Separating Maps of the Lattice E_8 and Triangulations of the Eight-Dimensional Torus

G. Dartois and A. Grigis

Laboratoire d'Analyse, Géométrie et Applications (UMR 7539),
Institut Galilée, Université Paris-Nord,
99 Avenue Jean-Baptiste Clément,
93430 Villetaneuse, France
{dartois, grigis}@math.univ-paris13.fr

Abstract. Our goal is to build triangulations of the eight-dimensional torus with 511 vertices, 511 being the smallest number of vertices of the known triangulations of the torus. We define the separating maps from the lattice E_8 to $\mathbb{Z}/511\mathbb{Z}$ and we establish the list of such maps up to symmetry.

Introduction

In [KL] Kühnel and Lassmann have described a triangulation of the n -dimensional torus $\mathbb{T}^n$, with $2^{n+1} - 1$ vertices. At present this is the smallest number of vertices among the known triangulations of $\mathbb{T}^n$. In [G] we built a new triangulation of $\mathbb{T}^4$ with 31 vertices. This triangulation is built starting from the Hurwitz lattice of integral quaternions.

Here we use the same approach in dimension 8 with the lattice $\Lambda = E_8$ associated to Cayley numbers. We use the notation Λ for the lattice, since in [B] the notation E_8 is reserved for the root system. We use Coxeter's notations α_n , β_n , and γ_n respectively for the simplex, the generalized octahedron or crosspolytope, and the hypercube of $\mathbb{R}^n$. We start from the well-known decomposition of $\mathbb{R}^8$ in α_8 's and β_8 's, with vertices on Λ (see [C] and Section 1 below). We do not follow the method of [G] which would consist in first building triangulations of $\mathbb{R}^8$ and secondly testing whether they pass to the torus. In fact, the number of triangulations of $\mathbb{R}^8$ is much too large and it is impossible to list them.

We therefore define in Section 2 the separating maps from Λ to $\mathbb{Z}/511\mathbb{Z}$. These maps are group homomorphisms which take distinct values on the elements of the root system.

By passing to the quotient, each separating map determines a torus $\mathbb{T}^8$ decomposed into α_8 's and β_8 's with 511 vertices.

The search for separating maps was made by computer. We say that two separating maps φ and $\lambda\varphi \circ R$ are equivalent, where λ is an invertible element of $\mathbb{Z}/511\mathbb{Z}$ and R is an element of the symmetry group of the lattice. We obtain 26 classes of separating maps. For both the lattice A_2 in dimension 2 and the lattice D_4 in dimension 4 there is a single class. These classes lead respectively to the triangulation of the two-dimensional torus with 7 vertices and to the triangulation of the four-dimensional torus with 31 vertices described in [G].

With a separating map we obtain a decomposition of $\mathbb{T}^8$ in α_8 's and β_8 's, and we try to get invariant triangulations of $\mathbb{T}^8$ with 511 vertices by cutting the β_8 's into simplices by the standard process. Among the 26 classes of separating maps, there is one for which this triangulation is impossible. For the 25 other ones we show in Section 3, by using a sorting algorithm, that it is possible to obtain triangulations of the torus. Even though the number of separating maps up to symmetry is reduced to 26, the number of induced triangulations is so large that it is impossible for us to sort them.

In Section 4 we show that similar known results in dimension 2 and 4 can be obtained by our approach.

The idea of using separating maps was already pointed out in [G], where it was applied to a description of the Kühnel–Lassmann triangulation of $\mathbb{T}^n$ using the lattice $\mathbb{Z}^n$. It could certainly be used in other situations. However, for $\mathbb{T}^3$ nothing new can be expected, as was shown in [G]. The application of this method to E_8 is a natural extension of [G], see Section 4.

The question arises as to why we require 511 vertices, rather than fewer of them. This is made clear in the paper. Even if a separating map from E_8 to $\mathbb{Z}/k\mathbb{Z}$ with $240 < k < 511$ existed, this separating map would determine by passing to the quotient a decomposed torus with k vertices. However, it is obvious that it would be impossible to obtain a corresponding triangulation, since the number of necessary diagonals would be incorrect. This is similar to the following fact for the torus T^2 : there exists a decomposition of T^2 into five square with five vertices, but it cannot generate a triangulation.

We obtain many invariant triangulations with 511 vertices. Consider any one of them. Its symmetry group contains a subgroup isomorphic to $\mathbb{Z}/511\mathbb{Z}$, the group of translations. It would be interesting to consider whether it had any other symmetries. However, we have such a large number of triangulations to consider that it does not seem appropriate to do so. Even the simpler problem of finding the symmetry group of the 26 decomposed tori which we have found, is not immediate and we postpone this question to a forthcoming paper.

1. Description of E_8

Descriptions of the lattice $\Lambda = E_8$ exist in [CS] and [C]. The root system E_8 is studied in [B]. The lattice can be presented in the odd coordinates system (in [C]) or in the even coordinates system (in [CS]). We choose the presentation in the even coordinates system. For the relation between Λ and the Cayley numbers see [C].

Definition 1.1. The lattice Λ consists in all the points

$$(x_1, \dots, x_8) \quad \text{where } x_i \in \mathbb{Z}, \quad \forall i \quad \text{or} \quad x_i \in \mathbb{Z} + \frac{1}{2}, \quad \forall i, \quad \sum x_i \equiv 0 \pmod{2}.$$

1.1. Norm

Consider the quadratic form

$$q(x) = \frac{1}{2} \langle x, x \rangle = \frac{1}{2} (x_1^2 + \dots + x_8^2).$$

This form takes integer values on Λ . The square norm of an element of Λ is $N(x) = q(x)$. We denote by U_i the set of elements of square norm equal to i , for any integer i . In particular, $U_0 = \{0\}$. We can describe U_i , $1 \leq i \leq 4$, following [C]. We denote as in [C] the unit vectors of the canonical basis of $\mathbb{R}^8$ by $l_1, \dots, l_8$; in [B] they are denoted by $\varepsilon_1, \dots, \varepsilon_8$.

The set U_1 consists of the units in the ring of Cayley numbers. It is equal to the root system E_8 . It contains 240 elements as follows:

- 112 elements $\pm l_1 \pm l_2$,
- 128 elements $\frac{1}{2}(\pm l_1 \pm l_2 \pm \dots \pm l_8)$ with an even number of minus signs.

The set U_2 consists of 2160 elements:

- 16 elements $\pm 2l_1$,
- 1120 elements $(\pm l_1 \pm l_2 \pm l_3 \pm l_4)$,
- 1024 elements $\frac{1}{2}(\pm 3l_1 \pm l_2 \pm \dots \pm l_8)$ with an odd number of minus signs.

The set U_3 contains 6720 elements:

- 1344 elements $\pm 2l_1 \pm l_2 \pm l_3$,
- 1792 elements $(\pm l_1 \pm l_2 \pm \dots \pm l_6)$,
- 3584 elements $\frac{1}{2}(\pm 3l_1 \pm 3l_2 \pm l_3 \pm \dots \pm l_8)$ with an even number of minus signs.

The set U_4 contains 17,520 elements: the 240 doubles of elements of U_1 and the following 17,280 elements:

- 8960 elements $\pm 2l_1 \pm l_2 \pm \dots \pm l_5$,
- 128 elements $(\pm l_1 \pm l_2 \pm \dots \pm l_8)$ with an odd number of minus signs,
- 7168 elements $\frac{1}{2}(\pm 3l_1 \pm 3l_2 \pm 3l_3 \pm l_4 \pm \dots \pm l_8)$ with an even number of minus signs,
- 1024 elements $\frac{1}{2}(\pm 5l_1 \pm l_2 \pm \dots \pm l_8)$ with an odd number of minus signs.

1.2. Quotient

We consider $\Lambda/2\Lambda$, it has $2^8 = 256$ elements or classes. These classes are as follows:

- the class of 0,
- 120 classes containing $\pm x$, $x \in U_1$,
- 135 classes each containing 16 elements of U_2 .

In fact these 135 last classes form the set E considered in Section 3 and are distributed as:

- the class containing $2l_1$,
- 70 classes containing elements like $l_1 + l_2 + l_3 + l_4$,
- 64 classes containing $\frac{1}{2}(3l_1 + l_2 + \dots + l_8)$.

More precisely the class containing $2l_1$ contains the 16 elements $\pm 2l_i$. A typical class among the 70 classes consists of the 8 elements $\pm l_1 \pm l_2 \pm l_3 \pm l_4$ with an even number of minus signs and of the 8 elements $\pm l_5 \pm l_6 \pm l_7 \pm l_8$ with an odd number of minus signs. Finally a typical class among the 64 classes consists of the 16 elements $\pm \frac{1}{2}(l_1 + l_2 + \dots - 3l_i + \dots + l_8)$.

1.3. Symmetries

Let σ be the symmetry $(x_1, x_2, \dots, x_8) \mapsto (-x_1, x_2, \dots, x_8)$. The lattice Λ is not invariant under σ . In [CS] the lattice $\overline{\Lambda} = \sigma(\Lambda)$ is called E_8 in the system of odd coordinates. In order to find the elements of $\overline{\Lambda}$ it is sufficient to exchange everywhere the expressions “with an even number of minus signs” and “with an odd number of minus signs.”

The symmetry group of Λ , denoted by W , is described in [B], [C], and [CS]. It is a reflection group because it is the Weyl group of the root system E_8 . The group W acts transitively on U_1 and U_2 . It also acts transitively on the 135 elements of $\Lambda/2\Lambda$ whose representative elements belong to U_2 . Its order is $|W| = 135 \times 8! \times 2^7$. One can show that W does not contain the symmetry group of the cube γ_8 . As a matter of fact the symmetry σ does not leave Λ invariant. However, if we add σ to W we get a group of order $135 \times 8! \times 2^8$ which leaves the set $\Lambda \cup \overline{\Lambda}$ invariant.

Later we will have to consider the group W_a generated by W and the translations by vectors belonging to the lattice 511Λ . This is a subgroup of the affine Weyl group. The following proposition can be deduced from [B]:

Proposition 1.2. *A fundamental domain of W_a is given by the alcove A defined by the following inequalities:*

$$\begin{aligned} |x_1| &< x_2 < x_3 < x_4 < x_5 < x_6 < x_7, \\ x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &< x_1 + x_8, \\ x_7 + x_8 &< 511. \end{aligned}$$

As a matter of fact this domain is obtained by dilating by a factor of 511 the alcove of the affine Weyl group described in [B].

1.4. The Tiling of $\mathbb{R}^8$

We now consider the Delaunay tiling associated to Λ . It is the dual tiling of Voronoi tiling. Recall that the Voronoi cell of an element γ of Γ is the set of points of $\mathbb{R}^8$ closer

to γ than any other point of Λ . The vertices of the Delaunay tiling are the points of the lattice Λ . This tiling is described by Coxeter in [C]. The cells are simplices α_8 and crosspolytopes β_8 . As a matter of fact we consider the polytope whose vertices are the 240 elements of U_1 . Its 7-faces are α_7 's and β_7 's. Each face α_7 forms together with O a cell α_8 of the tiling. On the other hand, each β_7 together with O and the element of U_2 symmetric to O with respect to the hyperplane containing the β_7 , is a cell β_8 of the tiling. Finally in this tiling O is a vertex of 2160 β_8 's and 17,280 α_8 's. Recall that the centers of these 2160 β_8 's, which are the deep holes close to O of the lattice, are the elements of $\frac{1}{2}U_2$ and the centers of the 17,280 α_8 's are the elements of $\frac{1}{3}U_4$ different from $\frac{2}{3}U_1$. The latter are the shallow holes of the lattice. In order to fix the idea we describe a typical α_8 : $\{l_1 + l_s \ (s = 2, \dots, 8), \frac{1}{2}(l_1 + \dots + l_8), O\}$ has center $\frac{5}{6}l_1 + \frac{1}{6}l_2 + \dots + \frac{1}{6}l_8$. Similarly, a typical β_8 is $\{l_1 \pm l_s \ (s = 2, \dots, 8), O, 2l_1\}$, with center l_1 .

In this tiling of $\mathbb{R}^8$, each α_8 has a common hyperface with nine β_8 's. Each β_8 has 2^7 hyperfaces in common with another β_8 . Each β_8 containing O has its center in $\frac{1}{2}U_2$. Each α_8 containing O has its center in $\frac{1}{3}U_4$, and the corresponding element of U_4 does not belong to $2U_1$.

Finally, we recall that according to Coxeter [C] the group W acts transitively on the 17,280 α_8 's containing O and that the subgroup which leaves such an α_8 invariant, also acts transitively on the eight vertices of this α_8 distinct from O .

2. Separating Maps

We have seen in Section 1.4 that the Delaunay tiling is a decomposition of $\mathbb{R}^8$ into polytopes α_8 and β_8 , invariant under translation by elements of the lattice Λ . In order to obtain a triangulation of $\mathbb{R}^8$, it is sufficient to cut the β_8 's into simplices. To do this we use the standard method which consists in choosing one of the eight diagonals and cutting the β_8 in 2^7 simplices, each of them containing the two extremities of the diagonal and the vertices of a face α_6 of the β_7 lying in the hyperplane orthogonal to the diagonal in its middle.

2.1. The Classes of β_8

We consider a β_8 of the Delaunay decomposition of $\mathbb{R}^8$ associated to Λ . It possesses eight diagonals. We translate each of these diagonals in order that an extremity coincides to O . The other end is a point of U_2 . The remarkable fact is that the 16 elements of U_2 so obtained are in the same class in $\Lambda/2\Lambda$. One can then define the class of the β_8 as the class of its diagonals. It is one of the 135 classes of $\Lambda/2\Lambda$ consisting of elements of U_2 . Two β_8 's of the same class then have the same directions of diagonals.

In order to build an invariant triangulation of $\mathbb{R}^8$ it is sufficient to choose, for each of the 135 classes of β_8 , one of the eight directions of the corresponding diagonals, and then to cut each β_8 in the standard way with the help of the chosen diagonal. In such a triangulation the link of O has 510 vertices: the 240 elements of U_1 and the 2×135 elements of U_2 which are extremities of diagonals of β_8 having O as the other extremity.

We then get 8^{135} invariant triangulations of $\mathbb{R}^8$. Of course they are not distinct modulo the action of the Weyl group W but to sort them, even up to symmetry, seems out of reach.

2.2. Definition of Separating Maps

We come to the triangulation of the torus. According to Grigis [G], to test if an invariant triangulation of $\mathbb{R}^8$ passes to the torus $\mathbb{T}^8$, it is sufficient to see if there exists a linear map φ from E_8 to $\mathbb{Z}/511\mathbb{Z}$ which takes different values on O and the 510 vertices of the link of O . As we cannot list the invariant triangulations of $\mathbb{R}^8$ and test them one after the other, we begin by selecting some linear maps from Λ to $\mathbb{Z}/511\mathbb{Z}$.

Definition 2.1. Let φ be a linear map from Λ to $\mathbb{Z}/511\mathbb{Z}$. We say that φ is a separating map if φ takes distinct values on all the elements of U_1 .

We have the following proposition:

Proposition 2.2. The map φ from Λ to $\mathbb{Z}/511\mathbb{Z}$ is a separating map if and only if it does not vanish on $U_1 \cup U_2 \cup U_3$.

Proof. We first remark that every element of $U_1 \cup U_2 \cup U_3$ can be written as the difference of two distinct elements of U_1 . As a matter of fact we have

$$\begin{aligned} l_1 + l_2 &= (l_1 + l_3) - (l_3 - l_2), \\ \frac{1}{2}(l_1 + \dots + l_8) &= (l_1 + l_2) - \frac{1}{2}(l_1 + l_2 - l_3 - \dots - l_8), \\ 2l_1 &= (l_1 + l_2) - (l_2 - l_1), \\ l_1 + l_2 + l_3 + l_4 &= (l_1 + l_2) - (-l_3 - l_4), \\ \frac{1}{2}(-3l_1 + l_2 + \dots + l_8) &= \frac{1}{2}(-l_1 - l_2 + l_3 + \dots + l_8) - (l_1 - l_2), \\ 2l_1 + l_2 + l_3 &= (l_1 + l_2) - (-l_1 - l_3), \\ l_1 + l_2 + l_3 + l_4 + l_5 + l_6 &= \frac{1}{2}(l_1 + l_2 + \dots + l_8) - \frac{1}{2}(-l_1 - \dots - l_6 + l_7 + l_8), \\ \frac{1}{2}(3l_1 + 3l_2 + l_3 + \dots + l_8) &= \frac{1}{2}(l_1 + l_2 + \dots + l_8) - (-l_1 - l_2). \end{aligned}$$

The proposition now follows immediately, since the difference between two elements of U_1 which are not opposed lives in $U_1 \cup U_2 \cup U_3$. $\square$

Separating maps are interesting for the following reason:

Proposition 2.3. If φ is a separating map, the torus $\mathbb{T}^8$ quotient of $\mathbb{R}^8$ by $\text{Ker } \varphi$ is cut into $511 \times 1920 \alpha_8$'s and $511 \times 135 \beta_8$'s. This decomposition has 511 vertices.

Proof. As a matter of fact, in the Delaunay tiling of $\mathbb{R}^8$ each α_8 or β_8 which has O as a vertex, lives on the torus because φ does not vanish either on U_1 or on U_2 . On the torus we have the 241 points image of U_0 and U_1 . The 270 other values of φ are

achieved on some elements of U_2 . The torus $\mathbb{T}^8$ then inherits an invariant decomposition which contains $(511 \times 17,280)/9 = 511 \times 1920$ α_8 's and $(511 \times 2160)/16 = 511 \times 135$ β_8 's. $\square$

With the help of a separating map we can build a combinatorial torus with 511 vertices, decomposed into α_8 's and β_8 's, locally isomorphic to the Delaunay tiling of $\mathbb{R}^8$ we have described above.

We define an equivalence relation between separating maps.

Definition 2.4. We say that two separating maps φ and ψ are equivalent if there exists a symmetry $w \in W$ and an invertible element λ of $\mathbb{Z}/511\mathbb{Z}$ such that $\psi = \lambda\varphi \circ w$.

If φ is a separating map, $\lambda\varphi \circ w$ is also a separating map so that we have defined an equivalence relation. This equivalence has a geometric version as we see in the following:

Proposition 2.5. *Two decomposed tori built with help of two separating maps are isomorphic if and only if the separating maps are equivalent.*

We consider two tori associated to two equivalent separating maps. It is clear that they are isomorphic. Conversely, we consider two isomorphic decomposed combinatorial torus $\mathbb{R}^8/\text{Ker } \varphi$ and $\mathbb{R}^8/\text{Ker } \psi$. There exists a symmetry w which maps $\text{Ker } \varphi$ onto $\text{Ker } \psi$ and ψ is then of the form $\lambda\varphi \circ w$, so that φ and ψ are equivalent.

2.3. Finding the Separating Maps

With the help of a bank of computers we were able to find and sort all the separating maps. We describe, without entering into programming subtleties, the strategy which we have used.

We represent a linear map φ from Λ to $\mathbb{Z}/511\mathbb{Z}$ by an element y of $\Lambda/511\Lambda$, by setting

$$\varphi(x) = \langle x, y \rangle = \sum_{i=1}^8 x_i y_i.$$

This is possible because the lattice E_8 is self-dual. There exist 511^8 such elements.

In order to be a separating map, φ must not vanish on $U_1 \cup U_2 \cup U_3$. The point y which represent φ must therefore avoid the hyperplane orthogonal to each element of this set. It must thus avoid $\frac{1}{2}(240 + 2160 + 6720) = 4560$ hyperplanes. As 4560 is very much larger than 511, it is not obvious at first sight that there exist separating maps.

If $w \in W$, then $\varphi \circ w$ is equivalent to φ and is represented by $w^{-1}(y)$. We then get all the separating maps by first searching for those which have their representative in the alcove A , then by allowing the group W to act. These steps were carried out by computer. The final result is the following:

Theorem 2.6. *There exist 26 equivalence classes of separating maps. Each of these classes counts 72 representatives in the alcove A .*

Table 1

1	(10, 18, 21, 22, 28, 47, 111, 247)
2	(2, 14, 17, 28, 50, 57, 75, 255)
3	$\frac{1}{2}(5, 41, 55, 61, 79, 95, 167, 509)$
4	$\frac{1}{2}(21, 35, 41, 43, 59, 95, 219, 495)$
5	$\frac{1}{2}(9, 35, 37, 59, 69, 75, 89, 535)$
6	$\frac{1}{2}(25, 33, 49, 51, 61, 113, 191, 501)$
7	(-1, 12, 24, 27, 31, 32, 106, 251)
8	$\frac{1}{2}(-7, 19, 21, 57, 73, 81, 103, 533)$
9	(4, 11, 14, 15, 31, 52, 64, 271)
10	$\frac{1}{2}(27, 37, 39, 63, 67, 125, 147, 527)$
11	(4, 10, 14, 15, 27, 30, 63, 275)
12	(4, 5, 11, 15, 38, 40, 52, 275)
13	(-10, 14, 17, 29, 30, 35, 103, 260)
14	$\frac{1}{2}(-9, 15, 19, 45, 85, 93, 107, 545)$
15	(-7, 10, 14, 26, 41, 51, 52, 279)
16	$\frac{1}{2}(7, 9, 19, 57, 81, 111, 197, 531)$
17	(4, 11, 22, 38, 42, 67, 79, 267)
18	(-10, 14, 15, 30, 41, 48, 51, 279)
19	$\frac{1}{2}(11, 21, 49, 55, 57, 145, 199, 523)$
20	(-9, 10, 16, 21, 24, 72, 89, 267)
21	(6, 12, 14, 17, 27, 31, 63, 282)
22	$\frac{1}{2}(31, 49, 65, 69, 71, 101, 159, 539)$
23	(-4, 11, 16, 20, 33, 34, 91, 273)
24	$\frac{1}{2}(5, 19, 21, 65, 95, 123, 131, 545)$
25	$\frac{1}{2}(-13, 47, 51, 67, 91, 97, 119, 557)$
26	$\frac{1}{2}(11, 25, 49, 53, 55, 145, 199, 531)$
	$\frac{1}{2}(11, 31, 65, 69, 77, 79, 197, 539)$
	(2, 3, 9, 18, 28, 41, 105, 272)
	$\frac{1}{2}(19, 23, 41, 71, 77, 85, 221, 531)$
	$\frac{1}{2}(-25, 37, 45, 51, 75, 111, 191, 539)$
	(5, 11, 15, 16, 28, 30, 66, 285)
	$\frac{1}{2}(25, 49, 51, 61, 83, 113, 179, 539)$

In Table 1 we give the list of representatives with minimal square norm, belonging to A , for each of these classes, numbered from 1 to 26. In general this representative is unique, but for some classes there are two or three representatives with the same minimal square norm.

There are two remarkable facts that we have found out, and for which we have no a priori proof, only the proof by brute force. The first fact is that every element representing a separating map has a square norm divisible by 511. This fact was also noticed in dimension 4 in [G], with 31 in place of 511 and D_4 in place of E_8 , and it is also true in dimension 2 with 7 and A_2 , see Section 4. The second fact relates to the number of representatives of a class in the alcove A . There are 432 invertible elements in $\mathbb{Z}/511\mathbb{Z}$ because $511 = 7 \times 73$ and $432 = 6 \times 72$. As -1 is invertible in $\mathbb{Z}/511\mathbb{Z}$ and belongs also to the group W , an equivalence class should count $216 \times |W|$ elements and therefore 216 representatives in A . In fact there are three times less elements because there exists a number n of order 6 in the multiplicative group of invertible elements of $\mathbb{Z}/511\mathbb{Z}$ such that if x is a representative, then $nx = w(x)$ with $w \in W_a$. For half of the

classes $n = 82$, for the other half $n = 138$. This fact also occurs in dimension 2, where we find one representative in the alcove in place of the expected three, and in dimension 4 where we find 5 representatives instead of 15, see Section 4.

We consider an equivalence class of a separating map, and its 72 representatives in the alcove A . We know that the group W_a is generated by the reflections associated to the walls of the alcove A . Otherwise if x is one of the 72 points considered above, all the others are of the form $y = \lambda w(x)$ with λ invertible and $w \in W$. Therefore if we consider the billiard trajectory in the alcove starting from O and passing through x , it passes after reflections on the walls through each of the 72 points and comes again in O . In fact this trajectory passes six times at each of the 72 points.

Finally we can estimate the rarity of separating maps. The total number of separating maps is $26 \times 72 \times |W|$. Compared with 511^8 the proportion is $1/(3.56 \times 10^9)$. This is so small that it explains why for a long time we missed them and were not sure of their existence.

3. Triangulation of Decomposed Tori

We consider a separating map φ from Λ to $\mathbb{Z}/511\mathbb{Z}$. It has a corresponding torus $\mathbb{T}^8$ decomposed into polytopes α_8 and β_8 . This decomposition is invariant under translations of the lattice. We say that φ generates a triangulation of $\mathbb{T}^8$ if there exists an invariant triangulation of $\mathbb{T}^8$ induced by the standard cutting of the β_8 's from the decomposition in α_8 's and β_8 's associated to φ .

Let E be the set of 135 classes of $\Lambda/2\Lambda$ containing elements of U_2 . Let F be the set of the 135 values up to sign not achieved by φ on the set $U_0 \cup U_1$. For each $x \in U_2$ such that $\varphi(x)$ is not taken on $U_0 \cup U_1$, we consider the couple $(c(x), \varphi(x))$ where $c(x)$ is the class of x in $\Lambda/2\Lambda$. We first remark that $(c(x), \varphi(x)) = (c(y), \varphi(y))$ implies $x = y$, because if $c(x) = c(y)$ and $x \neq y$, then $x - y \in 2U_1$ and $\varphi(x) \neq \varphi(y)$. We consider the relation $\mathcal{R}$ between E and F , made up of these couples.

Proposition 3.1. *There exists a triangulation of $\mathbb{T}^8$ generated by φ if and only if there exists a bijective map from E to F included in the relation $\mathcal{R}$.*

Proof. Consider an invariant triangulation induced from φ . It has 270 diagonals of β_8 starting from O , that is to say 135 directions. To each of them corresponds its class, i.e., an element of E . On the other hand, the second extremity of a diagonal beginning at O belongs to U_2 and φ takes at this point a value not achieved on the set $U_0 \cup U_1$, otherwise two points of the link of O would coincide on the torus, and we would not get a triangulation. Therefore to each direction of a diagonal corresponds an element of F . There thus exists a bijective map from E to F included in $\mathcal{R}$.

Conversely, if such a bijective map exists we consider the 135 couples $(c(x), \varphi(x))$ constituting its graph, the points x being in U_2 . We select the 135 diagonals joining O to these points x . For each β_8 of the decomposition of T^8 , we select the diagonal having the same direction as one of these 135's, and we triangulate the β_8 's in the standard way. We then get an invariant triangulation of T^8 . $\square$

Table 2

	0	1	2	3	4	5	6	7	8
1	0	0	0	12	15	27	48	27	6
2	0	0	0	3	24	30	48	27	3
3	0	0	0	6	15	33	54	27	0
4	0	0	0	12	9	33	48	33	0
5	0	0	3	3	9	33	69	18	0
6	0	0	3	6	18	33	33	39	3
7	0	0	3	3	15	33	57	18	6
8	0	0	3	6	12	39	42	27	6
9	3	0	0	3	18	39	39	27	6
10	0	0	3	3	6	48	54	12	9
11	0	0	0	6	12	42	45	27	3
12	0	0	0	3	15	33	63	21	0
13	0	3	0	0	27	21	57	27	0
14	0	0	0	0	24	33	48	30	0
15	0	0	0	0	30	24	48	33	0
16	0	0	0	6	12	39	51	24	3
17	0	0	0	9	9	42	42	30	3
18	0	0	3	3	9	42	54	18	6
19	0	0	0	6	21	36	33	33	6
20	0	0	0	6	12	39	51	24	3
21	0	0	0	9	15	30	45	36	0
22	0	3	0	0	24	36	33	30	9
23	0	0	0	3	15	42	45	30	0
24	0	0	3	3	30	30	21	36	12
25	0	0	6	0	18	33	39	36	3
26	0	0	3	9	6	36	51	24	6

The problem is then reduced to determining whether the relation $\mathcal{R}$ does or does not contain a bijective map. Before we try to solve this question, we establish two new tables. In Table 2 we have determined for each of the 26 classes of separating maps the number of couples in $E \times F$ which project on each element of E . Each entry of the table describes the number of elements of E which are projections of n couples. We observe immediately that in the case of the separating map number 9, there are three elements of E which are not projections of any couple. It is then clear that for this separating map number 9 there exists no one-to-one map included in the relation from E to F . We then get:

Proposition 3.2. *The separating maps of the class number 9 do not generate an invariant triangulation of T^8 .*

In the Table 3 we have determined the number of couples of $E \times F$ which project onto each element of F . We report in the table the number of elements of F which are projections of n couples. This time there is nothing remarkable which allows us to assert that the relation does not contain any bijective map.

We have used the following algorithm in order to find a bijective map.

Table 3

	4	5	6	7
1	6	54	63	12
2	9	48	66	12
3	9	51	60	15
4	9	48	66	12
5	12	42	69	12
6	9	51	60	15
7	9	48	66	12
8	6	57	57	15
9	9	57	57	12
10	3	60	57	15
11	3	54	69	9
12	3	63	51	18
13	9	54	63	9
14	9	51	57	18
15	0	66	54	15
16	3	57	63	12
17	3	54	69	9
18	0	51	84	0
19	3	54	69	9
20	3	60	57	15
21	9	42	75	9
22	0	66	54	15
23	3	60	57	15
24	0	60	66	9
25	3	54	69	9
26	3	54	69	9

Problem 3.3 (Sorting Problem). Let E and F have the same cardinal. Given a relation $\mathcal{R}$ between E and F satisfying

$$\begin{aligned} \forall x \in E, \quad F_x = \{y \in F; x\mathcal{R}y\} &\neq \emptyset & \text{and} \\ \forall y \in F, \quad E_y = \{x \in E; x\mathcal{R}y\} &\neq \emptyset, \end{aligned} \tag{1}$$

can we find a bijective map from E to F included in $\mathcal{R}$?

Here is the algorithm. For each $x \in E$ we set $p_x = \text{card } F_x$, for each $y \in F$ we set $n_y = \text{card } E_y$ and for each $(x, y) \in E \times F$ we set $\mu(x, y) = \inf(p_x, n_y)$.

Consider a couple (x_0, y_0) which minimize $\mu(x, y)$. In case of equality, we choose a couple which minimize $n_y + p_x$. We replace E by $E' = E \setminus \{x_0\}$, F by $F' = F \setminus \{y_0\}$, and $\mathcal{R}$ by $\mathcal{R}' = \mathcal{R} \cap (E' \times F')$.

We show that if $\mathcal{R}$ contains a bijective map f , then $\mathcal{R}'$ has the same property (1). If there exists $x \in E'$ such that F'_x is void, then we have $F_x = \{y_0\}$ and $\mu(x, y_0) = \mu(x_0, y_0) = 1$. As $x_0\mathcal{R}y_0$ and $x\mathcal{R}y_0$, we have $n(y_0) \geq 2$ and $p(x_0) = p(x) = 1$. This says that $F_{x_0} = F_x = \{y_0\}$ and that $\mathcal{R}$ does not contain any one-to-one map. If there exists $y \in F'$ such that E'_y is void, we have $E_y = \{x_0\}$ and $\mu(x_0, y) = \mu(x_0, y_0) = 1$. As $x_0\mathcal{R}y_0$ and $x_0\mathcal{R}y$, we have $p(x_0) \geq 2$ and $n(y_0) = n(y) = 1$. This says that $E_y = E_{y_0} = \{x_0\}$ and that $\mathcal{R}$ does not contain any onto map.

We can repeat the process until (1) is satisfied as long as there remain elements in E . If we have exhausted the elements of E , we have built a sequence of couples $(x_0, y_0) \cdots (x_{n-1}, y_{n-1})$ which form the graph of a bijective map included in $\mathcal{R}$.

We do not know the answer to the following question: if $\mathcal{R}$ contains a bijective map, does $\mathcal{R}'$ contain a bijective map? In other words we are not sure that the selected couple (x_0, y_0) belongs to the graph of a bijective map included in $\mathcal{R}$. We have an algorithm of reduced complexity, but we do not know whether it always gives, in the general case, a solution.

For our problem we have used this algorithm for each of the 25 classes of separating maps different from the number 9. Every time we have got a bijective map and even a large number.

We then get:

Theorem 3.4. *Each of the separating maps not equivalent to the number 9 generates invariant triangulations of T^8 with 511 vertices.*

The problem of classifying these triangulations seems out of reach, because even though only 25 types of separating maps generating triangulations of the torus exist, each of them generates such a big number of triangulations that it seems out of question to sort them even up to symmetry.

4. Dimension 2 and Dimension 4

We show here how the same strategy can be used in dimension 2 and in dimension 4. We find new proof of known results.

In dimension 2 the considered lattice is $\Lambda = A_2$, it is the triangular lattice. The Delaunay tiling is the triangulation of $\mathbb{R}^2$ by equilateral triangles. The separating maps are the linear maps from Λ to $\mathbb{Z}/7\mathbb{Z}$ which take different values on the six elements of U_1 , which are the six roots of A_2 . Because 2 is invertible in $\mathbb{Z}/7\mathbb{Z}$ one can represent the separating maps by points of $\Lambda/7\Lambda$. The symmetry group of A_2 is larger than the Weyl group of A_2 , in fact it is the Weyl group of G_2 . Therefore we consider the alcove A equal to the 7-dilated alcove of the affine Weyl group of G_2 . We find out that this alcove A contains in its interior only one point of Λ ; this point y represent a separating map. There exists only one equivalence class of separating maps. We remark that y has a square norm divisible by 7. In the alcove A the billard trajectory starting from O and passing through y passes six times at y before returning in O ; at the point B there is a right angle and the trajectory bounces onto itself. In Fig. 1 we show the lattice, the alcove, and the billard trajectory. The invariant triangulation of the torus induced by this separating map is the well-known triangulation of T^2 with seven vertices. The sublattice $\text{Ker } \varphi$ is the triangular lattice with edge Oy . Its Voronoi cell is a hexagon centered in O containing seven points of Λ , which are the elements of $U_0 \cup U_1$.

The dimension 4 case was considered in [G]. The lattice we consider is $\Lambda = D_4$, it is the Hurwitz lattice of quaternions. The associated Delaunay tiling is the decomposition of $\mathbb{R}^4$ into β_4 's, it is the regular tiling denoted $(3, 3, 4, 3)$ by Schläfli. The separating maps are the linear maps from Λ to $\mathbb{Z}/31\mathbb{Z}$ which take distinct values on the 24 elements

Fig. 1 The alcove A and the billard trajectory in dimension 2.

of U_1 which are the 24 roots of D_4 . We can represent the separating maps by points of $\Lambda/31\Lambda$ because 2 is invertible in $\mathbb{Z}/31\mathbb{Z}$. Here again the symmetry group of D_4 is equal to the Weyl group of F_4 and we consider the alcove A equal to the 31-dilated alcove of the affine Weyl group of F_4 . This alcove A is defined by the inequalities

$$x_2 > x_3 > x_4 > 0, \quad x_1 > x_2 + x_3 + x_4, \quad 2x_1 < 31.$$

There is only one equivalence class of separating maps and it has five representatives in the alcove A . The representative $(2, 3, 4, 8) \in \Lambda$, which was given in [G], is not in this alcove. The five representatives living in the alcove A are

$$\frac{1}{2}(17, 7, 5, 3), \quad \frac{1}{2}(29, 13, 9, 5), \quad (14, 4, 2, 1), \quad \frac{1}{2}(29, 15, 7, 1), \quad (15, 7, 2, 1).$$

We remark that all these elements have a square norm divisible by 31. It is easy to see also that every billard trajectory in A beginning at O and passing through one of these five points passes in fact six times at each of the five points before it reaches O again, and the trajectory is divided into 31 equal segments by these 5×6 points. Indeed, if x is a representative in A , $5x$ is equal to $w(x)$ with $w \in W_a$, and 5 is of order 6 in the multiplicative group of $\mathbb{Z}/31\mathbb{Z}$.

In [G] it is seen that there exists up to symmetry a unique invariant triangulation, generated by these separating maps. It is the triangulation G of T^4 with 31 vertices.

References

- [B] Bourbaki, N.: *Eléments de Mathématiques, Groupes et Algèbres de Lie*, Chapters 4–6, Masson, Paris, 1981.
- [CS] Conway, J. H., and Sloane, N. J. A.: *Sphere Packings, Lattices and Groups*, Grundlehren 290, Springer-Verlag, New York, 1988.
- [C] Coxeter, H. S. M.: *Twelve Geometric Essays*, Southern Illinois University Press, Carbondale, IL, 1968.
- [G] Grigis, A.: Triangulation du tore de dimension 4, *Geom. Dedicata* **69** (1998), 121–139.
- [KL] Kühnel, W., and Lassmann, G.: Combinatorial d-tori with a large symmetry group, *Discrete Comput. Geom.* **3** (1988), 169–176.
- [S] Senechal, M.: Tiling the torus and other spaces forms, *Discrete Comput. Geom.* **3** (1988), 55–72.

Received February 26, 1999, and in revised form June 9, 1999.