An Extension of a Theorem of Yao and Yao

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Abstract In this paper we study $N_d(k)$, the smallest positive integer such that any nice measure μ in $\mathbb{R}^d$ can be partitioned into $N_d(k)$ convex parts of equal measure so that every hyperplane avoids at least k of them. A theorem of Yao and Yao states that $N_d(1) \leq 2^d$. Among other results, we obtain the bounds $N_d(2) \leq 3 \cdot 2^{d-1}$ and $N_d(1) \geq C \cdot 2^{d/2}$ for some constant C . We then apply these results to a problem on the separation of points and hyperplanes.

Keywords Convex partition · Measure equipartition · Yao–Yao theorem

1 Introduction

A convex partition of $\mathbb{R}^d$ into n parts is a covering $\mathcal{P} = \{C_1, \dots, C_n\}$ of $\mathbb{R}^d$ consisting of closed convex bodies with pairwise disjoint interiors. We say that a hyperplane $H \subset \mathbb{R}^d$ avoids a set C if it does not intersect its interior.

The classical Yao–Yao theorem [12] states the following.

Theorem (Yao and Yao, 1985) *Let μ be a nice measure in $\mathbb{R}^d$, then there is a convex partition $\mathcal{P}$ of $\mathbb{R}^d$ into 2^d parts of equal μ -measure such that every hyperplane in $\mathbb{R}^d$ avoids at least one element of $\mathcal{P}$.*

In the original proof of this theorem the measure has to have a continuous density function bounded away from 0. Later, in [7], this was weakened to the condition that the measure of every hyperplane be 0. In this paper we ask μ to satisfy the original

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Table 1 Upper bounds for N_2

k	$N_2(k)$	k	$N_2(k)$	k	$N_2(k)$	k	$N_2(k)$
1	4	6	14	11	24	16	32
2	6	7	16	12	24	17	32
3	8	8	18	13	28	18	36
4	10	9	20	14	30	19	36
5	12	10	22	15	32	20	36

conditions, as in [12]. We say that a measure that satisfies these conditions is a *nice measure*

This theorem gives a partition in which almost every hyperplane intersects exactly $2^d - 1$ pieces. The proof gives a unique partition for each ordered orthonormal basis $(u_1, \dots, u_d)$. We extend this theorem to the case when every hyperplane is required to avoid 2 pieces.

Theorem 1 *Let μ be a nice measure in $\mathbb{R}^d$, then there is a convex partition $\mathcal{P}$ of $\mathbb{R}^d$ into $3 \cdot 2^{d-1}$ parts of equal μ -measure such that every hyperplane in $\mathbb{R}^d$ avoids at least two elements of $\mathcal{P}$.*

For $d = 2$ this follows from a theorem by Buck and Buck [4] which states that any nice measure μ in $\mathbb{R}^2$ can be divided into six parts of equal measure by three concurrent lines. Our method gives in this case a partition by three lines, two of which are parallel. The proof of Theorem 1 is given in Sect. 4. Some details of Yao and Yao's original proof are necessary, so a sketch of their methods is given in Sect. 3.

Let $N_d(k)$ be the smallest positive integer such that the following holds: *For every nice measure μ on $\mathbb{R}^d$ there exists a partition of $\mathbb{R}^d$ into $N_d(k)$ convex parts of equal measure such that every hyperplane avoids at least k parts.* We call such a partition a k -equipartition. Yao–Yao's theorem and Theorem 1 are equivalent to the bounds $N_d(1) \leq 2^d$ and $N_d(2) \leq 3 \cdot 2^{d-1}$.

There is another number which seems useful. Let $M_d(k, \alpha)$ be the smallest positive integer such that *for every nice measure μ on $\mathbb{R}^d$ there is a family of $M_d(k, \alpha)$ convex sets such that every hyperplane avoids at least k of them and they all have measure at least α .* Clearly we have

$$N_d(k) \geq M_d\left(k, \frac{1}{N_d(k)}\right). \quad (1)$$

For N_d there is a simple bound in dimension 2 that strictly improves the bounds that can be obtained by using Yao–Yao's theorem, Theorem 1 and the formulas of Sect. 5 for all $k \leq 15$ except $k = 1, 2, 7, 12$ (see Table 1).

Proposition 2 $N_2(k) \leq 2k + 2$.

For a fixed d , we can determine the asymptotic behavior of $N_d(k)$.

Theorem 3 $\lim_{k \rightarrow \infty} \frac{N_d(k)}{k} = 1$.

This means that the condition of equipartitioning a given measure is not very strong in the sense that, if there are enough parts, it can be done so that every hyperplane avoids almost all of them. This is the same behavior as one would expect from a random partition. These results are proven in Sect. 5. In Sect. 2 they are applied to an apparently simple but annoyingly resistant problem regarding separation of points and hyperplanes.

Problem Determine all pairs $(\alpha, \beta) \in \mathbb{R}_+^2$ such that for any finite set X of points in $\mathbb{R}^d$ and any finite set Y of hyperplanes in $\mathbb{R}^d$, there are sets $A \subset X$ and $B \subset Y$ such that:

- $|A| \geq \alpha|X|$,
- $|B| \geq \beta|Y|$,
- no two points in A are separated by a hyperplane in B .

One should note that this problem is similar to Theorem 18 in [5], and the proof method used for that problem also involves the Yao–Yao theorem. Trying to solve the question above we obtained a bound for $N_d(1)$ in terms of $h_d(t)$, the measure of a spherical cap of S^d with central angle t . This measure is computed with the usual probability measure in S^d .

Theorem 4 *Let $\alpha > 0$ be such that there is a family A of convex sets in $\mathbb{R}^d$ with the following properties:*

- *Every set in A has measure at most α ,*
- *every hyperplane avoids at least one set of A ,*
- *the sum of the measures of the sets in A is not greater than 1.*

Then

$$\alpha \leq 2 \cdot h_d\left(\frac{\pi}{4}\right) \leq 2 \cdot 2^{-\frac{d}{2}}.$$

The hypotheses of this theorem can be written simply as $1 \geq \alpha \cdot M_d(1, \alpha)$. Note that if A is a 1-equipartition, the conditions above hold. Thus, we can set $\alpha = \frac{1}{N_d(1)}$. This is also implied by (1).

As a consequence we obtain,

Corollary 5

$$N_d(1) \geq 2^{\frac{d}{2}-1}.$$

For the approximation of the cap measure see [2], for example. Thus, an exponentially large number of parts is needed. This answers a question by B. Bukh on whether the number of pieces needed is indeed super-polynomial. This kind of bound can be obtained for $N_d(k)$ for any k in terms of spherical caps of S^d . However, explicit approximations are hard to find. These results are a consequence of Proposition 6 and Theorem 9 below.

Remark The Yao–Yao theorem can be generalized in the following way: Given $a_1, a_2, \dots, a_{2^d} > 0$ such that $\sum a_i = \mu(\mathbb{R}^d)$, there is a partition of $\mathbb{R}^d$ into 2^d convex parts $\{C_1, C_2, \dots, C_{2^d}\}$ such that $\mu(C_i) = a_i$ for all i and every hyperplane avoids the interior of at least one C_i . Theorem 1 can also be generalized in the same way. This is made clear in the next sections.

2 The (α, β) Problem

The (α, β) problem deals with how well behaved points and hyperplanes are with each other in terms of separation. This problem was told to the authors by I. Bárány in its first version, mentioned in Sect. 1. It has the difficulty that it is not self-dual, so we work with a second version which does have this property.

Problem (Second version) Find all pairs $(\alpha, \beta) \in \mathbb{R}_+^2$ such that for any two nice centrally symmetric probability measures μ_1, μ_2 in S^d there are sets $A, B \subset S^d$ with $\mu_1(A) \geq \alpha, \mu_2(B) \geq \beta$ such that either

$$\begin{aligned} a \cdot b &\geq 0 \quad \text{for all } a \in A, b \in B \quad \text{or} \\ a \cdot b &\leq 0 \quad \text{for all } a \in A, b \in B. \end{aligned} \quad (2)$$

Here the condition of the measures being centrally symmetric is not really needed. If they are not then we can consider the measures given by $\mu'_1(A) = \frac{1}{2}(\mu_1(A) + \mu_1(-A))$ and $\mu'_2(A) = \frac{1}{2}(\mu_2(A) + \mu_2(-A))$ and obtain the same pairs (α, β) for these measures.

Let $\mathcal{C}'_d$ be the set of pairs (α, β) that satisfy the conditions of the original problem and $\mathcal{C}_d$ be the set of pairs that satisfy the conditions of the second version of the problem.

Note that $(\alpha, \beta) \in \mathcal{C}'_d$ if and only if $(\frac{\alpha}{2}, \frac{\beta}{2}) \in \mathcal{C}_d$. This is done simply by embedding $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ as a hyperplane not containing the origin. Then every point $a \in \mathbb{R}^d$ corresponds to the pair of points $\{a', -a'\}$ in S^d in the line Oa and every hyperplane $H \in \mathbb{R}^d$ corresponds to the pair of points $\{b', -b'\}$ in S^d such that b' is orthogonal to every line Ob with $b \in H$.

With this transformation we see that the original problem is essentially equivalent to one similar to the second version but with symmetric finite sets of points instead of probability measures. The change to measures follows from the fact that every nice measure can be approximated by linear combinations of Dirac measures and vice versa.

The sets A and B in this problem can (and will) be taken as the intersection of S^d and a convex cone in $\mathbb{R}^{d+1}$ with apex at the origin.

We shall denote by M^d the usual probability measure on S^d . Given $0 \leq t < \pi$ and $x \in S^d$, let $C(d, x, t)$ be the spherical cap of S^d with center x and central angle t . Denote by $h_d(t)$ its M^d -measure.

Given $A \subset S^d$, let $A^\top$ be the set of points $x \in S^d$ such that there exists $a \in A$ with $a \cdot x = 0$. Note that if A is connected, the largest set $B \subset S^d$ that satisfies (2) is one of the connected components of the complement of $A^\top$.

Given a set A of fixed measure, in order to bound the measure of B we need a variant of Lévy's isoperimetric inequality [8]. With our notation, Theorem 2.1 in [6] states the following.

Theorem *Let A be a closed subset of S^d and set $t > 0$ so that $M^d(A) = h_d(t)$. Then for every $\varepsilon > 0$, $M^d(A_\varepsilon) \geq h_d(t + \varepsilon)$, where A_ε is the set of points $x \in S^d$ with geodesic distance smaller than ε from x (i.e. the set of points for which there exists $a \in A$ with $\arccos(a \cdot x) < \varepsilon$).*

If $\varepsilon = \frac{\pi}{2}$ and A is connected, then $S^d \setminus A_\varepsilon$ is one of the two connected components of $S^d \setminus A^\top$. Therefore, if μ_1 and μ_2 are equal to M^d and $A, B \subset S^d$ satisfy (2) and $M^d(A) = h_d(t)$ for some $t > 0$, then $M^d(B) \leq h_d(\frac{\pi}{2} - t)$. This is the following proposition.

Proposition 6 *All points in $\mathcal{C}_d$ lie on or below the curve*

$$\left\{ \left(h_d(t), h_d\left(\frac{\pi}{2} - t\right) \right) : 0 \leq t \leq \frac{\pi}{2} \right\}.$$

This turns out to be best possible if $d = 1$.

Theorem 7

$$\mathcal{C}_1 = \left\{ (\alpha, \beta) : \alpha + \beta \leq \frac{1}{2} \right\}.$$

Proof Take α, β with $\alpha + \beta \leq \frac{1}{2}$. Suppose that for every arc segment A with $\mu_1(A) = \alpha$ we have $\mu_2(A^\top) > 2\alpha$. Let ϕ be the rotation of S^1 by an angle of $\frac{\pi}{2}$ and note that $A^\top = \phi(A) \cup \phi^{-1}(A)$. Since μ_1 and μ_2 are centrally symmetric then $\phi(A) = \phi^{-1}(A)$ and therefore $\mu_1(A) < \mu_2(\phi(A))$. Since this happens for every arc A with μ_1 -measure α , $\mu_1(S^1) < \mu_2(S^1)$, which is a contradiction. Therefore there exists an arc segment A that satisfies $\mu_2(A^\top) \leq 2\alpha$. If B is any of the two components of $S^1 \setminus A^\top$, then $\mu_2(B) \geq \beta$. This proves one of the inclusions, Proposition 6 gives the other. $\square$

The Yao–Yao type partition theorems can be used to find pairs in the (α, β) problem. The following lemma is the main tool for this purpose.

Lemma 8 *Let $0 \leq \rho \leq 1$. Suppose that for any nice measure μ_1 on S_d there exists a family F of closed connected subsets of S^d and a probability measure μ_F on F such that*

- $\mu_1(A) \geq \alpha$ for all $A \in F$,
- For every $b \in S^d$, the set $F_b = \{A \in F : A \cap \{b\}^\top \neq \emptyset\}$ is μ_F -measurable and $\mu_F(F_b) \leq \rho$.

Then $(\alpha, \frac{1-\rho}{2}) \in \mathcal{C}_d$.

Proof Let μ_1 and μ_2 be nice measures. Suppose that we can find F and μ_F for μ_1 as above, then by Fubini's Theorem

$$\begin{aligned}\int_F \mu_2(A^\top) d\mu_F &= \int_F \int_{S^d} \chi(A^\top) d\mu_2 d\mu_F = \int_{S^d} \int_F \chi(F_b) d\mu_F d\mu_2 \\ &= \int_{S^d} \mu_F(F_b) d\mu_2 \leq \rho,\end{aligned}$$

where χ is the characteristic function of a set. Thus, we can find A_0 such that $\mu_2(A_0^\top) \leq \rho$. This means that there is a set B such that $\mu_1(B) \geq \frac{1-\rho}{2}$ and the sign of $a \cdot b$ is constant for all $a \in A_0$ and $b \in B$. $\square$

Using this with $N_d(k)$ and $M_d(k, \alpha)$, we obtain the following.

Theorem 9 For any two positive integers k and d ,

$$\left(\frac{1}{2N_d(k)}, \frac{k}{2N_d(k)} \right) \in \mathcal{C}_d.$$

More generally, if $\alpha > 0$,

$$\left(\frac{\alpha}{2}, \frac{k}{2M_d(k, \alpha)} \right) \in \mathcal{C}_d.$$

Proof Given a hyperplane in $\mathbb{R}^{d+1}$ that does not pass through the origin, consider the radial projection $\mathbb{R}^d \rightarrow S^d$. Since the measure of every great circle in S^d has null μ -measure, this induces a probability measure $\mu^\top$ in $\mathbb{R}^d$. Fix a family of $M_d(k, \alpha)$ convex sets of measure α such that every hyperplane (in $\mathbb{R}^d$) avoids the interior of at least k of them. If we bring this family back to S^d , we obtain a new family $\mathcal{F}$ of $2M_d(k, \alpha)$ convex sets in S^d , each of measure $\frac{\alpha}{2}$. By choosing μ_F to be the uniform probability measure on F and applying Lemma 8, we are done. $\square$

We are now ready to prove Theorem 4.

Proof of Theorem 4 Note that the conditions of Theorem 4 can be written as $\frac{1}{M_d(1, \alpha)} \geq \alpha$. Thus, since $(\frac{\alpha}{2}, \frac{1}{2M_d(1, \alpha)}) \in \mathcal{C}_d$ by Theorem 9, we have $(\frac{\alpha}{2}, \frac{\alpha}{2}) \in \mathcal{C}_d$. Finally, by Proposition 6 we obtain $\frac{\alpha}{2} \leq h_d(\frac{\pi}{4})$, as we wanted. $\square$

It should be noted that the same method also implies lower bounds for $N_d(k)$ for any k .

Applying the results obtained in Sect. 5 for $N_d(k)$, we can show the following.

Corollary 10 For any two non-negative integers k_1 and k_2 , not both equal to 0, we have

$$\frac{1}{2} \left(\left(\frac{1}{2^d} \right)^{k_1} \left(\frac{1}{3 \cdot 2^{d-1}} \right)^{k_2}, 1 - \left(1 - \frac{1}{2^d} \right)^{k_1} \left(1 - \frac{1}{3 \cdot 2^{d-2}} \right)^{k_2} \right) \in \mathcal{C}_d.$$

Fig. 1 Bounds for $\mathcal{C}_2$ and $\mathcal{C}_3$

This gives in particular that $(\frac{1}{2^{d+1}}, \frac{1}{2^{d+1}}) \in \mathcal{C}_d$ and $(\frac{1}{3 \cdot 2^d}, \frac{1}{3 \cdot 2^{d-1}}) \in \mathcal{C}_d$. The fact that $(\frac{1}{2^{d+1}}, \frac{1}{2^{d+1}}) \in \mathcal{C}_d$ was obtained earlier in [1] using a similar method. In Fig. 1 there are plots of these points together with the bound obtained in Proposition 6 in dimensions 2 and 3.

Corollary 11 *There are pairs $(\alpha, \beta) \in \mathcal{C}_d$ arbitrarily close to $(0, \frac{1}{2})$.*

This last corollary comes from the fact that $\lim N_d(k)/k = 1$. However, as these pairs get close to $(0, \frac{1}{2})$ they are significantly worse than what Proposition 6 gives.

We can get better bounds if further conditions are imposed on one of the measures. Let $\mathcal{C}_d(\Delta)$ be the set of pairs (α, β) such that for any two measures μ_1, μ_2 on S^d such that μ_1 has density f with respect to M^d satisfying $\text{Lip}(f) \leq \Delta$, there are sets A, B in S^d satisfying $\mu_1(A) = \alpha$, $\mu_2(B) = \beta$ and either $a \cdot b \geq 0$ for all $a \in A, b \in B$ or $a \cdot b \leq 0$ for all $a \in B, b \in B$.

Theorem 12 *For every $0 < \lambda \leq 1$ and $0 < r < \frac{1-\lambda}{\Delta}$ we have*

$$\left(\lambda h_d(r), h_{d-1} \left(\frac{\pi}{2} - 2 \arcsin \left(\frac{\sin(r)}{\sin(\frac{1-\lambda}{\Delta} - r)} \right) \right) \right) \in \mathcal{C}_d(\Delta).$$

If r is close to 0, then the pairs obtained are close to

$$\left(\lambda h_d(r), h_{d-1} \left(\frac{\pi}{2} - c_2 r \right) \right)$$

for a constant c_2 depending on λ and Δ . That is, the difference in dimension with respect to the bounds of Proposition 6 is compensated by the constants. The idea of the proof is to use a small S^{d-1} in S^d and the Lipschitz condition to construct sets as in the proof of Lemma 8. If instead of this we use a hypercube (of dimension $d-1$) in S^d , we obtain bounds of the type $(\frac{c_1}{m^{d-1}}, \frac{1}{2} - \frac{c_2 d}{m})$. These are worse than the ones in Theorem 12 but are easier to grasp.

Fig. 2 The family for Theorem 12

Proof of Theorem 12 Since $\mu_1(S^d) = 1$, there must be a point $x_0 \in S^d$ such that $f(x_0) \geq 1$. Consider $R = \min(\frac{1-\lambda}{\Delta}, \frac{\pi}{2})$. Since $\text{Lip}(f) \leq \Delta$, it follows that $f(x) \geq \lambda$ for all $x \in S^d$ at distance at most R from x_0 . Consider $r \leq \frac{R}{4}$. Given two points $x, y \in S^d$, denote by $\text{dist}_{S^d}(x, y)$ their distance in the sphere. Namely, the angle they sustain at the origin. For each $x \in S^d$, define

$$S(x, r) = \{y \in S^d : \text{dist}_{S^d}(x, y) \leq r\}.$$

We construct the family F as follows (see Fig. 2):

$$F = \{S(x, r) : \text{dist}(x, x_0) = R - r\}.$$

We can note that

- Each set in F has measure at least $\lambda h_d(r)$
- $S(x, r)$ is the intersection of S^d with a ball with center x and radius $\sin(r)$.
- The locus of the centers of the balls in F is $S(x_0, R - r)$; the intersection of S^d with a ball with center x_0 and radius $\sin(R - r)$.

Consider μ_F the usual probability measure on $S(x_0, R - r)$. Then, every hyperplane through the center of S^d intersects a subset of F of size at most

$$1 - 2h_{d-1}\left(\frac{\pi}{2} - 2\arcsin\left(\frac{\sin(r)}{\sin(R-r)}\right)\right).$$

Then, the same argument for Lemma 8 finishes the proof. $\square$

3 Yao–Yao’s Original Proof

We give a sketch of the original proof by Yao and Yao since its spirit will be followed in the proof of our main theorem. This will also allow us to note some additional properties and fix notation.

Let $O(d)$ be the space consisting of $d \times d$ matrices u such that $u^T u = I$, the space $SO(d) \subset O(d)$ consists of matrices with determinant 1. A matrix $u \in O(d)$ can be expressed as $u = (u_1, \dots, u_d)$ where u_i is the i th row vector of u . In this way every u can be identified with an ordered orthonormal base of $\mathbb{R}^d$.

Fix a base $u = (u_1, \dots, u_d)$ of $\mathbb{R}^d$. If H is a hyperplane orthogonal to u_1 , define the open half-spaces

$$H^+ = \{x + tu_1 : x \in H, t > 0\},$$

$$H^- = \{x - tu_1 : x \in H, t > 0\}.$$

Let v be a unit vector in $\mathbb{R}^d$ not orthogonal to u_1 and let $p_v : \mathbb{R}^d \rightarrow H$ be the projection such that $p_v(x + tv) = x$ for all $x \in H$ and $t \in \mathbb{R}$. We can identify H with $\mathbb{R}^{d-1}$ by means of the base $u_2, \dots, u_d$. There is a natural way to define measures μ_v^+ and μ_v^- in H : For any measurable $S \subset H$, set $\mu_v^\pm(S) = \mu(p_v^{-1}(S) \cap H^\pm)$.

In [12] a center $c \in \mathbb{R}^d$ for μ relative to the base $u_1, \dots, u_d$ is defined as follows:

- If $d = 1$ then c is the point that splits $\mathbb{R}$ into two parts of equal μ -measure.
- If $d > 1$, let H be the hyperplane orthogonal to u_1 that splits $\mathbb{R}^d$ into two parts of equal μ -measure. Then c lies on H and there exists a unit vector v (with $u_1 \cdot v > 0$) such that c is a center for both μ_v^+ and μ_v^- relative to $u_2, \dots, u_d$.

Then it is proved that c exists and is unique. Inductively it is easy to see that this induces a partition of $\mathbb{R}^d$ into 2^d parts and each part is a cone bounded by d hyperplanes having c as its vertex. Half of these regions are ‘positive’ in the sense that they lie in H^+ and correspond to regions in H equipartitioning μ_v^+ , and the other half are ‘negative’. By induction, each of these halves is once again either positive or negative. So for every sequence of d elements in $\{+, -\}$ we have a unique region.

Using this we can show that v is unique. Set $H_1 = H$. Since c is a Yao–Yao center, v must be contained in the hyperplane H_2 through c that splits H^+ into 2 parts of equal measure. Note that $H_2 \cap H$ is orthogonal to u_2 and H_2 contains a face of the region R which is positive at every level. We can define hyperplanes H_i for $i \leq d$ similarly in such a way that each H_i is a face of R and show that they are unique independently of v . Since v is a unit vector with $v \cdot u_1 > 0$ and it is contained in $H_2 \cap \dots \cap H_d$, it has only one possible value.

Using the uniqueness of c and v , it follows that c and v vary continuously with u .

Note that this partition has the required properties. If a hyperplane intersects the line through c parallel to v in H^+ then it avoids one of the elements of the partition contained in H^- and vice versa.

If we want each element of the partition to have a pre-described value, then the same proof works by changing the choice of H appropriately.

4 Proof of Theorem 1

Problems involving partitions of measures are topological in nature. There is a standard way to approach such problems, known as the test map scheme. First, we parametrize a subset of possible partitions by a space X (called phase space), and

Fig. 3 The hyperplanes, centers, and projection vectors

construct a space Y (called target space) of parameters of a partition. These spaces are related by a natural function $f : X \rightarrow Y$ (called a test map). Ideally, there is group acting on both X and Y such that f is equivariant. The existence of the target partition is then reduced to showing that any equivariant function on those spaces always takes some value (e.g. [9, 11]). We follow this sketch and reduce the problem to showing that some equivariant functions always have a zero.

In our construction, we parametrize a set of partitions by $O(d)$, the space of all orthonormal basis of $\mathbb{R}^d$. There is an action of $(\mathbb{Z}_2)^d$ on $O(d)$, such that given $g \in (\mathbb{Z}_2)^d$ and $u \in O(d)$, gu is the result of changing the sign of some elements of u , depending on g . The target space and the action on it are much more elaborate and will be shown in the proof.

We start with the geometrical part of the proof of Theorem 1 and continue with the topological part.

Let $u = (u_1, \dots, u_d)$ be an orthonormal base of $\mathbb{R}^d$, we think of u_1 as the upwards direction. If H is a hyperplane orthogonal to u_1 , define the open half-spaces

$$H^+ = \{x + tu_1 : x \in H, t > 0\},$$

$$H^- = \{x - tu_1 : x \in H, t > 0\}.$$

Let H_1 and H_2 be the hyperplanes orthogonal to u_1 such that the sets $A = H_1^+$, $B = H_1^- \cap H_2^+$ and $C = H_2^-$ have equal μ -measure. Let $\mu_1 = \mu|A \cup B$ and $\mu_2 = \mu|B \cup C$.

Yao–Yao’s theorem applied to μ_1 gives a unique center $O_1 \in H_1$ and a unique projection vector v_1 pointing downwards (i.e. $u_1 \cdot v_1 < 0$).

Let $J_1 \subset H_1$ be the $(d-2)$ -dimensional flat through O_1 orthogonal to u_2 . Note that the hyperplane $K_1 = \{J_1 + tv_1 : t \in \mathbb{R}\}$ splits B into two parts of equal μ_1 -measure.

Define analogously $O_2 \in H_2$, v_2 pointing upwards, J_2 and K_2 (see Fig. 3). Since K_1 and K_2 each divide B into two parts of equal μ -measure, they intersect in a $(d-2)$ -dimensional flat $J \subset B$ parallel to J_1 and J_2 .

The centers O_1 and O_2 as well as the vectors v_1 and v_2 vary continuously with u .

Our aim is to find u such that the vectors v_1 and v_2 are parallel to the line O_1O_2 . We first show how, using this, we can construct the desired partition. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the corresponding Yao–Yao partition to μ_1 and μ_2 .

If this is the case, then we can use the partition $\mathcal{P}$ consisting of the elements of $\mathcal{P}_1$ contained in A , the elements of $\mathcal{P}_2$ contained in C and the non-empty elements $K \cap B$ such that $K \in \mathcal{P}_1$. Let l be the line that contains O_1 and O_2 . Every hyperplane avoids at least two elements of $\mathcal{P}$. This is because if it hits the line l in section A , it misses a section contained in B and one contained in C , if it hits l in B it misses a section in A and one in C and if it hits l in C it misses a section in A and one in B .

Now we use topological methods to show that there is a base u satisfying the conditions above. First we need some definitions.

Given $x = (x_1, \dots, x_n) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_n}$, we define $g_i(x)$ as the result of changing the sign of the i th coordinate of x . We always use g_i to denote this function independently of the target space, since it causes no confusion. We denote the j th coordinate of $x_i \in \mathbb{R}^{d_i}$ by $x_i^{(j)}$. Given $v \in \mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$, define $v^{(j)} = (v_1^{(j)}, \dots, v_{d-j}^{(j)})$ and $v^T \in \mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$ as $v^T = (v^{(1)}, \dots, v^{(d-1)})$.

An easier way to visualize this last construction is to consider a $(d-1) \times (d-1)$ matrix V induced by v in the following way. In the k th row write the coordinates of v_k followed by $k-1$ signs “ $\times$ ”,

$$V = \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_{d-2} \\ v_{d-1} \end{matrix} \begin{pmatrix} v_1^{(1)} & v_1^{(2)} & \cdots & v_1^{(d-2)} & v_1^{(d-1)} \\ v_2^{(1)} & v_2^{(2)} & \cdots & v_2^{(d-2)} & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{d-2}^{(1)} & v_{d-2}^{(2)} & \cdots & \times & \times \\ v_{d-1}^{(1)} & \times & \cdots & \times & \times \end{pmatrix}.$$

Then v^T is the set of vectors induced in the same way by the transpose V^T of V , namely

$$V^T = \begin{matrix} v^{(1)} \\ v^{(2)} \\ \vdots \\ v^{(d-2)} \\ v^{(d-1)} \end{matrix} \begin{pmatrix} v_1^{(1)} & v_2^{(1)} & \cdots & v_{d-2}^{(1)} & v_{d-1}^{(1)} \\ v_1^{(2)} & v_2^{(2)} & \cdots & v_{d-2}^{(2)} & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_1^{(d-2)} & v_2^{(d-2)} & \cdots & \times & \times \\ v_1^{(d-1)} & \times & \cdots & \times & \times \end{pmatrix}.$$

Let $r : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ be the projection such that $r(O_1) = r(O_2) = O$ and $\{r(u_2), \dots, r(u_d)\}$ is the canonical basis.

The affine hyperplane $r(J)$ is orthogonal to u_2 , so it is of the form $\{v : v \cdot u_2 = \lambda\}$ for some $\lambda \in \mathbb{R}$. Let $x \in \mathbb{R}^{d-2}$ and $y \in \mathbb{R}^{d-2}$ be the vectors consisting of the last $d-2$ coordinates of $r(v_1)$ and $r(v_2)$, respectively.

Let $h(u) = (x, y, \lambda) \in \mathbb{R}^{d-2} \times \mathbb{R}^{d-2} \times \mathbb{R}$, note that if $h(u) = 0$ for some u , then the vectors v_1 and v_2 are parallel to the line O_1O_2 and we are done.

The map h satisfies the following conditions:

- $h(g_1(u)) = (y, x, \lambda)$,
- $h(g_2(u)) = (x, y, -\lambda)$,
- $h(g_{i+2}(u)) = (g_i(x), g_i(y), \lambda)$ for $i = 1, \dots, d-3$.

Let $f : O(d) \rightarrow \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \dots \times \mathbb{R}^1$ be defined by

$$f(u) = ((x + y, \lambda), x - y, 0, \dots, 0).$$

Finding a zero of h is equivalent to finding a zero of f . We will prove something more general.

Claim Assume $f : O(d) \rightarrow \mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$ is a function such that whenever $f(u) = v$, then

- $f(g_1(u)) = g_2(v)$,
- $f(g_2(u)) = g_{d-1}(v^T)^T$,
- $f(g_{i+2}(u)) = g_i(v^T)^T$ for $i = 1, \dots, d-2$.

Then there exists $u \in O(d)$ such that $f(u) = 0$.

We use a similar proof method to the one Bárány used to prove the Borsuk–Ulam Theorem in [3]. This method is thoroughly explained in Chap. 2.2 of [9] and in [10].

Together with composition, we can think of $\{g_1, \dots, g_d\}$ as a set of generators of the group $\mathbb{Z}_2^d$. Given the conditions on f , there are natural group actions of $\mathbb{Z}_2^d$ on $O(d)$ and $\mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$ such that f is equivariant. However, the space $O(d)$ is too large for our needs, so instead we consider the restriction $f_1 = f|_{SO(d)}$ and the group actions of $\mathbb{Z}_2^{d-1}$ on $SO(d)$ and $\mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$ obtained by taking the group generated by $\{g_1 \circ g_d, \dots, g_{d-1} \circ g_d\}$. By $g_i \circ g_d$ we mean the composition of the functions g_i and g_d , which changes the sign of coordinates i and d in both the domain and the range of f . Since f is equivariant with all the g_i , it is also equivariant with their compositions.

The main idea of the proof is to show that, given two equivariant functions on these spaces, the parity of the number of orbits of $\mathbb{Z}_2^{d-1}$ in $SO(d)$ that are sent to 0 is the same. This is done via an equivariant homotopy, and it is the last step in the proof. Let us first find a function $f_0 : SO(d) \rightarrow \mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$ that sends exactly one orbit of $\mathbb{Z}_2^{d-1}$ to 0.

Define $f_0 : SO(d) \rightarrow \mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$ as the function given by

$$f_0(u) = (v_1, \dots, v_{d-1}),$$

where

- $v_1 = (u_3^{(1)}, \dots, u_d^{(1)}, u_2^{(1)})$,
- $v_2 = u_1^{(1)} \cdot (u_3^{(2)}, \dots, u_d^{(2)})$,
- $v_{i+2} = (u_3^{(i+2)}, \dots, u_{d-i}^{(i+2)})$ for $i = 1, \dots, d-3$.

This function is continuous. Let us show that it is also equivariant. If we show that f_0 is equivariant with all the g_i for $1 \leq i \leq d$, then it will be equivariant with the

subgroup generated by $g_i \circ g_d$ for $1 \leq i \leq d$. We will show each of the bullets in the conditions of f hold for f_0 separately.

- If we change the sign of u_1 , then only the sign of v_2 changes, so $f_0(g_1(u)) = g_2(v)$.
- If we change the sign of u_2 , then only the last coordinate of v_1 changes. Since v_1 is the only element in the range with dimension $d - 1$, this translates to $f_0(g_2(u)) = g_{d-1}(v^T)^T$.
- If we change the sign of u_{i+2} for $1 \leq i \leq d - 2$, then we are changing the sign of the i th coordinate of each v_j (only for those v_j that have at least i coordinates). This translates to $f_0(g_{i+2}(u)) = g_i(v^T)^T$ for all $1 \leq i \leq d - 2$.

Furthermore, if $f_0(u) = 0$, then u_i has to be the i th element of the canonical basis or its negative. Therefore f_0 has exactly 2^{d-1} zeros in $SO(d)$. Note as well that the differential Df_0 is non-degenerate in its zeros.

Let $F : SO(d) \times I \rightarrow \mathbb{R}^{d-1} \times \dots \times \mathbb{R}^1$ be the equivariant homotopy given by $F(u, t) = tf_1(u) + (1 - t)f_0(u)$ that takes f_0 to f_1 .

If we assume that F is generic enough, then the set $F^{-1}(0)$ consists of paths and cycles. If f_1 has no zeros, then all the paths of $F^{-1}(0)$ have their endpoints at points of the form $(u, 0)$ where u is a zero of f_0 . Therefore there must be a path connecting two such points, but this is impossible due to the group action.

It may happen that F is not generic enough. However, since f_0 has 0 as a regular value, we may perturb F slightly to a map H so that we have the following.

- H is a homotopy between two maps f'_0 and f'_1 , and 0 is a regular value of H .
- f'_1 has no zeros.
- f'_0 has only one orbit of points that give 0.

Thus, we may reach a contradiction using H instead of F .

5 Other Proofs

In this section we prove Proposition 2 and Theorem 3, which were stated in the introduction.

Proof of Proposition 2 We proceed inductively to construct $2k + 2$ regions $A_1, A_2, \dots, A_{2k+2}$, each of μ -measure $\frac{1}{2k+2}$, so that their boundaries are contained in the union of at most $k + 1$ lines. Let ℓ_1 be an oriented halving line (i.e. a line that splits $\mathbb{R}^2$ into two parts of equal μ -measure that has fixed right and left sides). Once that $\ell_1, \ell_2, \dots, \ell_{i-1}, \ell_i$, $A_1, A_2, \dots, A_{i-1}$ and $A_{k+2}, A_{k+3}, \dots, A_{k+1+(i-1)}$ have been constructed, let ℓ_{i+1} be the oriented halving line such that the regions

$$A_i = \{x \in \mathbb{R}^2 : x \text{ is right of } \ell_{i+1} \text{ and left of } \ell_i\} \setminus \bigcup_{j < i} A_j$$

$$A_{k+1+i} = \{x \in \mathbb{R}^2 : x \text{ is left of } \ell_{i+1} \text{ and right of } \ell_i\} \setminus \bigcup_{j < i} A_{k+1+j}$$

Fig. 4 Regions for Proposition 2

have μ -measure $\frac{1}{2k+2}$. To see that it is possible to find such ℓ_{i+1} , consider μ' the measure μ restricted to $\mathbb{R}^2 \setminus \bigcup_{j < i} (A_j \cup A_{k+1+j})$. Note that ℓ_i is a halving line of μ' , so we need ℓ_{i+1} to be a halving line of μ' such that $\mu'(A_i) = \frac{1}{2k+2}$. This implies that $\mu'(A_{k+1+i}) = \frac{1}{2k+2}$. Since μ and μ' coincide in A_i and A_{k+1+i} , we obtain the desired line.

We end up with something like the right side of Fig. 4. This is a partition as ℓ_1 and ℓ_{k+2} are equal as non-oriented lines, since at the point of constructing ℓ_{k+2} we have used all the measure. Since $\mathbb{R}^2 \setminus \bigcup_{j < i} (A_j \cup A_{k+1+j})$ consists of two convex components of equal measure for all i , every A_i is convex. If $\ell \subset \mathbb{R}^2$ is a line not parallel to any ℓ_i , then it intersects each of the lines $\ell_1, \dots, \ell_{k+1}$ once, at every intersection ℓ enters at most one new region. Thus ℓ intersects at most $k+2$ elements of the partition. This implies that it avoids at least k , as we wanted. $\square$

Proof of Theorem 3 Clearly $N_d(k) \geq d+k$, as there is a hyperplane through any given d points. The function N_d also satisfies

$$N_d(k_1 + k_2) \leq N_d(k_1) + N_d(k_2). \quad (3)$$

To see this, partition $\mathbb{R}^d$ by a hyperplane H that divides its measure in proportions $N_d(k_1) : N_d(k_2)$. We can find a k_1 -equipartition of one side and a k_2 -equipartition of the other side. We are left with $N_d(k_1) + N_d(k_2)$ parts of equal measure such that every hyperplane avoids k_1 parts on one side of H and k_2 parts in the other side of H .

We also have the asymptotically stronger equation

$$N_d(k_1 N_d(k_2) + k_2 N_d(k_1) - k_1 k_2) \leq N_d(k_1) N_d(k_2). \quad (4)$$

This can be shown by finding a k_1 -equipartition of $\mathbb{R}^d$ and further partition each of its pieces by k_2 -equipartitions. We are left with $N_d(k_1) N_d(k_2)$ parts of equal measure such that every hyperplane intersects at most $(N_d(k_1) - k_1)(N_d(k_2) - k_2)$ of them.

Starting with Yao–Yao’s theorem and iterating (4), a sequence of partitions can be found such that in the i th step we have 2^{di} parts of equal measure and every hyperplane intersects at most $(2^d - 1)^i$ of them. Therefore, a sequence k_i can be found in which $N_d(k_i)/k_i$ tends to 1. Then (3) implies

$$\lim_{k \rightarrow \infty} \frac{N_d(k)}{k} = 1. \quad \square$$

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References

1. Alon, N., Pach, J., Pinchasi, R., Radoičić, R., Sharir, M.: Crossing patterns of semi-algebraic sets. *J. Comb. Theory, Ser. A* **111**(2), 310–326 (2005)
2. Ball, K.: An elementary introduction to modern convex geometry. In: Levy, S. (ed.) *Flavors of Geometry*. Mathematical Sciences Research Institute Publications, vol. 31, pp. 1–58. Cambridge Univ. Press, Cambridge (1997)
3. Bárány, I.: Borsuk’s theorem through complementary pivoting. *Math. Program.* **18**, 84–88 (1980)
4. Buck, R.C., Buck, E.F.: Equipartition of convex sets. *Math. Mag.* **22**(4), 195–198 (1949)
5. Bukh, B., Hubbard, A.: Space crossing numbers. *Comb. Probab. Comput.* **21**(3), 358 (2012)
6. Figiel, T., Lindenstrauss, J., Milman, V.: The dimension of almost spherical sections of convex bodies. *Acta Math.* **139**, 53–94 (1977)
7. Lehec, J.: On the Yao–Yao partition theorem. *Arch. Math.* **92**, 366–376 (2009)
8. Lévy, P., Pellegrino, F.: *Problemes concrets d’analyse fonctionnelle*. Collection de monographies sur la théorie des fonctions, vol. 8. Gauthier-Villars, Paris (1951)
9. Matoušek, J.: *Using the Borsuk–Ulam Theorem*. Universitext. Springer, Berlin (2003)
10. Musin, O.R.: Borsuk–Ulam type theorems for manifolds. *Proc. Am. Math. Soc.* **140**(7), 2551–2560 (2012)
11. Soberón, P.: Balanced convex partitions of measures in $\mathbb{R}^d$. *Mathematika* **58**, 71–76 (2012)
12. Yao, A.C., Yao, F.F.E.: A general approach to d-dimensional geometric queries. In: *Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing*, pp. 163–168. ACM, New York (1985)