Tiling Convex Polygons with Congruent Equilateral Triangles

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Abstract We study the sets $\mathcal{T}_v = \{m \in \{1, 2, \dots\} : \text{there is a convex polygon in } \mathbb{R}^2 \text{ that has } v \text{ vertices and can be tiled with } m \text{ congruent equilateral triangles}\}$, $v = 3, 4, 5, 6$. $\mathcal{T}_3$, $\mathcal{T}_4$, and $\mathcal{T}_6$ can be quoted completely. The complement $\{1, 2, \dots\} \setminus \mathcal{T}_5$ of $\mathcal{T}_5$ turns out to be a subset of Euler's *numeri idonei*. As a consequence, $\{1, 2, \dots\} \setminus \mathcal{T}_5$ can be characterized with up to two exceptions, and a complete characterization is given under the assumption of the Generalized Riemann Hypothesis.

Keywords Convex pentagon · Equilateral triangle · Tiling · Dissection · Numeri idonei · Idoneal number · Generalized Riemann Hypothesis

1 Introduction and First Observations

It is well known and elementary that every polygon with v vertices (v -gon for short) in $\mathbb{R}^2$ can be dissected into m triangles for every $m \geq v - 2$. However, the question for all triples (v, m, k) such that every convex v -gon splits into m convex k -gons is solved only for edge-to-edge dissections [1]. Even the restriction to $k = 3$ (dissection into triangles) gives rise to attractive nontrivial problems, such as dissections into acute triangles [10], into triangles of the same area [6], and into mutually similar triangles [7]. Laczkovich [8] characterizes all convex polygons P that can be tiled with congruent copies of a given triangle T . In some sense, we investigate an inverse question for the particular case of an equilateral triangle. *Given integers $v \geq 3$ and*

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Fig. 1 Proof of Proposition 2

$m \geq 1$, is there a convex v -gon P in the Euclidean plane that has v vertices and admits a tiling with m congruent equilateral triangles?

We search for a complete characterization of the set

$$\mathcal{T} = \{(v, m) : v \in \{3, 4, \dots\}, m \in \{1, 2, \dots\}, \text{ and there exists a convex } v\text{-gon} \\ \text{that can be tiled with } m \text{ congruent equilateral triangles}\}.$$

Throughout this paper we shall consider tilings with equilateral triangles of edge length one, without loss of generality. Let us recall that P is tiled with triangles $T_1, \dots, T_m$ if $P = T_1 \cup \dots \cup T_m$ and the interiors of $T_1, \dots, T_m$ are mutually disjoint.

First elementary observations are summarized as follows (see also [3, Sect. 2]).

Proposition 1 *If a convex v -gon can be tiled with finitely many equilateral triangles, then $v \in \{3, 4, 5, 6\}$.*

Proof Suppose that a convex v -gon is dissected into equilateral triangles. Then its inner angles have sizes of $\frac{\pi}{3}$ or $\frac{2\pi}{3}$, and the sizes of the respective outer angles are $\frac{2\pi}{3}$ and $\frac{\pi}{3}$. Since the v outer angles sum up to 2π , we obtain $v \leq 6$. $\square$

Now determining $\mathcal{T}$ turns out to be equivalent to identifying the sets

$$\mathcal{T}_v = \{m : (v, m) \in \mathcal{T}\}, \quad v = 3, 4, 5, 6.$$

Proposition 2 (a) $\mathcal{T}_3 = \{p^2 : p \in \{1, 2, \dots\}\}$.

(b) $\mathcal{T}_4 = \{2, 3, \dots\}$.

Proof (a) If a triangle T is dissected into equilateral triangles with edge length 1, then T is equilateral itself and has an integer edge length, say p (see Fig. 1). Comparing the area of T and of its tiles shows that T is tiled with p^2 triangles.

(b) Clearly, $1 \notin \mathcal{T}_4$. The right-hand part of Fig. 1 illustrates that all numbers $2p$ and $2p + 1$, $p \in \{1, 2, \dots\}$, belong to $\mathcal{T}_4$. $\square$

2 Tiling Pentagons

Our results on tilings of pentagons are essentially based on a relation with the so-called idoneal numbers, which were introduced as *numeri idonei* by Euler. The Latin adjective “idoneus” stands for words such as “suitable”, “appropriate”, “adequate”, “capable”, or “convenient”. Euler used these numbers to find large primes. We refer to the excellent survey [5] for details on idoneal numbers. Here we cite only some properties that are important for our study of the set $\mathcal{T}_5$.

The set $\mathcal{I}$ of idoneal numbers can be characterized as

$$\mathcal{I} = \{1, 2, \dots\} \setminus \mathcal{E}, \quad \text{where } \mathcal{E} = \{ab + ac + bc : a, b, c \in \{1, 2, \dots\}, a > b > c\}.$$

This is not Euler’s original definition, but goes back to Rains (see [5, Remark 35(b)]).

Theorem 1

(a) All idoneal numbers that do not exceed 10000 are given by

$$\begin{aligned} &\mathcal{I} \cap \{1, \dots, 10000\} \\ &= \{ \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 12, \quad 13, \quad 15, \\ &\quad 16, \quad 18, \quad 21, \quad 22, \quad 24, \quad 25, \quad 28, \quad 30, \quad 33, \quad 37, \quad 40, \quad 42, \quad 45, \\ &\quad 48, \quad 57, \quad 58, \quad 60, \quad 70, \quad 72, \quad 78, \quad 85, \quad 88, \quad 93, \quad 102, \quad 105, \quad 112, \\ &\quad 120, \quad 130, \quad 133, \quad 165, \quad 168, \quad 177, \quad 190, \quad 210, \quad 232, \quad 240, \quad 253, \quad 273, \quad 280, \\ &\quad 312, \quad 330, \quad 345, \quad 357, \quad 385, \quad 408, \quad 462, \quad 520, \quad 760, \quad 840, \quad 1320, \quad 1365, \quad 1848 \}. \end{aligned}$$

(b) There are at most two idoneal numbers larger than 1848.

(c) If the Generalized Riemann Hypothesis is true, then there is no idoneal number larger than 1848.

(d) There is no idoneal number n such that $1848 < n < 2.5 \cdot 10^{10}$.

Part (a) of the theorem goes back to Euler. He conjectured that 1848 was the largest idoneal number or, in other words, that there existed exactly 65 idoneal numbers [5, Sect. 2.1]. Parts (b) and (c), which support Euler’s conjecture, are based on the work of Weinberger [9], [5, Theorem 22 and Corollary 23]. Numerical results of Jacobson, Ramachandran, and Williams imply (d) [4], [5, Sect. 2.7].

There are several references claiming that there is at most one idoneal number larger than 1848, thus sharpening Theorem 1(b) (see, e.g., the online articles on idoneal numbers in Wikipedia and MathWorld or the comments on sequence no. A000926 from The On-line Encyclopedia of Integer Sequences). Kani [5, Remark 24(a)] qualifies this as an error that seems to originate from [2].

Fig. 2 Proofs of Lemmas 1 and 3

We come back to tilings of pentagons.

Lemma 1 $\mathcal{T}_5 = \{p^2 - q^2 - r^2 : p, q, r \in \{1, 2, \dots\}, p > q + r\}$.

Proof Suppose that a convex pentagon P is dissected into equilateral triangles of edge length 1. Then the edges of P have integer lengths, and P has one inner angle of size $\frac{\pi}{3}$ and four of size $\frac{2\pi}{3}$. The legs of the angle of size $\frac{\pi}{3}$, together with the straight line spanned by the opposite edge of P , determine an equilateral triangle T (see the left-hand part of Fig. 2). The edge length p of T is a sum of lengths of edges of P , so $p \in \{1, 2, \dots\}$. P is obtained by cutting off two equilateral triangles T_q and T_r of edge lengths q and r , respectively, from T . Since q and r are edge lengths of P , it follows that $q, r \in \{1, 2, \dots\}$. Moreover, $p > q + r$ because p is the sum of q , r , and the length of P 's edge opposite to the angle of $\frac{\pi}{3}$. Since T splits into p^2 equilateral triangles of edge length 1 and T_q and T_r are tiled with q^2 and r^2 such triangles, a respective tiling of P must consist of $p^2 - q^2 - r^2$ tiles. This shows the inclusion " $\subseteq$ " in the lemma.

Conversely, given p, q, r as in the lemma, one can construct a pentagon P as in Fig. 2. The corresponding tiling of P is obtained from the tiling of T into p^2 pieces by cutting off the $q^2 + r^2$ pieces that tile T_q and T_r . This shows the inverse inclusion " $\supseteq$ ". $\square$

Lemma 1 goes back to [3, Sect. 3], where also another equivalent characterization of $\mathcal{T}_5$ is given.

Lemma 2 $\mathcal{E} \subseteq \mathcal{T}_5$, where $\mathcal{E}$ is as in the definition of idoneal numbers.

Proof Let $ab + ac + bc \in \mathcal{E}$ be given such that $a, b, c \in \{1, 2, \dots\}$ and $a > b > c$. At least two of the numbers a, b, c have the same parity.

Case 1: a and b have the same parity. We put

$$p = \frac{a+b}{2} + c, \quad q = \frac{a-b}{2}, \quad r = c.$$

Now $b > 0$ yields $p > q + r$, $a > b$ gives $q > 0$, and $c > 0$ shows that $r > 0$. Consequently, $ab + ac + bc = p^2 - q^2 - r^2 \in \mathcal{T}_5$ by Lemma 1.

Case 2: a and c have the same parity. Let

$$p = \frac{a+c}{2} + b, \quad q = \frac{a-c}{2}, \quad r = b.$$

We obtain $p > q + r$ from $c > 0$, $q > 0$ from $a > c$, and $r > 0$ from $b > 0$. Again, $ab + ac + bc = p^2 - q^2 - r^2 \in \mathcal{T}_5$ by Lemma 1.

Case 3: b and c have the same parity. We put

$$p = a + \frac{b+c}{2}, \quad q = \frac{b-c}{2}, \quad r = a.$$

Here inequality $p > q + r$ is equivalent to $c > 0$, $q > 0$ to $b > c$, and $r > 0$ to $a > 0$. As above, $ab + ac + bc = p^2 - q^2 - r^2 \in \mathcal{T}_5$. $\square$

Lemma 2 shows that

$$\{1, 2, \dots\} \setminus \mathcal{T}_5 \subseteq \mathcal{I}; \quad (1)$$

i.e., every positive integer m that does not allow a tiling of any convex pentagon with m congruent regular triangles is an idoneal number. This has the following consequences.

Theorem 2 *The set $\{1, 2, \dots\} \setminus \mathcal{T}_5$ of exceptional numbers contains at least 59 and at most 61 integers. More precisely,*

(a)

$$\begin{aligned} & \{1, \dots, 2.5 \cdot 10^{10} - 1\} \setminus \mathcal{T}_5 \\ &= \{ \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \quad \quad 8, \quad 9, \quad 10, \quad 12, \quad 13, \\ & \quad 16, \quad 18, \quad 21, \quad 22, \quad 24, \quad 25, \quad \quad \quad 30, \quad 33, \quad 37, \quad 40, \quad 42, \quad 45, \\ & \quad 48, \quad 57, \quad 58, \quad \quad \quad 70, \quad 72, \quad 78, \quad 85, \quad 88, \quad 93, \quad 102, \quad 105, \\ & \quad 120, \quad 130, \quad 133, \quad 165, \quad 168, \quad 177, \quad 190, \quad 210, \quad 232, \quad \quad \quad 253, \quad 273, \quad 280, \\ & \quad 312, \quad 330, \quad 345, \quad 357, \quad 385, \quad 408, \quad 462, \quad 520, \quad 760, \quad 840, \quad 1320, \quad 1365, \quad 1848 \}, \end{aligned}$$

(b) $\{1, 2, \dots\} \setminus \mathcal{T}_5$ contains at most two numbers larger than 1848, and

(c) if the Generalized Riemann Hypothesis is true, then $\{1, 2, \dots\} \setminus \mathcal{T}_5$ consists exactly of the 59 number presented under (a).

Proof Elementary numerical calculations based on Lemma 1 were used for determining the set $\{1, \dots, 1848\} \setminus \mathcal{T}_5$. It follows from (1) and Theorem 1(d) that $\{1849, \dots, 2.5 \cdot 10^{10} - 1\} \setminus \mathcal{T}_5 = \emptyset$. This gives (a).

Parts (b) and (c) are consequences of (1) and Theorem 1(b) and (c). $\square$

Note that one could disprove the Generalized Riemann Hypothesis by finding a number in $\{1849, 1850, \dots\} \setminus \mathcal{T}_5$. However, we do not expect that such a number exists.

In [3, Sect. 5] all tilings of convex pentagons with at most 60 congruent equilateral triangles are illustrated.

3 Tiling Hexagons

The set $\mathcal{T}_6$ is already completely determined in [3, Sect. 4]. Here we present a slightly different approach.

Lemma 3 $\mathcal{T}_6 = \{p^2 - q^2 - r^2 - s^2 : p, q, r, s \in \{1, 2, \dots\}, p > \max\{q + r, q + s, r + s\}\}$.

Proof The idea is as in the proof of Lemma 1 (see the right-hand part of Fig. 2). A convex hexagon H admits a tiling with equilateral triangles of edge length 1 if and only if H is obtained from a circumscribed equilateral triangle with integer edge length p by cutting off three equilateral triangles of edge lengths $q, r, s \in \{1, 2, \dots\}$, where $p > \max\{q + r, q + s, r + s\}$. The last inequality is due to the fact that all edges of H are of positive length. $\square$

Theorem 3 $\{1, 2, \dots\} \setminus \mathcal{T}_6$ consists of exactly 21 integers, namely

$$\mathcal{T}_6 = \{1, 2, \dots\} \setminus \{1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 15, 17, 20, 21, 23, 29, 36, 39, 41, 44, 84\}.$$

Proof Simple computations based on Lemma 3 allow us to find all $m \in \mathcal{T}_6$ such that $m \leq 184$. This way one finds the 21 exceptional numbers presented above.

It remains to show that $\mathcal{T}_6$ contains all integers $m \geq 185$. Note that a hexagon H with parameters p, q, r, s as in Lemma 3 admits a tiling with $p^2 - q^2 - r^2 - s^2$ equilateral triangles of unit edge length, which is composed of $p - q$ horizontal layers of tiles. Indeed, the triangle of edge length p can be tiled with p layers (as illustrated in Fig. 1), and the above q layers are removed (see Fig. 2). We restrict our consideration to the case $p - q = 13$; that is, the “vertical width” of H is 13. We put $p = 14 + l$ and $q = 1 + l, l = 0, 1, \dots$. Then the conditions from Lemma 3 are certainly satisfied if $r, s \in \{1, 2, \dots\}$ are chosen such that $r + s \leq 13$.

Suppose that r and s are fixed. Then the corresponding number $m = m_l \in \mathcal{T}_6$ satisfies

$$\begin{aligned} m = m_l &= (14 + l)^2 - (1 + l)^2 - r^2 - s^2 \\ &= (14^2 - 1^2 - r^2 - s^2) + 26l = m_0 + 26l, \end{aligned}$$

where m_0 corresponds to the case $l = 0$. Consequently, by varying $l \in \{0, 1, \dots\}$ we obtain $m \in \mathcal{T}_6$ for all integers $m \geq m_0$ that satisfy $(m \bmod 26) = (m_0 \bmod 26)$. Table 1 illustrates how r and s can be chosen in order to obtain $m \in \mathcal{T}_6$ for all $m \geq 185$. $\square$

The above proof shows that $\mathcal{T}_6$ can be realized with hexagons whose “vertical width” does not exceed 13. There exist numbers $m \in \mathcal{T}_6$ that cannot be realized with a

Table 1 Proof of Theorem 3

$m \bmod 26$	m_0	r	s	$m \bmod 26$	m_0	r	s
0	130	1	8	13	91	2	10
1	105	3	9	14	170	3	4
2	106	5	8	15	145	1	7
3	185	1	3	16	94	1	10
4	134	5	6	17	121	5	7
5	161	3	5	18	70	2	11
6	110	2	9	19	123	6	6
7	137	3	7	20	98	4	9
8	86	3	10	21	73	1	11
9	113	1	9	22	178	1	4
10	166	2	5	23	127	2	8
11	115	4	8	24	50	1	12
12	142	2	7	25	155	2	6

smaller width. The smallest example of that kind is $m = 2541$ (obtained by systematic numerical computations).

Again, the preprint [3] gives more details. In particular, [3, Sect. 5] lists all tilings of convex hexagons with fewer than 85 congruent equilateral triangles.

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