New Roses: Simple Symmetric Venn Diagrams with 11 and 13 Curves

Khalegh Mamakani · Frank Ruskey

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Abstract A symmetric n -Venn diagram is one that is invariant under n -fold rotation, up to a relabeling of curves. A simple n -Venn diagram is an n -Venn diagram in which at most two curves intersect at any point. In this paper, we introduce a new property of Venn diagrams called crosscut symmetry, which is related to dihedral symmetry. Utilizing a computer search restricted to diagrams with crosscut symmetry, we found many simple symmetric Venn diagrams with 11 curves. The question of the existence of a simple 11-Venn diagram has been open since the 1960s. The technique used to find the 11-Venn diagram is extended and a symmetric 13-Venn diagram is also demonstrated.

Keywords Venn diagram · Crosscut symmetry · Symmetric graphs · Hypercube

1 Introduction

Mathematically, an n -Venn diagram is a collection of n simple *closed* (Jordan) curves in the plane with the following properties: (a) Each of the 2^n different intersections of the open interiors or exteriors of all n curves is non-empty and connected; (b) there are only finitely many points where the curves intersect. If each of the intersections is of only two curves, then the diagram is said to be *simple*.

Venn diagrams can be viewed as plane graphs where the *edges* are those curve segments that lie between two adjacent intersection points. Furthermore, because the intersections of the interiors and exteriors in the definition are connected, they form the faces of this plane graph. The unbounded face that corresponds to the intersection of the exteriors of all curves is called the *outermost face*, and the face that corresponds

K. Mamakani · F. Ruskey
Department of Computer Science, University of Victoria, Victoria, BC, Canada
e-mail: ruskey@cs.uvic.ca

to the intersection of the interiors of all curves is called the *innermost face*. A k -face ($0 \leq k \leq n$) in an n -Venn diagram is a face which is in the interior of exactly k curves. In a *monotone* Venn diagram, every k -face is adjacent to at least one $(k - 1)$ -face (if $k > 0$) and it is also adjacent to at least one $(k + 1)$ -face (if $k < n$). Monotone Venn diagrams are precisely those that can be drawn with convex curves [1]. The diagrams under consideration in this paper are both monotone and simple.

An n -Venn diagram is *symmetric* if it is left fixed (up to a relabeling of the curves) by a rotation of the plane by $2\pi/n$ radians. Interest in symmetric Venn diagrams was initiated by Henderson [9] in a 1963 paper in which he showed that a symmetric n -Venn diagram could not exist unless n is a prime number (see also [14]). It is easy to draw symmetric 2- and 3-Venn diagrams using circles as the curves, but it was not until 1975 that Grünbaum [6] published a simple symmetric 5-Venn diagram, one that could be drawn using ellipses. Some 20 years later, in 1992, simple symmetric 7-Venn diagrams were discovered independently by Grünbaum [7] and by Edwards [4]. The total number of non-isomorphic simple 7-Venn diagrams that are convexly drawable is also known [12].

The construction of a simple symmetric 11-Venn diagram has eluded all previous efforts until now. We know of several futile efforts that involved either incorrect constructions, or unsuccessful computer searches. The main purpose of this paper is to demonstrate that there are simple symmetric 11- and 13-Venn diagrams.

It should be noted that if the diagrams are not constrained to be simple, then the problem has been solved. Hamburger [8] was the first to discover a (non-simple) symmetric 11-Venn diagram, and Griggs, Killian and Savage (GKS) [5] have shown how to construct symmetric, but highly non-simple, n -Venn diagrams for any prime n . These constructions, in a sense, are maximally non-simple since they involve points where all n curves intersect. Some progress toward “simplifying” the GKS construction is reported in [11], but their approach could never succeed in producing truly simple diagrams.

Part of the interest in Venn diagrams is due to the fact that their geometric dual graphs are planar spanning subgraphs of the hypercube; furthermore, if the Venn diagram is simple then the subgraph is maximal in the sense that every face is a quadrilateral. Symmetric drawings of Venn diagrams imply symmetric drawings of spanning subgraphs of the hypercube. Some recent work on finding symmetric structures embedded in the hypercube is reported in [10] and [3].

2 Crosscut Symmetry

We define a *crosscut* of a Venn diagram as a segment of a curve that extends from the innermost face to the outermost face and “cuts” (i.e., intersects) every other curve exactly once. Except for $n = 2$ and 3, where the symmetric 2-Venn and 3-Venn diagrams have 4 and 6 crosscuts, respectively. For $n > 3$, a symmetric n -Venn diagram either has n crosscuts or it has none. Referring to Fig. 1a, notice that each of the 7 curves has a single crosscut. Recall that in this paper all Venn diagrams are assumed to be simple and monotone.

Fig. 1 **a** A simple rotationally symmetric monotone 7-Venn diagram with a cluster colored (shaded). **b** The cylindrical representation of a cluster of the diagram, showing the reflective aspect of crosscut symmetry

Lemma 1 *If $n > 3$, then a symmetric n -Venn diagram has at most one crosscut per curve.*

Proof A curve of a Venn diagram touches a face at most once [2]. In a monotone Venn diagram, every curve touches the outermost face and the innermost face exactly once [1]. Thus, a curve in any Venn diagram cannot have three or more crosscuts, because that curve would touch the outermost face (and the innermost face) at least twice. Now suppose that some curve C in an n -Venn diagram V has two crosscuts. Then those crosscuts must start/finish at the same edge of C on the outer face, and finish/start at an edge of C on the innermost face of V . Thus, curve C contains exactly $2(n - 1)$ intersections with the other curves. If the Venn diagram V is symmetric, then there must be a total of $n(n - 1)$ intersection points. On the other hand, a simple symmetric Venn diagram has exactly $2^n - 2$ intersection points [13]. Since $n(n - 1) = 2^n - 2$ has a solution for $n = 1, 2, 3$, but not for $n > 3$, the lemma is proved. $\square$

A *clump* in a Venn diagram is a collection of faces that is bounded by a simple closed path of edges. The *size* of a clump is the number of faces that it contains. Aside from the innermost face and the outermost face, a rotationally symmetric n -Venn diagram can be partitioned into n congruent clumps, each of size $(2^n - 2)/n$; in this case, we call the clump a *cluster*—it is like a fundamental region for the rotation, but omitting the parts of the fundamental region corresponding to the full set and to the empty set. Referring again to Fig. 1a, a cluster has been shaded, and this cluster is redrawn in Fig. 1b. Notice that the cluster of Fig. 1b has a central shaded section, which has a reflective symmetry about the crosscut. The essential aspects of this reflective symmetry are embodied in the definition of crosscut symmetry.

Definition 1 Given a rotationally symmetric n -Venn diagram V , we label the curves as $C_1, C_2, \dots, C_n$ according to the clockwise order in which they touch the unbounded outermost face. Assume that the faces of V (except the innermost and outermost) can be partitioned into n clusters $S_1, S_2, \dots, S_n$ in such a way that each cluster S_k has the property that every curve intersects S_k in a segment and that S_k contains the crosscut for curve C_k . Let $L_{i,k}$ be the list of curves that we encounter as we follow C_i in the cluster S_k in clockwise order starting at the innermost face, and let $\ell_{i,k}$ denote the length of

$L_{i,k}$. We say that V is *crosscut symmetric* if for every cluster S_k , for any $i \neq k$, the list $L_{i,k}$ is palindromic; that is, for $1 \leq j \leq \ell_{i,k}$, we have $L_{i,k}[j] = L_{i,k}[\ell_{i,k} - j + 1]$.

Figure 1a shows a simple rotationally symmetric 7-Venn diagram that also has crosscut symmetry. In the Survey of Venn diagrams, it is known as M4 [13]. There is only one other simple 7-Venn diagram with crosscut symmetry; it is dubbed “Hamilton” by Edwards [4]. Hamilton also has “polar symmetry” but is not used here because it also has polar symmetry which might cause confusion with the crosscut symmetry. Cluster S_1 of the diagram M4 is shown in Fig. 1b where a segment of C_1 is the crosscut. The list of crossing curves for each curve in the cluster S_1 is shown below.

$$\begin{aligned} L_{1,1} &= [C_2, C_5, C_4, C_6, C_3, C_7], \\ L_{2,1} &= [C_3, C_1, C_3], \\ L_{3,1} &= [C_2, C_4, C_6, C_5, C_1, C_5, C_6, C_4, C_2], \\ L_{4,1} &= [C_5, C_3, C_5, C_1, C_5, C_3, C_5], \\ L_{5,1} &= [C_4, C_6, C_3, C_6, C_4, C_1, C_4, C_6, C_3, C_6, C_4], \\ L_{6,1} &= [C_7, C_5, C_3, C_5, C_1, C_5, C_3, C_5, C_7], \\ L_{7,1} &= [C_6, C_1, C_6]. \end{aligned}$$

Imagine a ray emanating from some point in the innermost face of an n -Venn diagram V and sweeping the surface of the diagram in a clockwise order. If V is simple and monotone, we can always deform the curves of V continuously such that at any moment the ray cuts each curve exactly once and no two intersection points of V lie on the ray at the same time [1]. Let v be the vector of curve labels along the ray where $v(1)$ is the outermost curve and $v(n)$ is the innermost curve. At each intersection point of V , a pair of adjacent curves crosses. Therefore, we can represent the whole diagram using a sequence of length $2^n - 2$ of curve crossings, where a crossing of curves $v(i)$ and $v(i + 1)$, for $1 \leq i < n$, is indicated by an entry of value i in the *crossing sequence*. See [12] for more details on this representation of simple monotone Venn diagrams. For a simple rotationally symmetric n -Venn diagram, the first $(2^n - 2)/n$ elements of the crossing sequence are sufficient for the representation of the entire diagram; the remainder of the crossing sequence is formed by $n - 1$ concatenations of this sequence. For example, if we start sweeping the surface of the Venn diagram of Fig. 1a a small distance from the left boundary of the cluster shown in the figure, then the first $18 = (2^7 - 2)/7$ elements of the crossing sequence could be the following list.

$$\underbrace{1, 3, 2, 5, 4}_{\rho} \underbrace{3, 2, 3, 4}_{\alpha} \underbrace{6, 5, 4, 3, 2}_{\delta} \underbrace{5, 4, 3, 4}_{\alpha^{r+}}$$

We write “could” here since crossings can sometimes occur in different orders and still represent the same diagram (e.g., taking $\rho = 3, 1, 5, 2, 4$ gives the same diagram); see the remark below. We encourage the reader to verify that this sequence is correct by referring to the crossing numbers shown to the right of the intersections in Fig. 1b.

Fig. 2 A cluster of a crosscut symmetric simple monotone n -Venn diagram (cylindrical representation)

Remark 1 If j, k is an adjacent pair in a crossing sequence $\mathcal{C}$ and $|j - k| > 1$, then the sequence $\mathcal{C}'$ obtained by replacing the pair j, k with the pair k, j is also a crossing sequence of the same diagram.

Theorem 1 A simple monotone rotationally symmetric n -Venn diagram is crosscut symmetric if and only if it can be represented by a crossing sequence of the form $\rho, \alpha, \delta, \alpha^{r+}$ where

- ρ is $1, 3, 2, 5, 4, \dots, n-2, n-3$.
- δ is $n-1, n-2, \dots, 3, 2$.
- α and α^{r+} are two sequences of length $(2^{n-1} - (n-1)^2)/n$ such that α^{r+} is obtained by reversing α and adding 1 to each element; that is, $\alpha^{r+}[i] = \alpha[|\alpha| - i + 1] + 1$.

Proof Let V be a simple monotone rotationally symmetric n -Venn diagram with crosscut symmetry. Assume that the curves of V are labeled $C_1, C_2, \dots, C_n$ according to their clockwise order of touching the outermost face. Consider a cluster of V where a segment of C_1 forms the crosscut. Each curve of the diagram touches the outermost and innermost face exactly once. Therefore, C_1 only intersects C_{n-1} at some point u on the left border of the cluster before intersecting the crosscut. Since the diagram is crosscut symmetric, $L_{n,1} = [C_{n-1}, 1, C_{n-1}]$ and so C_n must intersect C_{n-1} at some point v on the right border of the cluster immediately below C_1 . Because of the rotational symmetry of V , the point v is the image of some point s on the left border under the rotation of $2\pi/n$ about the center of the diagram. At the point s , the curves C_{n-1} and C_{n-2} intersect. Again, because of the crosscut symmetry, $L_{n-1,1} = [C_n, C_{n-2}, \dots, C_{n-2}, C_n]$, and so there is a corresponding point t on the right border where C_{n-1} and C_{n-2} intersect. Continuing in this way, we can see that the left and right border of a crosscut symmetric n -Venn diagram must have the “zig-zag” shape as illustrated in Fig. 2.

Consider two rays emanating from the center of the diagram that cut all the curves immediately before the left zig-zag border and immediately after the right zig-zag border of the cluster. Let ν and σ be the curve vectors along the rays as we move the rays in opposite directions toward the crosscut. We prove by induction that as long as

the two rays do not intersect the crosscut, at each moment σ is a cyclic shift of ν one element to right; therefore, if $\nu(i)$ and $\nu(i+1)$ intersect on the left side, then $\sigma(i+1)$ and $\sigma(i+2)$ must also intersect on the right side. Initially, ν and σ are

$$[C_{n-1}, C_n, C_{n-3}, C_{n-2}, \dots, C_2, C_3, C_1] \text{ and } [C_1, C_{n-1}, C_n, C_{n-3}, \dots, C_5, C_2, C_3]$$

respectively. According to the “zig-zag” shape of the borders, after $n-2$ crossings on the borders, ν and σ are

$$[C_n, C_{n-2}, C_{n-1}, C_{n-4}, \dots, C_4, C_2, C_1] \text{ and } [C_1, C_n, C_{n-2}, C_{n-1}, \dots, C_3, C_4, C_2].$$

Clearly, the $n-2$ crossings on the left border can be represented by the crossing sequence $\rho = 1, 3, 2, 5, 4, \dots, n-2, n-3$.

Now suppose for any of the previous k intersection points, σ is always a cyclic rotation of ν one element to the right and suppose the next crossing occurs between $\nu(i)$ and $\nu(i+1)$. Since $\nu(i) = \sigma(i+1)$ and $\nu(i+1) = \sigma(i+2)$ the next crossing on the right side of crosscut must occur between $\sigma(i+1)$ and $\sigma(i+2)$, for otherwise the diagram would not be crosscut symmetric. Thus, σ remains a cyclic shift of ν by one element to the right after the crossing. Therefore, by induction, each crossing of $\nu(i)$ and $\nu(i+1)$ corresponds to a crossing of $\sigma(i+1)$ and $\sigma(i+2)$. Let α represents the sequence of crossings that follows the first $n-2$ crossings that occur on the left border. Reversing the crossings of the right side of the crosscut, the entire crossing sequence of the cluster is

$$1, 3, 2, 5, 4, \dots, n-2, n-3, \alpha, n-1, n-2, \dots, 2, 1, \alpha^{r+}, n-2, n-1, \dots, 5, 6, 3, 4, 2,$$

where $\alpha^{r+}[i] = \alpha[|\alpha| - i + 1] + 1$. Removing the $n-1$ elements representing the intersection points of the right border, we will get the required crossing sequence $\rho, \alpha, \delta, \alpha^{r+}$ of the diagram.

To prove the converse, suppose we are given a rotationally symmetric n -Venn diagram V with the crossing sequence $\rho, \alpha, \delta, \alpha^{r+}$ as specified in the statement of the theorem. Then the crossing sequence of one cluster of V is $\rho, \alpha, \delta, \alpha^{r+}, \rho, n-1$ where $\rho = [1, 3, 2, 5, 4, \dots, n-2, n-3]$, and $\delta = [n-1, n-2, \dots, 3, 2]$. The ρ sequence indicates that the cluster has the “zig-zag”-shaped borders. Therefore, by Remark 1, the crossing sequence of the cluster can be transformed into an equivalent crossing sequence

$$A = [1, 3, 2, \dots, n-2, n-3, \alpha, n-1, n-2, \dots, 3, 2, 1, \alpha^{r+}, n-2, n-3, \dots, 3, 4, 2].$$

Thus, there is a crosscut in the cluster and

$$A[|A| - i + 1] = A[i] + 1, \quad 1 \leq i \leq \frac{|A| - n + 1}{2},$$

where $|A|$ is the length of A . Using reasoning similar to the first part of the proof, it can be shown that at each pair of crossing points corresponding to $A[i]$ and $A[|A| - i + 1]$,

the same pair of curves intersects. So, for each curve C in the cluster, as we move the rays along C in opposite directions, we encounter the same curves that intersect C and therefore the diagram is crosscut symmetric. $\square$

Given a cluster S_i of a crosscut symmetric n -Venn diagram where a segment of C_i is the crosscut, there are the same number of faces on both sides of the crosscut. Furthermore, let r be a k -face in S_i that lies in the exterior of C_i and interior to the curves in some set $\mathcal{K}$; then there is a corresponding $(k+1)$ -face r' that is in the interior of C_i and also interior to the curves in $\mathcal{K}$.

2.1 Simple Symmetric 11-Venn Diagrams

By Theorem 1, if we have the subsequence α of a crossing sequence, then we can construct the corresponding simple monotone crosscut symmetric n -Venn diagram. Therefore, for small values of prime n , an exhaustive search of α sequences may give us possible crosscut symmetric n -Venn diagrams. For example, for $n = 3$ and $n = 5$, the sequence α is empty and the only possible cases are the three circles Venn diagram and Grünbaum 5-ellipses. For $n = 7$, the valid cases of α are $[3, 2, 4, 3]$ and $[3, 2, 3, 4]$. Our search algorithm is of the backtracking variety; for each possible case of α , we construct the crossing sequence $S = \rho, \alpha, \delta, \alpha^{r+}$ checking along the way whether it currently satisfies the Venn diagram constraints, and then doing a final check of whether S represents a valid rotationally symmetric Venn diagram. Using this algorithm for $n = 11$, we found more than 200,000 simple monotone symmetric Venn diagrams, any one of which affirmatively settles the long-standing open question of the existence of a simple monotone symmetric 11-Venn diagram.

Figure 3 shows the first simple symmetric 11-Venn diagram discovered. It was discovered in March of 2012, so following Anthony Edwards' tradition of naming symmetric diagrams [4], we name it "Newroz" which means "the new day" or "the new sun" and refers to the first days of spring in Kurdish/Persian culture; for English speakers, Newroz sounds also like "new rose", perhaps also an apt description (Figs. 4, 5). Further illustrations of our 11-Venn diagrams may be found at <http://webhome.cs.uvic.ca/~ruskey/Publications/Venn11/Venn11.html>. Below is the α sequence for Newroz.

[323434543234345434545654565676543254346545
676787656543457654658765457656876546576567].

3 Iterated Crosscut Symmetry

Figure 6a shows the only other simple monotone 7-Venn diagram with crosscut symmetry discovered and named "Hamilton" by Anthony Edwards [4]. Consider a cluster of this Venn diagram shown in Fig. 6b. Hamilton has another interesting property which we call *iterated crosscut symmetry*. As can be seen from the figure, a property similar to crosscut symmetry can be observed in the two shaded areas on the left side of the main crosscut. The horizontally shaded area contains two faces separated by a segment of the yellow curve as the crosscut. The crosscut of the vertically shaded area

Fig. 3 Newroz, the first simple symmetric 11-Venn diagram

Fig. 4 A blowup of part of Newroz

is a segment of the green curve intersecting the two parallel curve segments exactly once inside the shaded area.

Figure 7 shows the structure of the left side of a cluster of a simple symmetric n -Venn diagram with iterated crosscut symmetry (Fig. 8). It consists of $\lfloor (n-2)/2 \rfloor$ stacked components, each one has its own crosscut symmetry, where the crosscut of the i -th component from the bottom of the stack is a segment of the curve with label $2(i+1)$.

Let $S_H = \rho_H, \alpha_H, \delta_H, \alpha_H^{r+}$ denote the crossing sequence of Hamilton where $\rho_H = 1, 3, 2, 5, 4, \alpha_H = 3, 2, 4, 3, \delta_H = 6, 5, 4, 3, 2$, and $\alpha_H^{r+} = 4, 5, 3, 4$. Using $\rho_H, \alpha_H, \delta_H$ as a seed and restricting the backtracking search to the cases with iterated crosscut symmetry, we found several examples of simple symmetric 11-Venn

Fig. 5 A cluster of Newroz, the first simple symmetric 11-Venn diagram

Fig. 6 Iterated crosscut symmetry: **a** Crosscut symmetric 7-Venn diagram “Hamilton”. **b** A cluster of Hamilton

diagrams with this property. Figure 9 shows one such 11-Venn diagram. The half of a cluster of this diagram is shown in Fig. 10.

Let $S_E = \rho_E, \alpha_E, \delta_E, \alpha_E^{r+}$ denote the crossing sequence of the diagram of Fig. 9. Then by Fig. 10, it is easy to verify that

$$\begin{aligned} \alpha_E = & 3, 2, 4, 3, 5, 4, 3, 2, 4, 3, 5, 4, 6, 5, 4, 3, 5, 4, 6, 5, 7, 6, 5, 4, 3, 2, 3, 4, \\ & 3, 4, 5, 4, 5, 6, 5, 4, 3, 6, 5, 6, 5, 4, 5, 4, 7, 6, 5, 4, 6, 5, 7, 6, 8, 7, 6, 5, \\ & 4, 3, 7, 8, 6, 7, 5, 6, 7, 8, 5, 6, 5, 6, 7, 6, 7, 4, 5, 6, 7, 6, 5, 6, 5, 4, 5, 4. \end{aligned}$$

Fig. 7 The underlying structure of a cluster of a simple symmetric Venn diagram with iterated crosscut symmetry

Fig. 8 The pattern of intersections within an iterated crosscut block

Definition 2 Let $\rho, \alpha, \delta, \alpha^{r+}$ be the crossing sequence of a crosscut symmetric n -Venn diagram V . We say that V has the *iterated crosscut property* if α can be decomposed as

$$\alpha = X_2, X_3, \dots, X_m \text{ with } m = (n-3)/2 \text{ and } X_i = \rho_i, \alpha_i, \delta_i, \alpha_i^{r+}$$

where

$$\begin{aligned} \rho_i &= 2i-1, 2i-2, \dots, 3, 2, \\ \delta_i &= 2i, 2i-1, \dots, 4, 3, \quad \text{and} \\ \alpha_i &\in \{3, 4, \dots, 2i-1\}^*. \end{aligned}$$

For example, the crossing sequence of the diagram illustrated in Fig. 9 has the crossing sequence shown below.

$$\overbrace{3, 2, 4, 3}^{\rho_2} \overbrace{5, 4, 3, 2}^{\delta_2} \overbrace{4, 3, 5, 4}^{\rho_3} \overbrace{6, 5, 4, 3}^{\alpha_3} \overbrace{5, 4, 6, 5}^{\delta_3} \overbrace{5, 4, 6, 5}^{\alpha_3^{r+}},$$

Fig. 9 An example of a simple symmetric 11-Venn diagram with iterated crosscut symmetry

$$\begin{array}{c}
 \overbrace{7, 6, 5, 4, 3, 2}^{\rho_4} \overbrace{3, 4, 3, 4, 5, 4, 5, 6, 5, 4, 3, 6, 5, 6, 5, 4, 5, 4, 7, 6, 5, 4, 6, 5, 7, 6}^{\alpha_4} \\
 \overbrace{8, 7, 6, 5, 4, 3}^{\delta_4} \overbrace{7, 8, 6, 7, 5, 6, 7, 8, 5, 6, 5, 6, 7, 6, 7, 4, 5, 6, 7, 6, 5, 6, 5, 4, 5, 4}^{\alpha_4^{r+}}
 \end{array}$$

3.1 Simple Symmetric 13-Venn Diagrams

It is natural to ask whether it is possible to find a simple symmetric 13-Venn diagram that has iterated crosscut symmetry. The answer is yes; applying the backtracking search using $\rho_E, \alpha_E, 11, \delta_E$ as a seed, we found more than 30,000 cases of simple symmetric 13-Venn diagrams that also have iterated crosscut symmetry.

The first simple symmetric 13-Venn diagram found with iterated crosscut symmetry is shown in Fig. 11. The top four components of the iterated crosscut structure of the diagram, all together forming the first 93 faces of the cluster, come from the first half cluster of the symmetric 11-Venn diagram of Fig. 10. This is because the α sequence of the 11-Venn diagram forms the first 84 elements of the α sequence of the 13-Venn diagram (see Table 1). The last component of the symmetric 13-Venn diagram, which contains $315 - 93 = 222$ faces, is shown in Fig. 12.

Fig. 10 Left side of the cluster of a simple symmetric 11-Venn diagram with iterated crosscut symmetry

4 Final Thoughts

Define a symmetric Venn diagram to have *dihedral symmetry* if it is left invariant by flipping it over (and possibly rotating it after the flip). It is interesting how close our diagrams are to having dihedral symmetry. For example, by removing $n(n-1)/2 = 5$ edges from Newroz, the remaining diagram can be drawn with dihedral symmetry (Fig. 13a); the edges to be removed are the diagonal edges that intersect a vertical bisector of the figure. By slightly modifying those removed edges, we can get a Venn diagram with dihedral symmetry (Fig. 13b). However, in the diagram of Fig. 13b, there are curves that intersect at infinitely many points, and there are exactly $n(n-3)/2 = 4$ points where 3 curves meet. In a similar fashion, any Venn diagram with crosscut symmetry can be transformed into a diagram with dihedral symmetry (Fig. 14).

Conjecture 1 There is no simple Venn diagram with dihedral symmetry if $n > 3$.

It is natural to wonder whether our method can be used to find a simple symmetric n -Venn diagrams, for prime numbers $n > 13$. The sizes of the α sequences for $n = 11, 13$, and 17 are $84,304$, and $3,840$, respectively. This ten-fold increase in the size of the alpha sequence, and hence the depth of the backtracking tree, suggests that it is not surprising that our current methods fail for $n = 17$.

One natural approach is to try to exploit further symmetries of the diagram. In the past, attention has focused on what is known as polar symmetry—in the cylindrical

Fig. 11 The first discovered simple symmetric 13-Venn diagram

Table 1 The α sequence of the simple symmetric 13-Venn diagram of Fig. 11

$\alpha_T =$	3, 2, 4,	3, 5, 4, 3, 2, 4,	3, 5, 4, 6, 5, 4, 3, 5,	4, 6, 5, 7, 6, 5, 4, 3, 2,	3, 4,
	3, 4, 5,	4, 5, 6, 5, 4, 3,	6, 5, 6, 5, 4, 5, 4, 7,	6, 5, 4, 6, 5, 7, 6, 8, 7,	6, 5,
	4, 3, 7,	8, 6, 7, 5, 6, 7,	8, 5, 6, 5, 6, 7, 6, 7,	4, 5, 6, 7, 6, 5, 6, 5, 4,	5, 4,
	9, 8, 7,	6, 5, 4, 3, 2, 3,	4, 3, 4, 5, 4, 5, 6, 5,	4, 3, 5, 4, 6, 5, 4, 5, 6,	7, 6,
	5, 4, 5,	6, 5, 6, 7, 6, 5,	6, 7, 6, 7, 8, 7, 6, 5,	4, 3, 5, 4, 6, 5, 7, 6, 5,	4, 6,
	5, 7, 6,	8, 7, 8, 7, 6, 5,	4, 5, 6, 7, 6, 5, 4, 7,	6, 8, 7, 6, 5, 7, 6, 5, 8,	7, 6,
	9, 8, 7,	6, 5, 4, 8, 7, 8,	7, 6, 7, 6, 5, 9, 8, 7,	6, 8, 7, 6, 5, 9, 8, 7, 6,	10, 9,
	8, 7, 6,	5, 4, 3, 7, 8, 9,	10, 6, 7, 8, 9, 7, 8, 9,	10, 6, 7, 8, 7, 8, 9, 8, 9,	5, 6,
	7, 8, 9,	10, 7, 8, 9, 6, 7,	8, 6, 7, 8, 9, 7, 8, 5,	6, 7, 8, 7, 6, 5, 6, 7, 8,	9, 8,
	9, 7, 8,	6, 7, 5, 6, 7, 8,	6, 7, 5, 6, 4, 5, 6, 7,	8, 9, 8, 7, 8, 7, 6, 7, 8,	7, 6,
	7, 6, 5,	6, 7, 8, 7, 6, 5,	6, 7, 5, 6, 4, 5, 6, 7,	6, 5, 6, 5, 4, 5, 4,	

representation, this means that the diagram is not only rotationally symmetric but that it is also symmetric by reflection about a horizontal plane, followed by a rotation; an equivalent definition, which we use below is that there is an axis of rotation through the equator that leaves the diagram fixed. In the past, polar symmetry did not help in finding 11-Venn diagrams (in fact, no polar symmetric Venn diagrams are yet known),

Fig. 12 The last component of the first half of the cluster of the 13-Venn diagram of Fig. 11

Fig. 13 **a** Part of two adjacent clusters of Newroz with the crosscut drawn vertically on the *left* and on the *right*. The zig-zag starts at the diagonal in the NW corner, and exits at the diagonal in the SE corner. **b** A slightly modified version, showing dihedral symmetry

but it is natural to wonder whether it might be fruitful to search for diagrams that are both polar symmetric and crosscut symmetric. However, the theorem proven below proves that there are no such diagrams for primes larger than 7.

Fig. 14 Newroz drawn with the crosscuts drawn along rays emanating from the center of the diagram, and with as much dihedral symmetry as possible

Of the four faces incident to an intersection point in a simple Venn diagram, a pair of non-adjacent faces is in the interior of the same number of curves and the number of containing curves of the other two differs by two. Define a k -point in a simple monotone Venn diagram to be an intersection point that is incident to two k -faces (and thus also one $(k - 1)$ -face and one $(k + 1)$ -face).

Lemma 2 *Given a cluster of a simple symmetric monotone n -Venn diagram which also has crosscut symmetry, the number of k -points on the left side of the crosscut, for $1 \leq k < n$ is*

$$\frac{1}{n} \left(\binom{n-1}{k} + (-1)^{k+1} \right).$$

Proof Let R_k denote the number of k -faces on the left side of the crosscut. There is a one-to-one correspondence between the k -faces on the left side and $(k + 1)$ -face

on the right side of the crosscut and we know that the total number of k -faces in the cluster is $\binom{n}{k}/n$. Therefore, we can compute the number of k -faces on the left side of the crosscut from the following recursion.

$$R_k = \begin{cases} 1 & \text{if } k = 1 \\ \binom{n}{k}/n - R_{k-1} & \text{if } 1 < k < n - 1. \end{cases} \quad (1)$$

Unfolding the recurrence (1), we have $nR_k = \sum_{0 \leq j \leq k-1} (-1)^j \binom{n}{k-j} = \binom{n-1}{k} + (-1)^{k+1}$. Every k -point in a simple monotone Venn diagram indicates the ending of one k -face and starting of another one. Therefore, we have the same number of k -points and k -faces on the left side of the crosscut, so both of these are counted by R_k . $\square$

Theorem 2 *There is no monotone simple symmetric n -Venn diagram with crosscut and polar symmetry for $n > 7$.*

Proof Let V be a monotone simple symmetric n -Venn diagram which has been drawn in the cylindrical representation with both polar and crosscut symmetry, and let S be a cluster of V with crosscut C . Since the diagram is polar symmetric, S remains fixed under a rotation of π radians about some axis through the equator. Under the polar symmetry action, a crosscut must map to a crosscut, and since there are n crosscuts and n is odd, one crosscut must map to itself, and so one endpoint of this axis can be taken to be the central point, call it x , of some crosscut (the other endpoint of the axis will then be midway between two crosscuts).

Now, consider a horizontal line ℓ that is the equator (and thus must include x). Let $m = (n - 1)/2$. The line ℓ cuts every m -face on the left side of C and every $(m + 1)$ -face on the right side of the crosscut. Because of the crosscut symmetry of V , each $(m + 1)$ -face on the equator (and so in the interior of C) corresponds to the image of some m -face on the equator (and so in the exterior of C) with the same number of bounding edges. Therefore, for V to be polar symmetric, every m -face on the left side of the crosscut must be symmetric under a flip about the horizontal line ℓ , and thus each such face must have an even number of bounding edges. The number of bounding edges cannot be 2, because otherwise it is not a Venn diagram. Thus an m -face on ℓ must contain at least one $(m - 1)$ -point. One of those $(m - 1)$ -points might lie on C and thus not be counted by R_{m-1} , but all other points are counted and so we must have $R_m \leq R_{m-1} + 1$. However,

$$\begin{aligned} R_m - R_{m-1} &= \frac{1}{n} \left(\binom{2m}{m} - \binom{2m}{m-1} \right) + 2(-1)^{m+1} \\ &= \frac{1}{n} (c_{m-1} + 2(-1)^{m+1}), \end{aligned}$$

where c_m is a Catalan number. An easy calculation then shows that $R_m \leq R_{m-1} + 1$ only for the primes $n \in \{2, 3, 5, 7\}$. $\square$

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