M satisfying $\ell(v, M) = k$. Let $M' = \text{star}(v, M)$. Then

$M' \in \mathcal{T}$ is a cone, so

$$1 = \rho_d(M') = \sum_{\text{vertices } w \text{ in } M'} d(\text{link}(w, M')). \quad (1)$$

It is easy to see that, for each vertex w of M' , $\ell(w, M') \leq k$. Moreover, if $\ell(w, M') = k$, then $\text{link}(w, M')$ is combinatorially equivalent to $\text{link}(v, M') = \text{link}(v, M)$. Define

$$V_1 = \{\text{vertices } w \text{ in } M' \text{ with } \ell(w, M') < k\},$$

$$V_2 = \{\text{vertices } w \text{ in } M' \text{ with } \ell(w, M') = k\}$$

and let $n = \#V_2$. By induction, for all $w \in V_1$,

$$d(\text{link}(w, M')) = e(\text{link}(w, M')).$$

For all $w \in V_2$,

$$d(\text{link}(w, M')) = d(\text{link}(v, M')) = d(\text{link}(v, M)).$$

Substituting into formula (1), we have

$$1 = n \cdot d(\text{link}(v, M)) + \sum_{w \in V_1} e(\text{link}(w, M')).$$

On the other hand, we already know

$$1 = \chi(M') = \sum_{\text{vertices } w \text{ in } M'} e(\text{link}(w, M')) = n \cdot e(\text{link}(v, M)) + \sum_{w \in V_1} e(\text{link}(w, M')).$$

Hence we must have

$$d(\text{link}(v, M)) = e(\text{link}(v, M)),$$

as desired. □

References

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