On the Chromatic Number of $\mathbb{R}^4$

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Abstract The lower bound for the chromatic number of $\mathbb{R}^4$ is improved from 7 to 9. Three graphs with unit distance embeddings in $\mathbb{R}^4$ are described. The first is a 7-chromatic graph of order 14 whose chromatic number can be verified by inspection. The second is an 8-chromatic graph of order 26. In this case the chromatic number can be verified quickly by a simple computer program. The third graph is a 9-chromatic graph of order 65 for which computer verification takes about one minute.

Keywords Unit distance graph · Chromatic number

1 Introduction

The *chromatic number* of the n -dimensional Euclidean space, denoted $\chi(\mathbb{R}^n)$, is the minimum number of colors with which the entire space $\mathbb{R}^n$ can be painted so that every two points at distance one have different colors. The problem has a long and celebrated history and is one of the most famous unsolved problems in discrete geometry.

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A *unit distance graph* in $\mathbb{R}^n$ is a graph whose vertices are points in $\mathbb{R}^n$ and whose edges are the pairs of vertices that are at distance one from each other. Assuming the axiom of choice, a result of de Bruijn and Erdős [5] implies that the chromatic number of $\mathbb{R}^n$ is equal to the maximum chromatic number of any finite unit distance graph in $\mathbb{R}^n$.

The structure of $\mathbb{R}^n$ does not change under scaling. Therefore, instead of forbidding two points which are at distance one from having the same color, we can equivalently use any $r > 0$ as the forbidden distance. This simple observation will be useful later, since the unit distance graphs we will construct will have simpler vertex coordinates if regarded as r -distance graphs for an appropriate choice of r .

2 Background

It is almost immediate that $\chi(\mathbb{R}) = 2$, since we can 2-color the real line by assigning, for each integer k , the color red to the interval $[2k - 1, 2k)$ and blue to the interval $[2k, 2k + 1)$. Under this coloring, no pair of points at distance one have the same color.

The situation becomes more complex in higher dimensions. Although the origins of the problem are unclear, Edward Nelson is usually credited with having been the first who raised the question for the plane in 1950. A few years earlier, Hadwiger [7] published a related result, showing that any cover of the plane by five congruent closed sets contains a unit distance in one of the sets. Soifer [14] offers an extensive history of the problem.

It is known that

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$

The lower bound is due to Moser and Moser [10], who constructed a unit distance graph with 7 vertices and chromatic number 4. The upper bound can be obtained by tiling the plane with monochromatic regular hexagons. It is easy to check that if one takes the hexagons of diameter slightly less than one, then there exists a periodic 7-coloring of the plane which avoids monochromatic pairs of points that are at unit distance apart. These bounds have not been improved for more than 60 years.

Our ignorance is even more profound in higher dimensional spaces. The following bounds are known:

$$\begin{aligned} 6 &\leq \chi(\mathbb{R}^3) \leq 15, \\ 7 &\leq \chi(\mathbb{R}^4) \leq 54. \end{aligned}$$

The lower bound for $\chi(\mathbb{R}^3)$ is due to Nechushtan [11], while the upper bound was discovered by Coulson [4] and independently by Radoičić and Tóth [12]. The lower bound for $\chi(\mathbb{R}^4)$ was initially proved by Cantwell [2] in 1996, and simpler proofs were subsequently found by Ivanov [8] and Cibulka [3]. The upper bound appears in [12].

The gap between the lower and upper estimates for $\chi(\mathbb{R}^n)$ grows exponentially in n . For $n \rightarrow \infty$, the following bounds are known:

$$(1.239 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n.$$

The lower bound is due to Raĭgorodskiĭ [13], who refined a prior estimate, $(1.207 + o(1))^n \leq \chi(\mathbb{R}^n)$, due to Frankl and Wilson [6]. The upper bound follows from a result of Larman and Rogers [9].

In this note, we improve the lower bound for $\chi(\mathbb{R}^4)$. As mentioned earlier, Cantwell [2] proved that $\chi(\mathbb{R}^4) \geq 7$. His proof is quite elaborate and hinges on the existence of a finite order unit distance graph of chromatic number 7 that can be embedded in $\mathbb{R}^4$. Cantwell did not aim to optimize the size of his unit distance graph, which has more than 2000 vertices.

In 2008, Cibulka [3] used some of Cantwell's ideas to find a much smaller unit distance graph of chromatic number 7 that can be embedded in $\mathbb{R}^4$: his graph has 32 vertices. Cibulka did not provide an explicit embedding.

In 2006, Ivanov [8], unaware of Cantwell's result, again proved the bound $\chi(\mathbb{R}^4) \geq 7$ using a completely new construction; Ivanov's graph has 33 vertices.

In this paper, we improve the current lower bound by constructing three $4\sqrt{2}$ -distance graphs with vertices in $\mathbb{R}^4$. Specifically, we present

- a $4\sqrt{2}$ -distance graph G with 14 vertices and chromatic number $\chi(G) = 7$,
- a $4\sqrt{2}$ -distance graph H with 26 vertices and chromatic number $\chi(H) = 8$,
- a $4\sqrt{2}$ -distance graph K with 65 vertices and chromatic number $\chi(K) = 9$.

3 $\chi(\mathbb{R}^4) \geq 7$

At its core, our method uses the ideas of Cantwell and Cibulka.

Let the *standard configuration* $\mathcal{S}_5$ be the set of ten points in $\mathbb{R}^5$ with two coordinates equal to 4 and the remaining three coordinates equal to 0, that is,

$$\mathcal{S}_5 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}. \quad (1)$$

where

$$\begin{aligned} u_1 &= (4, 4, 0, 0, 0), & u_2 &= (4, 0, 4, 0, 0), & u_3 &= (4, 0, 0, 4, 0), \\ u_4 &= (0, 0, 4, 4, 0), & u_5 &= (0, 4, 0, 4, 0), & u_6 &= (0, 4, 4, 0, 0), \\ u_7 &= (4, 0, 0, 0, 4), & u_8 &= (0, 4, 0, 0, 4), & u_9 &= (0, 0, 4, 0, 4), \\ u_{10} &= (0, 0, 0, 4, 4). \end{aligned}$$

The reason for considering this set as the starting point of our construction is clear: 30 out of the 45 distances determined by pairs of points in $\mathcal{S}_5$ are $4\sqrt{2}$. This high density of equal distances is going to be useful later.

We can embed $\mathcal{S}_5$ in $\mathbb{R}^4$ since all of its points lie in the 4-dimensional hyperplane $x_1 + x_2 + x_3 + x_4 + x_5 = 2$. After an appropriate isometry, we can map $\mathcal{S}_5$ into $\mathbb{R}^4$ to obtain the following set

$$\mathcal{T}_5 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}. \quad (2)$$

where

$$\begin{aligned} v_1 &= (-4, 0, 0, 0), & v_2 &= (0, -4, 0, 0), & v_3 &= (0, 0, -4, 0), \\ v_4 &= (4, 0, 0, 0), & v_5 &= (0, 4, 0, 0), & v_6 &= (0, 0, 4, 0), \\ v_7 &= (-2, -2, -2, 2\sqrt{5}), & v_8 &= (-2, 2, 2, 2\sqrt{5}), \\ v_9 &= (2, -2, 2, 2\sqrt{5}), & v_{10} &= (2, 2, -2, 2\sqrt{5}). \end{aligned}$$

It is straightforward to check that $\|u_i - u_j\| = \|v_i - v_j\|$ for all $1 \leq i < j \leq 10$. At this point, the $4\sqrt{2}$ -distance graph induced by the points in $\mathcal{T}_5$ has chromatic number 5.

An *octahedron* is a set of six points in $\mathbb{R}^n$ that is isometric to the set consisting of the points $[\pm 4, 0, 0]$, $[0, \pm 4, 0]$ and $[0, 0, \pm 4]$. It is not hard to check that the set $\mathcal{T}_5$ defined above contains five octahedra:

$$\begin{aligned} O_1 &= \{v_1, v_2, v_6, v_7, v_8, v_9\} \\ O_2 &= \{v_1, v_3, v_5, v_7, v_8, v_{10}\} \\ O_3 &= \{v_2, v_3, v_4, v_7, v_9, v_{10}\} \\ O_4 &= \{v_4, v_5, v_6, v_8, v_9, v_{10}\} \\ O_5 &= \{v_1, v_2, v_3, v_4, v_5, v_6\} \end{aligned} \tag{3}$$

For a given octahedron embedded in $\mathbb{R}^4$ there exist two *special points* that are at distance $4\sqrt{2}$ from all six vertices of the octahedron. Indeed, in the case where the octahedron has vertices located at $[\pm 4, 0, 0, 0]$, $[0, \pm 4, 0, 0]$ and $[0, 0, \pm 4, 0]$ both points $[0, 0, 0, \pm 4]$ have the desired property.

For each of the first four octahedra given in (3) we compute the coordinates of one such special point.

$$\begin{aligned} v_{11} &= (-1 - \sqrt{5}, -1 - \sqrt{5}, 1 + \sqrt{5}, 1 + \sqrt{5}) \\ v_{12} &= (-1 - \sqrt{5}, 1 + \sqrt{5}, -1 - \sqrt{5}, 1 + \sqrt{5}) \\ v_{13} &= (1 - \sqrt{5}, -1 + \sqrt{5}, -1 + \sqrt{5}, -1 + \sqrt{5}) \\ v_{14} &= (1 - \sqrt{5}, 1 + \sqrt{5}, 1 + \sqrt{5}, 1 + \sqrt{5}) \end{aligned}$$

We are now in position to prove our first result.

Theorem 3.1 *Let G be the graph whose vertex set is given by*

$$V(G) := \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\},$$

where v_i , $1 \leq i \leq 14$, have been previously defined. Two vertices are adjacent if the corresponding points are at distance $4\sqrt{2}$ from each other. Then $\chi(G) = 7$.

Proof The graph G has 57 edges

$$E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{1, 11\}, \{1, 12\}, \{2, 3\}, \{2, 4\}, \\ \{2, 5\}, \{2, 8\}, \{2, 9\}, \{2, 11\}, \{2, 12\}, \{2, 14\}, \{3, 6\}, \{3, 7\}, \{3, 8\}, \{3, 9\}, \{3, 11\}, \{3, 12\}, \\ \{3, 13\}, \{4, 5\}, \{4, 6\}, \{4, 8\}, \{4, 10\}, \{4, 11\}, \{4, 14\}, \{5, 7\}, \{5, 9\}, \{5, 10\}, \{5, 12\}, \{5, 14\}, \\ \{6, 7\}, \{6, 8\}, \{6, 10\}, \{6, 11\}, \{6, 13\}, \{7, 9\}, \{7, 10\}, \{7, 12\}, \{7, 13\}, \{8, 9\}, \{8, 10\}, \{8, 11\}, \\ \{8, 13\}, \{8, 14\}, \{9, 10\}, \{9, 12\}, \{9, 13\}, \{9, 14\}, \{10, 13\}, \{10, 14\}, \\ \{11, 14\}, \{12, 14\}, \{13, 14\}\}$$

It can be checked that G has independence number $\alpha(G) = 3$ and that the only maximum independent sets are $\{v_4, v_{12}, v_{13}\}$, $\{v_5, v_{11}, v_{13}\}$, $\{v_{10}, v_{11}, v_{12}\}$ and $\{v_{11}, v_{12}, v_{13}\}$.

Suppose that G is 6-colorable. Since $\alpha(G) = 3$, it follows that no four vertices can receive the same color. There are 14 vertices and 6 colors, so by the pigeonhole principle there are two colors, each of which has to be used for an (independent) set of three vertices. However, this is impossible since every pair of maximum independent sets have nonempty intersection. This proves that $\chi(G) \geq 7$.

On the other hand, there exist proper 7-colorings of this graph. One such example is presented below.

Vertex	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
Color	1	2	3	3	4	2	5	1	6	7	4	7	4	5

□

4 $\chi(\mathbb{R}^4) \geq 8$

The above construction suggests that there exist $4\sqrt{2}$ -distance graphs whose vertices are points in $\mathbb{R}^4$ with coordinates in the quadratic extension field $\mathbb{Q}(\sqrt{5})$ and having high chromatic number. This intuition turns out to be correct as we will next present a relatively small unit distance graph which improves the lower bound for $\chi(\mathbb{R}^4)$ from 7 to 8.

Theorem 4.1 *There exists a graph H with chromatic number 8 which can be represented as a unit distance graph in $\mathbb{R}^4$.*

The theorem follows from the construction of a graph of order 26, given below. We begin by first building a graph with 35 vertices and an automorphism group of order 24. Then our graph H is obtained by deleting nine of the 35 vertices.

The construction is based on a set of seven points in $\mathbb{R}^4$ listed in the Table 1. For each of these seven points, we let the symmetric group S_4 act on the coordinates, generating a total of 35 distinct points. The number of distinct points generated by this action is also given in the table, along with the coordinates of each of the seven points in $\mathbb{R}^4$. Two vertices are adjacent if the distance between them is exactly $4\sqrt{2}$.

Table 1 Points used in the construction of the graph in Theorem 4.1

Type 1	$1 + 1 \cdot \sqrt{5}$,	$1 + 1 \cdot \sqrt{5}$,	$1 + 1 \cdot \sqrt{5}$,	$1 + 1 \cdot \sqrt{5}$	1
Type 2	$4 + 0 \cdot \sqrt{5}$,	$0 + 0 \cdot \sqrt{5}$,	$0 + 0 \cdot \sqrt{5}$,	$0 + 0 \cdot \sqrt{5}$	4
Type 3	$-4 + 0 \cdot \sqrt{5}$,	$0 + 0 \cdot \sqrt{5}$,	$0 + 0 \cdot \sqrt{5}$,	$0 + 0 \cdot \sqrt{5}$	4
Type 4	$0 + 2 \cdot \sqrt{5}$,	$-2 + 0 \cdot \sqrt{5}$,	$-2 + 0 \cdot \sqrt{5}$,	$-2 + 0 \cdot \sqrt{5}$	4
Type 5	$1 - 1 \cdot \sqrt{5}$,	$-1 + 1 \cdot \sqrt{5}$,	$-1 + 1 \cdot \sqrt{5}$,	$-1 + 1 \cdot \sqrt{5}$	4
Type 6	$0 + 2 \cdot \sqrt{5}$,	$-2 + 0 \cdot \sqrt{5}$,	$2 + 0 \cdot \sqrt{5}$,	$2 + 0 \cdot \sqrt{5}$	12
Type 7	$1 + 1 \cdot \sqrt{5}$,	$1 + 1 \cdot \sqrt{5}$,	$-1 - 1 \cdot \sqrt{5}$,	$-1 - 1 \cdot \sqrt{5}$	6
Total					35

The graph H is then obtained by deleting nine of these 35 vertices, leaving a graph of order 26 with 156 edges. The nine points to be deleted consist of two vertices of type 4:

$$\begin{aligned} & (0 + 2 \cdot \sqrt{5}, -2 + 0 \cdot \sqrt{5}, -2 + 0 \cdot \sqrt{5}, -2 + 0 \cdot \sqrt{5}) \\ & (-2 + 0 \cdot \sqrt{5}, 0 + 2 \cdot \sqrt{5}, -2 + 0 \cdot \sqrt{5}, -2 + 0 \cdot \sqrt{5}) \end{aligned}$$

one vertex of type 7:

$$(1 + 1 \cdot \sqrt{5}, 1 + 1 \cdot \sqrt{5}, -1 - 1 \cdot \sqrt{5}, -1 - 1 \cdot \sqrt{5})$$

and six vertices of type 6. The six deleted vertices of type 6 are those that have $2 \cdot \sqrt{5}$ as either the first or the second coordinate.

To verify the claim that the chromatic number of H is at least 8, we used several different computer programs to verify the absence of proper 7-colorings of H . The simplest of these are based on the following simple recursive procedure that does an exhaustive search for a K -coloring of a graph of order N . It employs a global variable *color*, an array of order N , which records the color of each vertex v for $1 \leq v \leq N$. The search is initiated with the call $DFS(1)$.

procedure $DFS(v)$

Local variable: $FCSet$ - the set of feasible colors for vertex v .

if $v > N$ **then**

 All vertices have been colored, report G is K -colorable.

 Exit.

end if

$FCSet \leftarrow \{1 \cdots K\}$.

for $u \leftarrow 1$ to $v - 1$ **do**

if u is adjacent to v **then**

$FCSet = FCSet - color(u)$

end if

end for

```

for  $c \in FCSet$  do
   $color(v) \leftarrow c$ 
  DFS( $v+1$ )
end for
end procedure

```

In order to decrease the time it takes to complete an exhaustive search for a 7-coloring, one can employ the usual degree saturation procedure. In this case, an ordering of the vertices is generated by picking vertices adjacent to the maximum number of previously selected vertices, and in case of ties, by picking the vertex of largest degree. This idea was first used in the context of chromatic numbers by Breláz [1]. Taking advantage of this, a 2013 era computer can easily complete the exhaustive search for a 7-coloring of the 26-vertex graph in a fraction of a second.

The adjacency matrix of this graph, along with a computer program for verifying its chromatic number, can be found at the web site [15].

5 $\chi(\mathbb{R}^4) \geq 9$

Using the same conventions as with the previous graph, including the action of S_4 on the coordinates, we construct a $4\sqrt{2}$ -distance graph K in $\mathbb{R}^4$ having order 65, size 588, and chromatic number 9. The coordinates of the vertices of K are given in the Table 2.

Using the programs provided at [15], one can verify, in under one minute, that there are no proper 8-colorings of K , thereby establishing that the chromatic number is at least 9. A proper coloring of K using 9 colors is available at [15]. Hence we have the following theorem.

Theorem 5.1 *The graph K has chromatic number 9 and can be represented as a unit distance graph in $\mathbb{R}^4$.*

Table 2 Points used in the construction of the graph in Theorem 5.1

$1 + 1 \cdot \sqrt{5},$	$1 + 1 \cdot \sqrt{5},$	$1 + 1 \cdot \sqrt{5},$	$1 + 1 \cdot \sqrt{5}$	1
$-4 + 0 \cdot \sqrt{5},$	$0 + 0 \cdot \sqrt{5},$	$0 + 0 \cdot \sqrt{5},$	$0 + 0 \cdot \sqrt{5}$	4
$4 + 0 \cdot \sqrt{5},$	$0 + 0 \cdot \sqrt{5},$	$0 + 0 \cdot \sqrt{5},$	$0 + 0 \cdot \sqrt{5}$	4
$0 + 2 \cdot \sqrt{5},$	$-2 + 0 \cdot \sqrt{5},$	$-2 + 0 \cdot \sqrt{5},$	$-2 + 0 \cdot \sqrt{5}$	4
$1 + 1 \cdot \sqrt{5},$	$-1 + 1 \cdot \sqrt{5},$	$-1 + 1 \cdot \sqrt{5},$	$-1 + 1 \cdot \sqrt{5}$	4
$-1 - 1 \cdot \sqrt{5},$	$-1 - 1 \cdot \sqrt{5},$	$1 + 1 \cdot \sqrt{5},$	$1 + 1 \cdot \sqrt{5}$	6
$1 + 1 \cdot \sqrt{5},$	$1 + 1 \cdot \sqrt{5},$	$-3 + 1 \cdot \sqrt{5},$	$-3 + 1 \cdot \sqrt{5}$	6
$-2 + 0 \cdot \sqrt{5},$	$2 + 0 \cdot \sqrt{5},$	$2 + 0 \cdot \sqrt{5},$	$0 + 2 \cdot \sqrt{5}$	12
$0 + 0 \cdot \sqrt{5},$	$-2 + 2 \cdot \sqrt{5},$	$2 + 2 \cdot \sqrt{5},$	$0 + 0 \cdot \sqrt{5}$	12
$-1 + 1 \cdot \sqrt{5},$	$-3 - 1 \cdot \sqrt{5},$	$-1 + 1 \cdot \sqrt{5},$	$3 + 1 \cdot \sqrt{5}$	12
				65

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