LR Characterization of Chirotopes of Finite Planar Families of Pairwise Disjoint Convex bodies

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Abstract We extend the classical LR characterization of chirotopes of finite planar families of points to chirotopes of finite planar families of pairwise disjoint convex bodies: a map χ on the set of 3-subsets of a finite set I is a chirotope of finite planar families of pairwise disjoint convex bodies if and only if for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of 3-subsets of J is a chirotope of finite planar families of pairwise disjoint convex bodies. Our main tool is the polarity map, i.e., the map that assigns to a convex body the set of lines missing its interior, from which we derive the key notion of arrangements of double pseudolines, introduced for the first time in this paper.

Keywords Convexity · Discrete geometry · Projective planes · Pseudoline arrangements · Chirotopes

1 Introduction

The term planar in the title makes reference to real two-dimensional projective planes. We review what we need of the basics of real two-dimensional projective planes

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Fig. 1 **a** A cross surface represented by a *circular diagram* with antipodal boundary points identified; **b** a pseudoline; **c** a double pseudoline with one of its core pseudolines drawn in *dashed* and with its disk side in *white*; **d** an oriented double pseudoline with one of its core oriented pseudolines drawn in *dashed*

and especially the notion of convex body before introducing the notion of chirotope, explaining the main result of the paper and the main lines of its proof.

1.1 Cross Surfaces and Projective Planes

We assume that the reader is familiar with basic notions of algebraic and combinatorial topology like homeomorphism, homotopy, fundamental group, covering, etc., found, for example, in [35, Chaps. 0 and 1]. The following standard notions, basic results and terminology associated with projective planes will be used throughout the paper; they are mainly taken from [28, 45, 48, 49]

- (1) A *closed (open) topological disk* or *closed (open) two-cell* is a topological space homeomorphic to the unit closed (open) disk of $\mathbb{R}^2$. An *orientation* of a topological disk is a one-to-one parametrization of the topological disk by the unit disk of $\mathbb{R}^2$, defined up to direct homeomorphism, and an *oriented topological disk* is a topological disk endowed with an orientation. Orientations will be indicated in our drawings by a little oriented circle in the interior of the disk or by an arrow on its boundary.
- (2) A *cross surface*¹ is a topological space homeomorphic to the “standard” cross surface $\mathbb{RP}^2$, quotient of the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ under identification of antipodal points; cross surfaces will be represented in our drawings by circular diagrams with antipodal boundary points identified, as illustrated in Fig. 1a.
- (3) An *open crosscap* or *open Möbius strip* is a topological space homeomorphic to a cross surface with one point or one closed topological disk deleted; an open crosscap is a noncompact surface and its one-point compactification (the space obtained by adding to the crosscap a point at infinity) is a cross surface.
- (4) A *pseudocircle* is a simple closed curve embedded in a cross surface; the connected components of the complement of a pseudocircle in its underlying cross surface are called its *open sides*, or simply its *sides*. An *oriented pseudocircle* is a pseudocircle endowed with an *orientation* (i.e., a one-to-one parametrization of the pseudocircle by $\mathbb{S}^1$, defined up to direct homeomorphism), indicated in our

¹ We follow the Conway’s proposition to call a sphere with one crosscap a *cross surface*; cf. [22].

drawings by an arrow; as usual the intersection of two oriented pseudocircles is the intersection of their unoriented versions.

- (5) A *pseudoline* is a non-separating pseudocircle and a *double pseudoline* or *pseudo-oval* is a separating pseudocircle; cf. Fig. 1b, c. There is a unique isomorphism class of pseudolines, i.e., given two pseudolines, one is the image of the other by a homeomorphism of their underlying cross surfaces; in particular the complement of a pseudoline is an open two-cell. Similarly for double pseudolines: there is a unique isomorphism class of double pseudolines and the complement of a double pseudoline has two connected components (an open two-cell and an open cross-cap). The *core* pseudolines of a double pseudoline are the pseudolines contained in its crosscap side; cf. Fig. 1c, d.
- (6) A *projective plane* is a topological point-line incidence geometry $(\mathcal{P}, \mathcal{L})$ whose point space $\mathcal{P}$ is a cross surface, whose line space $\mathcal{L}$ is a subspace of the space of pseudolines of $\mathcal{P}$, and whose incidence relations are the membership relations; as usual the dual of a point p of a projective plane is denoted p^* and is defined as its set of incident lines. The *duality principle* for projective planes asserts that the dual $(\mathcal{L}, \mathcal{P}^*)$ of a projective plane $(\mathcal{P}, \mathcal{L})$ is still a projective plane, i.e., $\mathcal{L}$ is a cross surface and $\mathcal{P}^*$, the set of p^* as p ranges over $\mathcal{P}$, is a subspace of the space of pseudolines of $\mathcal{L}$. In particular the dual of a finite set of points is an *arrangement of pseudolines*, i.e., a finite set of pseudolines (living in the same cross surface) that intersect pairwise in exactly one point; the basics of pseudoline arrangements used in the paper are reviewed in Appendix A. A projective plane is isomorphic to its bidual via the map that assigns to a point its dual and to a line the set of duals of its points.
- (7) The standard projective plane is defined as the standard cross surface $\mathbb{RP}^2$ together with the image of the space of great circles of $\mathbb{S}^2$ under the canonical projection $\mathbb{S}^2 \rightarrow \mathbb{RP}^2$. (Equivalently the standard projective plane can be defined as the projective completion of the Euclidean plane.) The standard projective plane is isomorphic to its dual via the map φ that assigns to the point $(u, v, w) \in \mathbb{S}^2$ the great circle with equation $ux + vy + wz = 0$ and that assigns to the great circle with equation $ux + vy + wz = 0$, for $(u, v, w) \in \mathbb{S}^2$, the pencil of circles through the point (u, v, w) .

A *convex body* is a closed subset of the point space of a projective plane whose intersection with any line of the plane is a (necessarily closed) line segment; the *polar* of a convex body U , denoted $U^\diamond$, is the set of lines of the plane missing the interior of the convex body and its *dual*, denoted U^* , is the set of lines of the plane intersecting the body but not its interior, *tangents* for short. For example, for $(u, v, w) \in \mathbb{S}^2$ and $h \in (0, 1)$, the disk in the standard projective plane with equation

$$|ux + vy + wz| \geq (1 - h^2)^{1/2} \quad (1)$$

is a convex body, its polar is the disk with equation $|ux + vy + wz| \geq h$, and its dual is the circle with equation $|ux + vy + wz| = h$. Similarly for finitely generated (pointed and full-dimensional) cones of the standard projective plane: the polar of the cone generated by the vectors $(u_i, v_i, w_i) \in \mathbb{S}^2$, $w_i > 0$, is the polyhedral cone

intersection of the half-spaces $u_i x + v_i y + w_i z \geq 0$, $z \geq 0$. As illustrated in these examples, a convex body of a projective plane is a closed topological disk, its polar is a convex body of the dual projective plane, and its dual is the boundary of its polar, hence a double pseudoline. Furthermore, polarity extends to oriented convex bodies: the polar of an oriented convex body has a natural orientation, inherited from the orientation of the body, compatible with the reorientation operation. (In the case where there is exactly one tangent through each boundary point and only one touching point per tangent the orientation of the polar inherited from the orientation f is simply defined as the extension to the unit closed disk of the map that assigns to $u \in \mathbb{S}^1$ the tangent to the convex body through the boundary point $f(u)$ of U . The general case follows once it is observed that the set of boundary points through which passes a proper interval of tangents and the set of proper line segments included in the boundary are both countable.) Last but not least, we take for granted that, up to homeomorphism, the dual arrangement of a pair of disjoint convex bodies of a projective plane is the unique arrangement of two double pseudolines that intersect transversely in four points and induce a cellular decomposition of their underlying cross surface.

Theorem 1 *A convex body of a projective plane is a topological disk, its polar is a convex body of the dual projective plane, and its dual is the boundary of its polar (hence a double pseudoline). Furthermore, up to homeomorphism, the dual arrangement of a pair of disjoint convex bodies of a projective plane is the unique cellular arrangement of two double pseudolines that intersect transversely in four points; in particular, two disjoint convex bodies share exactly four common tangents, the arrangement of these four tangents is simple, and the set of lines missing the two bodies is nonempty.*

Proof No proofs of these basic properties are available in the literature on convexity in projective planes that we became aware [7–9, 11, 12, 14–16, 36, 39, 52]. For completeness we offer proofs in Appendix C. $\square$

Figure 2a shows a pair of disjoint convex bodies with the arrangement of their four common tangents. Each body is indexed, oriented and marked with an interior point. Figure 2b shows its dual arrangement. The automorphism group of the dual arrangement is trivial, the permutation group of a 2-set or the dihedral group of order 8 (group of automorphisms of the square) depending on whether orientation and indexing of the pseudocircles are both taken into account, the orientation of the pseudocircles

Fig. 2 **a** Two disjoint oriented convex bodies with their common tangents and **b** their dual arrangement

Fig. 3 **a** The rhombicuboctahedron arrangement; **b** an indexed and oriented version of the rhombicuboctahedron arrangement; **c** the cube (or hemi-cube) arrangement; **d** an arrangement of three double pseudolines obtained from the hemi-cube arrangement by a splitting mutation

is taken into account but not their indexing, or neither the orientation nor the indexing are taken into account. Observe that the dual arrangement does not encode the nature of the contacts between the convex bodies and their common tangents. Thereafter, the four common tangents of two disjoint convex bodies will be called their *bitangents*.

1.2 Definitions and Main Results

Throughout the paper, we use the words *configuration of convex bodies* for a finite family of pairwise disjoint convex bodies of a projective plane and we use, unless specified otherwise, the words *arrangement of double pseudolines* for a finite family of double pseudolines of a cross surface with the property that its subfamilies of size two are homeomorphic to the dual arrangement of a (hence any) configuration of two convex bodies; cf. Theorem 1. The *rhombicuboctahedron* or *hemi-rhombicuboctahedron arrangement* is the arrangement of double pseudolines composed of the three circles of the standard projective plane with centers $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and radius $\arccos 1/2$ or, to say it differently, with equations are $|x| = 1/2$, $|y| = 1/2$ and $|z| = 1/2$. Its face poset is that of the projective version of the rhombicuboctahedron (hence the name), one of the 13 Archimedean solids. The *cube* or *hemi-cube arrangement* is the arrangement of double pseudolines composed of the 3 circles of the standard projective plane with equations $|x| = 1/\sqrt{3}$, $|y| = 1/\sqrt{3}$ and $|z| = 1/\sqrt{3}$. Its face poset is obtained from that of the projective version of the cube (the hemi-cube) by replacing its 1-cells by digons; cf. Fig. 3. We extend in the natural way the basic terminology of arrangements of pseudolines to arrangements of double pseudolines. In particular we use the following terminology.

- (1) A vertex of an arrangement is *ordinary* if exactly two curves of the arrangement meet at that vertex. An arrangement is *simple* if all vertices of it are ordinary. Three vertices of the arrangement of Fig. 3d are ordinary; three are not. The rhombicuboctahedron arrangement is simple; the hemi-cube arrangement is not.
- (2) A *mutation* is a homotopy of arrangements during which only one of the curves of the arrangement is moving and only one of the incidence relations between the moving curve and the vertices of the cell complex induced by the other curves changes its value, swapping from false to true (first case) or from true to false (second case): In the first case we speak of a *merging* mutation and in second case

Fig. 4 **a** The first barycentric subdivision of an arrangement of two double pseudolines and **b** its flag diagram together with a pair τ_1, τ_2 of generators of its automorphism group, implicitly defined by the images of the flag F ; **c** the first barycentric subdivision of the hemi-cube arrangement and **d** its flag diagram together with a triple τ_1, τ_2, τ_3 of generators of its automorphism group, implicitly defined by the images of the flag F

we speak of a *splitting* mutation. Figure 3c and d shows arrangements of three double pseudolines that are related by mutations of merging and splitting.

- (3) The *flag diagram* of an arrangement is the 3-valent graph on its set of *flags* (maximal simplices of its first barycentric subdivision) whose edges are the pairs of adjacent flags, each edge being labeled by the numeral 0, 1 or 2 depending on whether the flags of the edge differ by their 0-, 1-, or 2-cells; one can also think a flag diagram as the Cayley graph of the group generated by the 0-, 1- and 2-flag operators, denoted σ_0, σ_1 and σ_2 in the sequel, which are the involutive operators on the set of flags that exchange two adjacent flags that differ by their 0-, 1-, or 2-cells, respectively. Figure 4a and b shows the (geometric version of the) first barycentric subdivision and the flag diagram of an arrangement of two double pseudolines. Figure 4c and d shows this for the hemi-cube arrangement.

- (4) An *indexed arrangement of oriented double pseudolines* is a one-to-one map that assigns to each index of a finite set of indices an oriented double pseudoline of a cross surface such that the image of the map is an arrangement of oriented double pseudolines.
- (5) The *isomorphism class* of an arrangement is its set of homeomorphic images: in other words, two arrangements are called *isomorphic* if one is the image of the other by a homeomorphism of their underlying cross surfaces. The *isomorphism class* of an indexed arrangement of oriented double pseudolines is defined in a similar way.
- (6) Let Δ be a finite abstract simplicial complex. A Δ -*chirotope* is a map on Δ that assigns to the simplex J an isomorphism class of arrangements of oriented double pseudolines indexed by J with the property that if J' is a subset of J then $\chi(J')$ is a subarrangement of $\chi(J)$. The $\chi(J)$, $J \in \Delta$, are called the *entries* of the Δ -chirotope χ . A k -*chirotope on the indexing set I* is a Δ -chirotope whose domain Δ is the complex of subsets of size at most k of I , and a *chirotope* is the restriction of a 3-chirotope to the set of 3-subsets of its domain.
- (7) For any indexed arrangement of oriented double pseudolines Γ and any simplicial complex Δ on the indexing set of Γ the Δ -*chirotope* of Γ is the map χ_Γ on Δ that assigns to $J \in \Delta$ the isomorphism class of the subarrangement of Γ indexed by J .

Arrangements of double pseudolines are conveniently represented by their flag diagrams, in view of the following two properties:

- (1) two arrangements are isomorphic if and only if their flag diagrams are isomorphic, cf. [2, Appendix 4.7]; and
- (2) the group of automorphisms of an arrangement (by definition quotient of the group of self-homeomorphisms of the arrangement by its subgroup of self-homeomorphisms isotopic to the identity map) is isomorphic to the group of automorphisms of its flag diagram or, equivalently, to the centralizer of the flag operators in the group of permutations of the flags. Note that an automorphism is defined by the image of one flag since the face poset of an arrangement is flag-connected.

Example 1 The automorphism group of an arrangement of two double pseudolines is the dihedral group of order 8, the group of automorphisms of the square. The automorphisms τ_1 and τ_2 defined by $\tau_1(F) = \sigma_1(F)$ and $\tau_2(F) = \sigma_0(F)$ where F is any one of the 8 flags of the tetragon intersection of the crosscap sides of the double pseudolines of the arrangement are an example of pair of generators of this group, for which $\tau_1^2 = \tau_2^2 = 1$ and $(\tau_2\tau_1)^4 = 1$ is a complete set of relations; cf. Fig. 4a and b.

Example 2 The automorphism group of the hemi-cube arrangement is the permutation group of a 4-set. The automorphisms τ_1 , τ_2 and τ_3 defined by $\tau_1(F) = \sigma_1(F)$, $\tau_2(F) = \sigma_0(F)$, and $\tau_3(F) = \sigma_1\sigma_2\sigma_1\sigma_2(F)$ where F is any one of the 3×8 flags of the three tetragons of the arrangement (each tetragon is the intersection of the crosscap sides of a pair of double pseudolines) are an example of triple of generators of this group, for which $\tau_1^2 = \tau_2^2 = \tau_3^2 = 1$, $(\tau_1\tau_2)^4 = 1$, $\tau_1\tau_3 = \tau_3^2\tau_1$ and $\tau_2\tau_3 = \tau_3(\tau_1\tau_2)^2$ is a complete set of relations; cf. Fig. 4c and d.

Besides (appropriately labeled) flag diagrams, two other codings of indexed arrangements of oriented double pseudolines are used in the paper. Both are defined using the idea of *signed indices*, namely the original indices $i_1, i_2, \dots, i_n$ and their complements $\bar{i}_1, \dots, \bar{i}_n$; the original indices are said to be positive, their complements are said to be negative, and the complement of a negative index is its positive version; cf. [40, p. 12]. Indexed arrangements of oriented double pseudolines are now extended to negative indices by assigning to a negative index the reoriented version of the double pseudoline assigned to its complement. In this introduction we only give the definition of one of these two codings, namely the coding by *side cycles*.

Let Γ be an indexed arrangement of oriented double pseudolines. Its coding by side cycles assigns to each (positive and negative) index of Γ two circular words on the set of indices: the first one is called its *side cycle of disk type* and the second one is called its *side cycle of crosscap type*. The side cycle of disk type assigned to the index i is the circular sequence of indices of the double pseudolines crossed by the side wheel of a sidecar rolling on Γ_i , side wheel on the disk side of Γ_i , that are (locally) oriented away from Γ_i . Similarly the side cycle of crosscap type assigned to the index i is the circular sequence of indices of the double pseudolines crossed by the side wheel of a sidecar rolling on Γ_i , side wheel on the crosscap side of Γ_i , that are (locally) oriented away from Γ_i . Note that the side cycles of disk (crosscap) type assigned to an index and to its complement are reverse to one another and that for simple arrangements the side cycle of disk type assigned to an index is the complement of its side cycle of crosscap type and vice versa. We show in Sect. 4 that the isomorphism class of an indexed arrangement of oriented double pseudolines depends only on its side cycles.

Example 3 The side cycles of disk type and crosscap type of the rhombicubeoctahedron arrangement of Fig. 3b are

$$\begin{array}{ll} 1 : 2233\overline{2233}, & 1 : \overline{2333}2233, \\ 2 : 3311\overline{3311}, & \text{and} \quad 2 : \overline{3311}3311, \\ 3 : 1122\overline{1122}, & 3 : \overline{1122}1122. \end{array}$$

Example 4 The side cycles of disk type and crosscap type of the hemi-cube arrangement of Fig. 5a are

$$\begin{array}{ll} 1 : 32\overline{23}3\overline{223}, & 1 : \overline{233}223\overline{32}, \\ 2 : \overline{311}33\overline{11}3, & \text{and} \quad 2 : \overline{13311}3\overline{31}, \\ 3 : \overline{211}22\overline{11}2, & 3 : 122\overline{11}22\overline{1}. \end{array}$$

Similarly the side cycles of disk type and crosscap type of the arrangement of Fig. 5b (obtained from that of Fig. 5a by a splitting mutation) are

$$\begin{array}{ll} 1 : 32\overline{23}3\overline{223}, & 1 : \overline{233}223\overline{32}, \\ 2 : \overline{311}33\overline{11}3, & \text{and} \quad 2 : 3\overline{1311}3\overline{31}, \\ 3 : \overline{211}22\overline{11}2, & 3 : 212\overline{11}22\overline{1}. \end{array}$$

Note that these two arrangements have the same side cycles of disk type but differ in their side cycles of crosscap type.

Fig. 5 **a** The first barycentric subdivision of the one-skeleton of an indexed and oriented version of the hemi-cube arrangement: each edge of the subdivision is labeled with the index of the signed supporting curve of the edge that is, locally on the edge, oriented away from the vertex of the arrangement to which the edge is incident; **b** an indexed and oriented version of the arrangement of Fig. 3d

We are now ready to state the first main result of the paper. It is a direct extension of the rank three case or pseudoline case of the Folkman–Lawrence LR characterization (LR for local realizability) of chirotopes of arrangements of pseudohyperplanes [21].

Theorem 2 *The map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its chirotope is one-to-one and its range is the set of maps χ on the set of 3-subsets of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of 3-subsets of J is a chirotope of arrangements of double pseudolines. In other terms, the map which assigns to an isomorphism class of indexed arrangements of double pseudolines its 3-chirotope is one-to-one and that which assigns its 5-chirotope is (one-to-one and) onto.*

The main lines of its proof are the following:

Concerning the range part we proceed in three steps. First, we extend the notion of double pseudoline arrangements by relaxing the condition on the arrangement which says that the genus of its underlying nonorientable surface is 1 while retaining locally in the vicinity of a curve of the arrangement, the notions of disk side and crosscap side. (The underlying surface of a subarrangement of size at least 2 is the one obtained by gluing topological disks along the boundaries of a closed tubular neighborhood of the curves of the subarrangement in the underlying surface of the whole arrangement. Thus, a subarrangement does not necessarily live in a surface whose genus is that of the underlying surface of the whole arrangement. By convention the underlying surface of a subarrangement of size zero or one is a cross surface.) Second, we show that *the map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its 5-chirotope is one-to-one and onto*. Third, we characterize among these arrangements *those living in a cross surface as those whose subarrangements of size at most 5 live in a cross surface*. To prove that the arrangements living in a cross surface are those whose subarrangements of size at most 5 live in a cross surface it can be argued that the mutation graphs of the latter are connected or that for any pair FF' of distinct faces of an arrangement living in a cross surface, there exists a subarrangement of size at most 3 whose faces containing F and F' are distinct, the

separation property for short. Thus, a byproduct of our study is the following direct extension of the Ringel homotopy theorem for arrangements of pseudolines [46].

Theorem 3 *Any two arrangements of double pseudolines of the same size and living in the same cross surface are homotopic via a finite sequence of mutations followed by an isotopy; in other words, mutation graphs are connected.*

Further analysis of the separation property leads us to prove that an arrangement of double pseudolines whose subarrangements of size at most 4 live in cross surfaces, lives in a cross surface or its subarrangements of size 4 belong to a well-defined class of few tens of arrangements. Therefore, a computer check of the conjecture that the arrangements of double pseudolines living in cross surfaces are those whose subarrangements of size at most 4 live in cross surfaces is doable with modest computing resources. This computer check will be the subject of another paper. That's all for the range part.

Concerning the one-to-one part we proceed by induction on the number of double pseudolines, the crucial case being the base case of 4 double pseudolines and, more specifically, the base case of 4 double pseudolines with a chosen one whose intersections with each of the others are ordinary and occur in consecutive runs, *Türkenbund* or *martagons*² for short. Fortunately the list of martagons on 4 double pseudolines is easily calculated by hand from the exhaustive list of simple arrangements of 3 double pseudolines which in turn is (less) easily calculated by hand using the connectedness of mutation graphs. It turns out that there are only two martagons on four double pseudolines and that each depends only on its chirotope.

We come now to the definition of chirotopes of configurations of convex bodies. Our definition is a natural extension of the classical definition of chirotopes of configurations of points of the standard projective plane; cf. Appendix B. As for arrangements of double pseudolines, indexed configurations of oriented convex bodies are extended to negative indices by assigning to a negative index the reoriented version of the convex body assigned to its complement.

Let Δ be an indexed configuration of oriented convex bodies of a projective plane $(\mathcal{P}, \mathcal{L})$, let τ be a line of $(\mathcal{P}, \mathcal{L})$, let $\mathcal{R}_\tau$ be the equivalence relation on $\mathcal{P}$ generated by the pairs of points belonging to a same line segment of $\Delta \cap \tau$, and let $\omega_\tau : \mathcal{P} \rightarrow \mathcal{P}/\mathcal{R}_\tau$ be the associated quotient map. We define

- (1) the *cocycle of Δ at τ* or the *cocycle of τ with respect to Δ* or the *cocycle of the pair (Δ, τ)* as the homeomorphism class of the image of the pair (Δ, τ) under ω_τ , i.e., the set of $(\varphi\omega_\tau\Delta, \varphi\omega_\tau\tau)$ as φ ranges over the set of homeomorphisms with domain $\mathcal{P}/\mathcal{R}_\tau$;
- (2) a *bitangent cocycle* or *zero-cocycle* as a cocycle at a bitangent;
- (3) the *isomorphism class of Δ* as the set of configurations that have the same set of bitangent cocycles as Δ , hence the same set of cocycles as Δ (use a simple perturbation argument); and

² “Da stehn sie also, die Geschwisterkinder, links blüht der Türkenbund, blüht wild, blüht wie nirgends, und rechts, da steht die Rapunzel, und Dianthus superbus, die Prachtnelke, steht nicht weit davon.” Gespräch im Gebirg, Paul Celan.

- (4) the *chirotope* of Δ as the map that assigns to each 3-subset J of the indexing set of Δ the isomorphism class of the subfamily indexed by J .

Figure 6 depicts the bitangent cocycles of an indexed configuration of three oriented convex bodies. Observe that the cocycle of a tangent to a body does not encode the nature, line segment or point, of the intersection between the tangent and the body since the map ω_τ reduces this intersection to a point. A cocycle is conveniently represented by its *signature*: a set of words on the signed indices plus the extra symbol $\bullet$ that is defined as follows. Let $\mathcal{D}_\tau$ be the closed 2-cell obtained by cutting $\mathcal{P}/\mathcal{R}_\tau$ along the line $\tau/\mathcal{R}_\tau$, let $\nu_\tau : \mathcal{D}_\tau \rightarrow \mathcal{P}/\mathcal{R}_\tau$ be the induced canonical projection, let Σ_τ be the set of connected components of the pre-images under ν_τ of the (indexed and oriented) convex bodies of Δ (the cardinality of Σ_τ is twice the number of convex bodies intersected by τ plus the number of bodies missed by τ), let ε be an orientation of $\mathcal{D}_\tau$ and let $\bar{\varepsilon}$ be its opposite. The *signature of the pair* (Δ, τ) or the *signature of* Δ at τ is then defined as the pair of *signatures of the triples* $(\Delta, \tau, \varepsilon)$ and $(\Delta, \tau, \bar{\varepsilon})$ where the *signature of a triple* $(\Delta, \tau, \varepsilon)$ is the set of indices of the elements of Σ_τ with orientation ε contained in the interior of $\mathcal{D}_\tau$ plus the circular sequence of indices of the elements of Σ_τ with orientation ε encountered when walking along the boundary of $\mathcal{D}_\tau$ according to the orientation ε with the convention that the indices indexing points are replaced by the extra symbol $\bullet$. Since the signature of the triple $(\Delta, \tau, \bar{\varepsilon})$ is obtained from that of the triple $(\Delta, \tau, \varepsilon)$ by replacing each of its elements by the reversal of its complement (with the convention that $\bullet$ is its own complement) the signature of (Δ, τ) can be represented by any of its two elements. Clearly the cocycle of a pair (Δ, τ) depends only on its signature and vice-versa. Figure 7 depicts the bitangent cocycles of configurations of two and three convex bodies together with their signatures.

We are now ready to state the second main result of the paper. Its onto part is called the *geometric representation theorem for arrangements of double pseudolines* thereafter.

Theorem 4 *The map that assigns to an indexed configuration of oriented convex bodies the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of oriented convex bodies. Furthermore the induced quotient map is one-to-one and onto, i.e., any arrangement of double pseudolines is isomorphic to the dual arrangement of a configuration of convex bodies.*

The main lines of its proof are the following:

Compatibility and one-to-one parts are easy consequences of two basic properties of cocycles: first, the injectivity of the map that assigns to each cell of the dual arrangement of an indexed configuration of two oriented convex bodies the cocycle of the configuration at some (hence any) element of the cell, and, second, the injectivity of the map that assigns to a bitangent cocycle of an indexed family of at least three oriented convex bodies the sub-cocycles obtained by removing in turn each of the convex bodies. Concerning the onto part we show that the property of being isomorphic to the dual arrangement of a configuration of convex bodies is invariant under mutation (mutation graphs being connected the result follows). To this end we first show that the isomorphism class of the dual arrangement of an indexed configuration

Fig. 6 **a** The dual configuration of an indexed and oriented version of the hemi-cube arrangement together with its bitangents (these bitangents are all tritangents and are labeled A , B , C and D to ease the correspondence between the diagrams); **b** its bitangent cocycles with their signatures

of oriented convex bodies depends only on the isomorphism class of its (appropriately) indexed arrangement of bitangents, its *Rapunzel* or *raiponce* for short. Then we easily characterize the class of raiponces, using the *enlargement theorem for pseudoline arrangements* of Goodman et al. [28]; cf. Appendix A. And finally we explain how to push back a mutation at the level of raiponces (the resulting operation is not a mutation). Combining Theorems 2 and 4 we get the result announced in the abstract, namely:

Theorem 5 *The map that assigns to an isomorphism class of indexed configurations of oriented convex bodies its chirotope is one-to-one and its range is the set of maps χ on the set of 3-subsets of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of 3-subsets of J is a chirotope of configurations of convex bodies.*

1.3 Organization of the Paper

The paper is organized as follows. In Sect. 2 we prove that mutation graphs are connected (Theorem 3) and we use this connectedness result to compute the isomorphism classes of simple arrangements of three double pseudolines and the martagons on three and four double pseudolines. In Sect. 3 we show, again using the connected-

Fig. 7 The bitangent cocycles on the indexing sets $\{1, 2\}$ and $\{1, 2, 3\}$: each bitangent cocycle is labeled at its *bottom left* with its signature and at its *bottom right* with its number of reoriented and reindexed versions: thus the number of bitangent cocycles on a given set of two indices is exactly the number (4) of bitangents of a pair of disjoint convex bodies, and the number of bitangent cocycles on a given set of three indices is $8 + 4 \times 24 = 104$

ness of mutation graphs proved in Sect. 2, that any arrangement of double pseudolines is isomorphic to the dual arrangement of a configuration of convex bodies (onto part of Theorem 4). In Sect. 4 we prove that the isomorphism class of an indexed arrangement of oriented double pseudolines depends only on its chirotope (Theorem 2) and we prove that the map that assigns to a configuration of convex bodies the isomorphism class of its dual arrangement is compatible with the isomorphism relation on configurations of convex bodies and that the induced quotient map is one-to-one (Theorem 4). In Sect. 5 we introduce the arrangements of double pseudolines living in nonorientable surfaces of arbitrary genus and we prove the LR characterization of chirotopes of indexed arrangements of oriented double pseudolines living in cross surfaces (Theorem 2). Still in Sect. 5 we offer results in strong support of the conjecture that the arrangements living in cross surfaces are those whose subarrangements of size at most 4 live in cross surfaces. In Sect. 6 we discuss *arrangements of pseudocircles* (as natural extensions of both arrangements of pseudolines and arrangements of double pseudolines), *crosscap or Möbius arrangements* and their *fibrations* (as dual arrangements of affine configurations of convex bodies with, in particular, a positive answer to a question of Goodman and Pollack about the realizability of their double permutation sequences by affine configurations of pairwise disjoint convex bodies). We conclude in the seventh and last section with a list of open problems suggested by this research.

2 Homotopy Theorem

In this section we prove that any two arrangements of double pseudolines with the same number of double pseudolines and living in the same cross surface are homotopic via a finite sequence of mutations followed by an isotopy; cf. Theorem 3. We proceed into two steps:

- (1) firstly, in order to benefit from Ringel’s homotopy theorem for arrangements of pseudolines, we embed the collection of isomorphism classes of simple arrangements of pseudolines into the collection of isomorphism classes of arrangements of double pseudolines; the embedding is canonical and is based on the notion of *thin* arrangement of double pseudolines;
- (2) secondly (and this is the core of our proof) we introduce a ‘pumping’ device to come down to the case of arrangements of pseudolines.

We also provide representatives of the isomorphism classes of simple arrangements of three double pseudolines and we use these representatives to compute the full list of martagons on three and four double pseudolines. Recall that martagons are arrangements that play a special rôle in the proof that the isomorphism class of an indexed arrangement of oriented double pseudolines depends only on its chirotope.

2.1 Thin Arrangements of Double Pseudolines

A simple arrangement of double pseudolines is *thin* if the crosscap sides of its double pseudolines are free of vertices. A thin arrangement of double pseudolines Γ^* is a *double* of a simple arrangement of pseudolines Γ (or Γ is a *core* arrangement of pseudolines of Γ^*) if there exists a one-to-one correspondence between Γ and Γ^* such that any pseudoline of Γ is a core pseudoline of its corresponding double pseudoline in Γ^* . For example the rhombicuboctahedron arrangement is thin and is the double of the octahedron arrangement, the unique simple arrangement of 3 pseudolines; cf. Fig. 8. The following two lemmas are simple consequences of the definitions.

Lemma 6 *The map that assigns to a simple arrangement of pseudolines its set of doubles induces a one-to-one and onto correspondence between the set of isomorphism classes of simple arrangements of pseudolines and the set of isomorphism classes of thin arrangements of double pseudolines.*

Fig. 8 **a** The octahedron arrangement and **b** its double, the rhombicuboctahedron arrangement

Fig. 9 **a** An arrangement of two double pseudolines; **b** its 2-sheeted unbranched covering

Lemma 7 Let Γ and Γ' be two simple arrangements of pseudolines and let Γ^* and Γ'^* be double versions of Γ and Γ' . Assume that Γ and Γ' are connected by a sequence of two mutations (a merging mutation followed by its ‘symmetric’ splitting mutation) during which the moving pseudoline is γ_i . Then Γ^* and Γ'^* are homotopic via a sequence of sixteen mutations during which the only moving double pseudoline is Γ_i^* .

2.2 The Pumping Lemma

We come now to the statement of our pumping lemma and to the proof of our homotopy theorem.

Lemma 8 (Pumping Lemma) Let Γ be a simple arrangement of double pseudolines, and $\gamma \in \Gamma$. Assume that there is a vertex of the arrangement Γ lying in the crosscap side of γ . Then there is a triangular two-cell of the arrangement Γ contained in the crosscap side of γ with a side supported by γ .

Proof Let $\mathcal{P}$ be the underlying cross surface of Γ and let $p : \tilde{\mathcal{P}} \rightarrow \mathcal{P}$ be a 2-sheeted unbranched covering of $\mathcal{P}$. For example the two relations

$$\begin{cases} \alpha_1 \alpha_2 = 1, \\ \alpha_2 \alpha_1 = 1 \end{cases}$$

define a 2-sheeted unbranched covering of the cross surface defined by the relation $\alpha\alpha = 1$; cf. [37,42]. The two lifts under p of a curve τ of Γ are denoted τ_+ and τ_- , and the set of lifts of the curves of Γ is denoted $\tilde{\Gamma}$. Figure 9a shows a subarrangement of two double pseudolines and Fig. 9b shows its 2-sheeted unbranched covering. We note that two curves of $\tilde{\Gamma}$ have exactly 0 or 2 intersection points depending on whether they are the lifts of the same curve in Γ , or not. By convention if B is one of the two intersection points of two crossing curves of $\tilde{\Gamma}$ then the other one is denoted B_* , as illustrated in Fig. 9b. Let C be the cylinder of $\tilde{\mathcal{P}}$ bounded by γ_+ and γ_- . We introduce the following terminology.

- (1) A γ -curve supported by $\gamma' \in \Gamma$, $\gamma' \neq \gamma$, is a maximal subcurve of γ'_+ or γ'_- contained in the cylinder C . Observe that there are four γ -curves supported

Fig. 10 The three possible arrangements of two γ -curves

Fig. 11 The admissible γ -triangle Δ encloses the admissible γ -triangle T

- by γ' (two per lift of γ') and that a γ -curve has an endpoint on γ_+ and the other one on γ_- . The γ -curve with endpoint B on γ_+ is denoted $\text{curve}_\gamma(B)$.
- (2) An *arrangement of γ -curve* is a set of γ -curves embedded in the cylinder C . The cell decomposition of the cylinder C induced by an arrangement of two γ -curves depends only on the number of intersection points, as depicted in Fig. 10.
  - (3) A γ -*triangle* is a triangular face of the arrangement of two crossing γ -curves with a side supported by γ_+ ; the vertex of a γ -triangle not on γ_+ is called its *apex* and the side of a γ -triangle supported by γ_+ is called its *base side*. The interior and the exterior of the base side of a γ -triangle T , considered as a subset of γ_+ , are denoted $\text{Int}_\gamma(T)$ and $\text{Ext}_\gamma(T)$, respectively.
  - (4) A γ -triangle is *admissible* if one of its two sides with the apex as an endpoint is an edge of $\tilde{\Gamma}$.
  - (5) An admissible γ -triangle $\Delta = XYY'$ with apex X and edge side XY is said to enclose an admissible γ -triangle $T = ABB'$ with apex A and edge side AB if T is included in Δ and walking along the base side of Δ from Y to Y' we encounter B' before B , thus the arrangement of the four γ -curves $\text{curve}_\gamma(Y)$, $\text{curve}_\gamma(Y')$, $\text{curve}_\gamma(B)$, $\text{curve}_\gamma(B')$ is, up to homeomorphism, one of those implicitly depicted in Fig. 11a in case $B \neq Y'$ or one of those implicitly depicted in Fig. 11b in case $B = Y'$. $\square$

Lemma 9 *There is at least one admissible γ -triangle.*

Fig. 12 Relative positions of an admissible γ -triangle T and its derived admissible γ -triangle T'

Proof Since by assumption there is a vertex of Γ in the crosscap side of the double pseudoline γ , there is a γ -triangle, say $T = ABB'$ with apex A . Let A' be the vertex of $\tilde{\Gamma}$ that follows B' on the side $B'A$ of T . Then A' is the apex of an admissible γ -triangle $T' = A'B'B''$ with edge side $A'B'$. This proves that there is at least one admissible γ -triangle. $\square$

Now let $T = ABB'$ be an admissible γ -triangle with apex A and edge side AB , let A' be the vertex of $\tilde{\Gamma}$ that follows B' on the side $B'A$ of T , and let $T' = A'B'B''$ be the admissible γ -triangle with apex A' and with edge side $A'B'$. A simple use of the Jordan curve theorem leads to the following three lemmas concerning the relative positions T with respect to T' , possibly in the presence of a third admissible γ -triangle Δ enclosing T . Figure 12a–d illustrates these lemmas.

Lemma 10 Assume that $T = T'$. Then T is a triangular two-cell of $\tilde{\Gamma}$.

Lemma 11 Assume that $T \neq T'$ and that $B'' \in \text{Int}_\gamma(T)$. Then $\text{curve}_\gamma(B'')$ crosses the side $B'A$ of T exactly once (at A') and $\text{Int}_\gamma(T')$ is contained in $\text{Int}_\gamma(T)$.

Lemma 12 Assume that $T \neq T'$ and that $B'' \in \text{Ext}_\gamma(T)$. Then

- (1) $\text{curve}_\gamma(B')$ and $\text{curve}_\gamma(B'')$ cross twice (at A' and A'_*) on the side $B'A$ of T ,
- (2) $\text{Int}_\gamma(T)$ and $\text{Int}_\gamma(T')$ are disjoint,
- (3) B'_* and $B''_* \in \text{Ext}_\gamma(T) \cap \text{Ext}_\gamma(T')$, and
- (4) walking along $\text{Ext}_\gamma(T) \cap \text{Ext}_\gamma(T')$ from B'' to B we encounter successively the points B''_* and B'_* .

Furthermore if Δ encloses T then Δ encloses T' .

Consider now the sequence of admissible γ -triangles $T_0, T_1, T_2, \dots$ defined inductively by $T_0 = T$ and $T_{k+1} = T'_k$ for $k \geq 0$. A simple combination of Lemmas 12 and 11 leads to the conclusion that the sequence T_k is stationary. According to Lemma 10 the pumping lemma follows.

Remark 1 The proof of the pumping lemma involves only subarrangements of size at most 6; cf. Fig. 12d. A slightly more careful analysis shows that only the subarrangements of size at most 5 are relevant. This key feature is exploited in Sect. 5 to extend the classical LR characterization of chirotopes of arrangements of pseudolines to chirotopes of arrangements of double pseudolines; cf. Theorem 2.

Remark 2 The pumping lemma asserts that a certain instance of the problem of sweeping a spherical arrangement of pseudocircles crossing pairwise in 0 or 2 points has a positive answer. This problem is studied in full generality by Snoeyink and Hershberger [50] and, as pointed to us by an anonymous referee, the pumping lemma can be derived from their results. (It is necessary to use both Theorem 3.1 and Lemma 5.2 of [50].)

We are now ready for the proof of our homotopy theorem.

Theorem 3 *Any two arrangements of double pseudolines of the same size and living in the same cross surface are homotopic via a finite sequence of mutations followed by an isotopy; in other words, mutation graphs are connected.*

Proof Clearly any arrangement of double pseudolines is homotopic, via a finite sequence of splitting mutations, to a simple one. Now by a repeated application of the pumping lemma we see easily that any simple arrangement of double pseudolines is homotopic, via a finite sequence of mutations, to a simple thin one. It remains to use Lemmas 6, 7 and the homotopy theorem of Ringel for arrangements of pseudolines to conclude the proof. $\square$

For the sake of completeness, we mention that one of the standard ways to prove the Ringel's homotopy theorem for arrangements of pseudolines is to show that any arrangement of pseudolines is homotopic, via a finite sequence of mutations followed by an isotopy, to a cyclic arrangement of pseudolines using *avant la lettre* the following specialization to arrangements of pseudolines of our pumping lemma for arrangements of double pseudolines (think of a pair of pseudolines as a pinched double pseudoline).

Lemma 13 (*Pumping Lemma for Arrangements of Pseudolines*) *Let Γ be a simple arrangement of pseudolines, let $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$, and let $M(\gamma, \gamma')$ be one of the two two-cells of the subarrangement $\{\gamma, \gamma'\}$. Assume that there exists a vertex*

of the arrangement Γ lying in $M(\gamma, \gamma')$. Then there exists a triangular two-cell of the arrangement Γ contained in $M(\gamma, \gamma')$ with a side supported by γ' and a vertex contained in $M(\gamma, \gamma')$.

Proof The proof is standard and will not be repeated here; see e.g. [10]. $\square$

Remark 3 The proof of the pumping lemma for arrangements of pseudolines involves only subarrangements of size 4. This observation will be used in Sect. 5 to give a new proof of the classical LR characterization of chirotopes of indexed arrangements of oriented pseudolines. (For historical comments on the various proofs of the LR characterization of chirotopes of indexed arrangements of oriented pseudolines and, more generally, pseudo-hyperplanes, we refer to [3–5].)

Remark 4 At this point it is natural to ask if the space of one-extensions of an arrangement of double pseudolines is connected under mutations, as is the space of one-extensions of an arrangement of pseudolines [2, 25, 53]. (A one-extension of an arrangement of n pseudolines Γ is a arrangement of $n + 1$ pseudolines Γ' of which Γ is a sub-arrangement.) A positive answer to that question, providing the key to a practical enumeration algorithm for simple arrangements of at most 5 double pseudolines, is given in [20]. The proof presented in [20] of this connectedness result is based on an enhanced version of the pumping lemma which says that, given a double pseudoline γ of an arrangement Γ with the property that the vertices of the arrangement Γ lying on the curve γ are ordinary, either there are (at least) two fans contained in the crosscap side of the double pseudoline γ with base sides supported by γ or there are no vertices of the arrangement contained in the crosscap side of γ . The enhanced version of the pumping lemma can be easily proved using the geometric representation theorem for arrangements of double pseudolines. It will be interesting to have a direct proof of it since, as explained in [20], the geometric representation theorem for arrangements of double pseudolines can be derived from it.

2.3 Martagons

The exhaustive list of isomorphism classes of simple arrangements of three double pseudolines is depicted in Fig. 13. This list was first established by hand, using the connectedness of the corresponding mutation graph. The adjacency list representation of this graph is the following:

C_{04}	adjacent to	C_{07}
C_{07}	:	C_{04}, C_{15}, C_{18}
C_{15}	:	$C_{07}, C_{25_1}, C_{25_2}$
C_{18}	:	C_{07}, C_{25_1}, C_{37}
C_{22}	:	C_{25_2}
C_{25_1}	:	$C_{15}, C_{18}, C_{32}, C_{33}, C_{43}$
C_{25_2}	:	$C_{15}, C_{22}, C_{33}, C_{36}$

Fig. 13 Representatives of the 13 isomorphism classes of simple arrangements of three double pseudolines. In this figure each isomorphism class is labeled at its *bottom left* with a symbol to name the arrangement and at its *bottom right* with the order of its automorphism group

C_{32}	:	C_{25_1}
C_{33}	:	C_{25_1}, C_{25_2}
C_{36}	:	C_{25_2}
C_{37}	:	C_{18}, C_{43}, C_{64}
C_{43}	:	C_{25_1}, C_{37}
C_{64}	:	C_{37}

where C_α denotes the arrangement whose 2-sequence of its numbers of 2-cells of size 2 and 3 is α . Such a sequence identifies a unique isomorphism class of arrangements, with one exception: the sequence 25 identifies two isomorphism classes (which have also the same numbers of two-cells of size 4, 5, 6, etc). To distinguish them we use the sequences 25_1 and 25_2 , where the subscript stands for the order of the automorphism group of the corresponding arrangement. The orders of the automorphism groups of the arrangements are reported at the bottom right of the arrangements in Fig. 13.

Thus there are 13 isomorphism classes of arrangements of three double pseudolines and 216 isomorphism classes of indexed arrangements of three oriented double pseudolines on a given set of three indices (and not 214 as indicated by error in [20]). This latter number is computed as the sum

$$\sum_{k \geq 1} \frac{3!2^3}{k} g_k,$$

where g_k is the number of arrangements with group of automorphisms of order k . For the number of isomorphism classes of arrangements of four double pseudolines and for the number of isomorphism classes of simple arrangements of five double pseudolines we refer to [20].

Using the exhaustive list of simple arrangements of three double pseudolines we now compute the martagons on three and four double pseudolines. Recall the definition of martagons. An arrangement of $n \geq 3$ double pseudolines Γ is called a *martagon with respect to* a double pseudoline γ of Γ if the vertices of the arrangement on the curve γ are ordinary and if for any $\gamma' \in \Gamma$, $\gamma' \neq \gamma$, no pair of distinct elements v, v' of

$$\bigcup_{\gamma'' \in \Gamma: \gamma'' \neq \gamma', \gamma} \gamma'' \cap \gamma \quad (2)$$

is separated on the curve γ by a pair of distinct elements u, u' of $\gamma' \cap \gamma$; in other words, the four intersection points of γ' and γ are ordinary and appear consecutively on the curve γ . For example the arrangements C_{22} and C_{32} of Fig. 13 are martagons with respect to the curved double pseudoline. Figure 14 depicts examples of martagons on

Fig. 14 Martagons with respect to the double pseudoline that do not intersect the *dashed* pseudoline, *red* in colored pdf, on three and four double pseudolines. In this figure each double pseudoline whose crosscap side is free of vertices is simply represented by one of its core pseudolines (Color figure online)

three and four double pseudolines. Observe that the subarrangements of size three of M_1 are C_{22} (3 times) and C_{04} , and those of M_2 are C_{22} and C_{32} , both 2 times. The reader will have no difficulties adding to these examples martagons of arbitrary size. We leave the verification of the following lemma to the reader.

Lemma 14 *The only martagons on three and four double pseudolines are the arrangements of Fig. 14.*

3 Geometric Representation Theorem

In this section we prove the *Geometric Representation Theorem* for double pseudoline arrangements announced in the introduction: any arrangement of double pseudolines is isomorphic to the dual arrangement of a configuration of convex bodies. The main idea of the proof is to show that the property on the set of arrangements of double pseudolines of being *the dual arrangement of a configuration of convex bodies*, is stable under mutations. The main ingredients of the proof are

- (1) the connectedness of mutation graphs;
- (2) the coding of the isomorphism class of an indexed arrangement of oriented double pseudolines by its family of *node cycles*;
- (3) the *raiponces*: we name thus the (appropriately) indexed arrangements of bitangents of indexed configurations of oriented convex bodies;
- (4) the existence of a projective plane extension for any arrangement of pseudolines [28].

3.1 Nodes and Node Cycles of an Arrangement

Let Γ be an indexed arrangement of oriented double pseudolines and let $v(\Gamma)$ be the indexed family of vertices of Γ defined by the following three conditions:

- (1) the indexing set of $v(\Gamma)$ is the set of unordered pairs $ij(=ji)$ of signed indices of Γ with the property that $i \neq \bar{j}$;
- (2) the $v_\alpha(\Gamma)$, $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, are the four intersection points of the double pseudolines Γ_i and Γ_j ;
- (3) walking along the double pseudoline Γ_i we encounter the $v_\alpha(\Gamma)$, $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, in cyclic order $v_{ij}(\Gamma)$, $v_{\bar{i}j}(\Gamma)$, $v_{i\bar{j}}(\Gamma)$, $v_{\bar{i}\bar{j}}(\Gamma)$, as illustrated in Fig. 15a.

The reader will easily check that the family $v(\Gamma)$ is well-defined.

The set of *nodes* of Γ , denoted $\mathcal{V}(\Gamma)$, is the quotient of the indexing set of $v(\Gamma)$ under the relation “to index the same vertex of Γ ” and the indexed family of *node cycles* of Γ , denoted $\mathcal{C}(\Gamma)$, is the indexed family of circular sequences of nodes of Γ that correspond to the circular sequences of vertices of Γ encountered when walking along the double pseudolines of Γ , each circular sequence being indexed by the index of the double pseudoline on which is done the walk. Note that the cycles assigned to an index and its complement are reverse to one another. For example for the hemi-cube arrangement of Fig. 15b one has $\mathcal{V}(\Gamma) = \{A, B, C, D\}$, $\mathcal{C}_1(\Gamma) = ABCD$, $\mathcal{C}_2(\Gamma) = ACBD$, and

Fig. 15 Indexed families of vertices of indexed arrangements of two and three oriented double pseudolines

$\mathcal{C}_3(\Gamma) = ABDC$ where

$$\begin{aligned} A &= \{12, \overline{13}, \overline{23}\}, & B &= \{\overline{12}, 13, \overline{23}\}, \\ C &= \{\overline{12}, \overline{13}, 23\}, & D &= \{\overline{12}, \overline{13}, 23\}. \end{aligned}$$

Similarly for the arrangement of Fig. 15c, obtained from the hemi-cube arrangement of Fig. 15b by a splitting mutation, one has $\mathcal{V}(\Gamma) = \{A, B, C, D, E, F\}$, $\mathcal{C}_1(\Gamma) = EFBCD$, $\mathcal{C}_2(\Gamma) = EACBD$, and $\mathcal{C}_3(\Gamma) = AFBDC$ where

$$\begin{aligned} A &= \{\overline{23}\}, \\ E &= \{12\}, \\ F &= \{\overline{13}\}, \\ B &= \{\overline{12}, 13, \overline{23}\}, \\ C &= \{\overline{12}, \overline{13}, 23\}, \\ D &= \{\overline{12}, \overline{13}, 23\}. \end{aligned}$$

The family $\mathcal{C}(\Gamma)$ turns out to be a coding of the isomorphism class of Γ .

Theorem 15 *Two indexed arrangements of oriented double pseudolines are isomorphic if and only if they have the same indexed family of node cycles.*

Proof Let Γ be an indexed arrangement of oriented double pseudolines, let $\mathcal{F}(\Gamma)$ be the set of flags of the cell poset $X(\Gamma)$ of Γ and let $\sigma_i(\Gamma) : \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\Gamma)$, $i \in \{0, 1, 2\}$, be its flag operators. The *node*, *index* and *side* of a flag $F \in \mathcal{F}(\Gamma)$ are

- (1) the node of Γ corresponding to the zero-cell of F ;
- (2) the index of the supporting double pseudoline of the one-cell of F that is outgoing at the zero-cell of F ;
- (3) the symbol μ or its complement $\overline{\mu}$ depending on whether the two-cell of F is contained in the crosscap side of the supporting double pseudoline of the one-cell of F or is contained in its disk side.

Fig. 16 The first barycentric subdivision of an indexed arrangement of two oriented double pseudolines where each flag is labeled with its node, index and side

Table 1 Table of the operator $\widehat{\sigma}_1(\Gamma)$ in the case where Γ is an indexed arrangement of two oriented double pseudolines with signed indexing set $\{i, \bar{i}, j, \bar{j}\}$

F	$(\{ij\}, i, \mu)$	$(\{ij\}, i, \bar{\mu})$	$(\{ij\}, \bar{i}, \mu)$	$(\{ij\}, \bar{i}, \bar{\mu})$
$\widehat{\sigma}_1(\Gamma)(F)$	$(\{ij\}, j, \mu)$	$(\{ij\}, \bar{j}, \mu)$	$(\{ij\}, j, \bar{\mu})$	$(\{ij\}, \bar{j}, \bar{\mu})$

Figure 16 shows the first barycentric subdivision of an indexed arrangement of two oriented double pseudolines where each flag is labeled, using the obvious convention, with its node, index and side. Let I be the set of positive indices of Γ , let $\widehat{\mathcal{F}}(\Gamma) = \{(A, \nu, \eta) \mid \nu \in I \cup \bar{I}, A \in \mathcal{C}_\nu(\Gamma), \eta \in \{\mu, \bar{\mu}\}\}$, let $\omega(\Gamma) : \mathcal{F}(\Gamma) \rightarrow \widehat{\mathcal{F}}(\Gamma)$ be the (one-to-one and onto) map that assigns to the flag F the triple composed of the node, index and side of F and, for $i \in \{0, 1, 2\}$, let

$$\widehat{\sigma}_i(\Gamma) = \omega(\Gamma)\sigma_i(\Gamma)\omega(\Gamma)^{-1}.$$

Table 1 gives the table of the operator $\widehat{\sigma}_1(\Gamma)$ in the case where Γ is an arrangement of two double pseudolines.

Clearly two arrangements of oriented double pseudolines Γ and Γ' are isomorphic if and only if for any $i \in \{0, 1, 2\}$ the operators $\widehat{\sigma}_i(\Gamma)$ and $\widehat{\sigma}_i(\Gamma')$ coincide. Therefore proving our theorem comes down to proving that the operators $\widehat{\sigma}_i(\Gamma)$, $i \in \{0, 1, 2\}$, depend only on the indexed family $\mathcal{C}(\Gamma)$. We define $\bar{\mu} = \mu$. Clearly

- (1) $\widehat{\sigma}_2(\Gamma)(A, \nu, \eta) = (A, \nu, \bar{\eta})$;
- (2) $\widehat{\sigma}_0(\Gamma)(A, \nu, \eta) = (A', \bar{\nu}, \eta)$ where A' is the successor of A in the cycle $\mathcal{C}_\nu(\Gamma)$.

Thus it remains to explain why $\widehat{\sigma}_1(\Gamma)$ depends only on $\mathcal{C}(\Gamma)$. (Actually it depends only on $\mathcal{V}(\Gamma)$.) For $J \subseteq I$ with at least two elements let $\Gamma|J$ be the restriction of Γ to J and let $i_J : \widehat{\mathcal{F}}(\Gamma|J) \rightarrow \widehat{\mathcal{F}}(\Gamma)$ be the induced canonical injection (note that i_J is the identity map on the two last coordinates). For $F \in \widehat{\mathcal{F}}(\Gamma)$, let $U(F)$ be the set of

Fig. 17 Cyclic arrangements of 3, 4, 5 and 6 pseudolines with their central cells marked with a little black bullet (red in colored pdf) (Color figure online)

$F_J = i_J \widehat{\sigma}_1(\Gamma|J)(i_J)^{-1}(F)$ where J ranges over the set of 2-subsets of I composed of the index of F and one of the indices occurring in its node, and endow $U(F)$ with the dominance relation $\prec_F$ defined by $F_J \prec_F F_K$ if $F_K = (\widehat{\sigma}_2(\Gamma)(F_J))_{J\Delta K}$ where as usual $J\Delta K$ denotes the set symmetric difference operator. Clearly $\prec_F$ is a total order and $\widehat{\sigma}_1(\Gamma)(F) = \min_{\prec_F} U(F)$. It follows that we can restrict our attention to the case where the size of the set of indices is two. The theorem follows. $\square$

Remark 5 In the preliminary versions [30,31] of the paper we used the notations $v_{ij1}(\Gamma)$, $v_{ij2}(\Gamma)$, $v_{ij3}(\Gamma)$ and $v_{ij4}(\Gamma)$ for the vertices $v_{ij}(\Gamma)$, $v_{\bar{i}j}(\Gamma)$, $v_{i\bar{j}}(\Gamma)$ and $v_{\bar{i}\bar{j}}(\Gamma)$ of the arrangement Γ . The new notations are better in that they are compatible with the operation of changing sign.

3.2 Raiponces

Recall that a *cyclic arrangement* of pseudolines is a simple arrangement of pseudolines with the property that the maximum of its two-cell sizes is its number of pseudolines. The simple arrangements of size at most 5 are cyclic. Figure 17 shows cyclic arrangements of three, four, five and six pseudolines. The isomorphism class of a cyclic arrangement of pseudolines depends only of its number of pseudolines; in particular the number of two-cells realizing the maximum of their sizes is 2, 4, 3 or 1 depending on whether the number of pseudolines of the arrangement is 2, 3, 4, or larger than 4. A two-cell realizing the maximum of the two-cell sizes of a cyclic arrangement of pseudolines is called a *central cell* of the arrangement. We isolate a simple lemma that will be repeatedly used in the sequel.

Lemma 16 *Let L be a cyclic arrangement of $n \geq 3$ pseudolines, let ∇ be a central cell of L and let $L_1, L_2, \dots, L_n$ be the circular list of pseudolines of L encountered when walking along the boundary of ∇ . Let K be a pseudoline such that (1) K is tangent to ∇ at the intersection point of L_1 and L_2 and (2) the family $L' = L \setminus \{L_1, L_2\} \cup \{K\}$ is an arrangement of pseudolines. Then (1) L' is cyclic and (2) ∇ is contained in a central cell ∇' of L' such that walking along the boundary of ∇' we encounter the pseudolines of L' in the circular order $K, L_3, L_4, \dots, L_n$.* $\square$

We are now ready to define the raiponces.

A *raiponce* L on a finite set of indices I is a simple indexed arrangement of pseudolines such that

Fig. 18 **a** A raiponce on the indexing set $\{i, j\}$; **b** a raiponce on the indexing set $\{1, 2, 3\}$ composed of 4 pseudolines: $A = L_{12} = L_{1\bar{3}} = L_{2\bar{3}}$, $B = L_{\bar{1}2} = L_{13} = L_{2\bar{3}}$, $C = L_{1\bar{2}} = L_{\bar{1}3} = L_{2\bar{3}}$, $D = L_{\bar{1}\bar{2}} = L_{1\bar{3}} = L_{2\bar{3}}$, **c** a raiponce on the indexing set $\{1, 2, 3\}$ composed of 6 pseudolines: $A = L_{2\bar{3}}$, $E = L_{12}$, $F = L_{\bar{1}3}$, $B = L_{\bar{1}2} = L_{13} = L_{2\bar{3}}$, $C = L_{1\bar{2}} = L_{\bar{1}3} = L_{2\bar{3}}$, $D = L_{\bar{1}\bar{2}} = L_{1\bar{3}} = L_{2\bar{3}}$

- (1) the indexing set of L is the set of unordered pairs $ij (= ji)$ of signed indices of I with the property that $i \neq \bar{j}$;
- (2) for any $i \in I$ and any $j \in I$, $i \neq j$, the subarrangement of L whose pseudolines are the L_α , $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, is an arrangement of four pseudolines; we denote by $\nabla_{i,j}$ its unique oriented two-cell such that walking along its boundary we encounter the pseudolines L_α , $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, in the circular order $L_{ij}, L_{\bar{i}j}, L_{i\bar{j}}, L_{\bar{i}\bar{j}}$, as illustrated in Fig. 18a; note that $\nabla_{i,j}$ and $\nabla_{j,i}$ are by construction disjoint and that their closures share two vertices but no edge;
- (3) for any $i \in I$ the subarrangement of L whose pseudolines are the L_α , $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, $j \in I \setminus i$, is cyclic and walking along the boundary of one of its oriented central cells we encounter for any $j \in I \setminus i$ the pseudolines L_α , $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, in the circular order $L_{ij}, L_{\bar{i}j}, L_{i\bar{j}}, L_{\bar{i}\bar{j}}$; this oriented central cell, denoted ∇_i , is necessarily the intersection of the $\nabla_{i,j}$, $j \in I \setminus i$. The indexed family of ∇_i is called the family of *central cells* of L .

Let L be a raiponce, let $\mathcal{V}(L)$ be the quotient of the indexing set of L under the relation “to index the same pseudoline of L ” and let $\mathcal{C}(L)$ be the indexed family of circular sequences of elements of $\mathcal{V}(L)$ encountered when walking along the (oriented) boundaries of the central cells of L , each sequence being indexed with the index of the central cell on the boundary of which is done the walk. An element of $\mathcal{C}(L)$ will be called a *cycle* of L . The reader will easily check that the families of cycles of the raiponces of Fig. 18a–c coincide with the families of cycles of the indexed arrangements of oriented double pseudolines of Fig. 15a–c.

Now let Δ be an indexed configuration of oriented convex bodies with the property that its arrangement of bitangents is simple and let Δ^* be its dual (indexed and oriented) arrangement. Clearly the indexed family $v(\Delta^*)$ of vertices of Δ^* —see Sect. 3.1 for its definition—is a raiponce on the indexing set of Δ , called the *raiponce* of Δ thereafter. The following lemma claims that any raiponce is the raiponce of an indexed configuration of oriented convex bodies and that the map that assigns to an indexed configuration of convex bodies the isomorphism class of its dual arrangement can be

factorized through the map that assigns to an indexed configuration of oriented convex bodies its raiponce. The proof is easy.

Lemma 17 *Let L be a raiponce on the indexing set I , let ∇ be its indexed family of central cells, let $\mathcal{G}$ be a projective plane extension of L , and let $\mathcal{R}(L, \mathcal{G})$ be the class of indexed configurations of oriented convex bodies Δ of $\mathcal{G}$ with indexing set I such that for any $i \in I$*

- (1) Δ_i is inscribed in the central cell ∇_i , and
- (2) the orientations of Δ_i and ∇_i are coherent.

Then

- (1) $\mathcal{R}(L, \mathcal{G})$ is nonempty, and
- (2) for any $\Delta \in \mathcal{R}(L, \mathcal{G})$, the raiponce of Δ is L and the isomorphism class of its dual arrangement Δ^* depends only on L .

Proof The first point is clear since by construction the closures of the ∇_i intersect pairwise in at most two vertices. Similarly the second point is clear since by construction $\mathcal{V}(\Delta^*) = \mathcal{V}(L)$ and $\mathcal{C}(\Delta^*) = \mathcal{C}(L)$. $\square$

A completion of a raiponce L is an indexed configuration of oriented convex bodies whose raiponce is L , and a *primal representation* of an indexed arrangement of oriented double pseudolines Γ is a raiponce L with the property that the isomorphism class of the dual arrangements of its completions is the isomorphism class of Γ . For example the raiponces of Fig. 18a–c are primal representations of the indexed arrangements of oriented double pseudolines of Fig. 15a–c, respectively. According to the previous discussion the properties ‘to be the dual arrangement of a family of pairwise disjoint convex bodies’ and ‘to have a primal representation’ are equivalent. The next step is devoted to the proof that this last property is stable under mutations.

Remark 6 The dual arrangement of the family of central cells of a primal representation of an arrangement of double pseudolines is, up to homeomorphism, obtained from the arrangement of double pseudolines by shrinking its digons into edges. (Here the duality is defined with respect to any projective plane extension of the primal representation.)

3.3 Stability Under Mutations

Theorem 18 *Let Γ and Γ' be two indexed arrangements of oriented double pseudolines related by a mutation. Then Γ has a primal representation if and only if Γ' has a primal representation.*

Before embarking on the proof we isolate a simple property of primal representations. The proof is easy.

Lemma 19 *Let L be a primal representation of an indexed arrangement of oriented double pseudolines Γ , let ∇ be its indexed family of central cells, let σ be a one-cell of Γ supported by the curve Γ_i , let v_α and v_β be endpoints of σ . Then L_α and L_β are consecutive pseudolines of the boundary of ∇_i and for any index $j \neq i$ of the indexing set of Γ one has*

Fig. 19 Dictionary between the relative positions of an edge σ supported by the double pseudoline Γ_j and a double pseudoline Γ_i of an indexed arrangement Γ of oriented double pseudolines and the relative positions of the corresponding central cells ∇_i and ∇_j of a primal representation of Γ

- (1) σ is contained in the crosscap side of Γ_j if and only if the arrangement of pseudolines $L_\alpha, L_\beta, L_{ij}, L_{i\bar{j}}, L_{\bar{i}j},$ and $L_{\bar{i}\bar{j}}$ is the one depicted in Fig. 19a;
- (2) σ is contained in the disk side of Γ_i if and only if the arrangement of pseudolines $L_\alpha, L_\beta, L_{ij}, L_{i\bar{j}}, L_{\bar{i}j},$ and $L_{\bar{i}\bar{j}}$ is the one depicted in Fig. 19b.

Proof of Theorem 18 Let L be a primal representation of an arrangement of oriented double pseudolines Γ , and consider a mutation connecting Γ to an arrangement Γ' . Our goal is to show that Γ' has a primal representation L' . Without loss of generality one can assume that Γ is the dual arrangement of a completion Δ of L .

We first examine the case of a merging mutation.

Let Σ be the complex of adjacent triangular two-cells of Γ involved in the merging mutation and let $\tilde{\Sigma}$ be one of its two lifts in a two-covering of the underlying cross surface. We consider the set of vertices of $\tilde{\Sigma}$ as an arrangement Ψ of oriented pseudolines and we introduce the subarrangement Ψ_0 composed of the three vertices of the boundary $\partial\tilde{\Sigma}$ of $\tilde{\Sigma}$ and the one level ℓ of Ψ_0 with respect to its unique two-cell σ_0 with cyclic boundary; note that ℓ is by construction a pseudoline and that any pseudoline in L not in Ψ crosses ℓ in at most three points. Figure 20 depicts the complex Σ of a merging mutation, the subarrangement Ψ_0 with its cyclic two-cell σ_0 marked, and the one-level ℓ of Ψ_0 with respect to σ_0 .

Fig. 20 **a** The complex Σ of triangular two-cells involved in the merging mutation connecting Γ to Γ' ; **b** the arrangement Ψ_0 composed of the vertices of the boundary of the complex $\tilde{\Sigma}$; and **c** the one-level ℓ of the arrangement Ψ_0

Let L' be the indexed family of pseudolines defined by

$$L'_\tau = \begin{cases} \ell & \text{if } L_\tau \text{ is a vertex of } \Sigma; \\ L_\tau & \text{otherwise,} \end{cases} \quad (3)$$

where τ ranges over the indexing set of L .

Lemma 20 *We claim that*

- (1) L' is a simple arrangement of pseudolines;
- (2) L' is a primal representation of the arrangement Γ' .

Proof Let K be the set of indices of the supporting double pseudolines of the one-cells of Σ , let $K' \subseteq K$ be the set of three indices of the three supporting double pseudolines of the three sides of the boundary of Σ , and let $w \in K'$ be the index of the moving curve of the mutation. We denote by $\hat{w}$ the vertex of the boundary of Σ opposite the side supported by Γ_w , and for any $t \in K \setminus \{w\}$ we denote by $\hat{t}$ the vertex of Σ where the double pseudolines Γ_t and Γ_w intersect. Let Ψ_0^+ be the arrangement Ψ_0 augmented with the line $\hat{t}$ if $t \in K \setminus K'$; the arrangement Ψ if $t = w$; the arrangement Ψ_0 otherwise. We denote by L^* the sub-raiponce of L obtained by deleting the L_{ij} with $i \in K \setminus K'$. The indexed families of central cells of L and L^* are denoted ∇ and ∇^* , respectively. Finally let $\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m$ be the sequence of vertices $\neq \hat{w}$ of Σ ordered along Γ_w .

Applying Lemma 19 to the one-cells of $\tilde{\Sigma}$ (and using induction on the size of $K \setminus K'$) we see easily that the relative position of ∇_t in the arrangement Ψ_0^+ depends only on whether the triangular two-cells of Σ are contained in the disk side $\mathcal{D}(\Gamma_t)$ of Γ_t or contained in the crosscap side $\mathcal{M}(\Gamma_t)$ of Γ_t , as indicated in Fig. 21. Furthermore one can also check that

- (1) for any $t \in K \setminus \{w\}$ the pseudoline ℓ is tangent to ∇_t at the intersection point of $\hat{w}$ and $\hat{t}$;
- (2) the pseudolines in $\Psi \setminus \Psi_0$ cross the pseudoline ℓ all in three points or all in one point;
- (3) the arrangement Ψ is cyclic;

Fig. 21 Relative position of ∇_t in the arrangement Ψ_0^+

- (4) the relative position of ∇_w^* in the arrangement Ψ_0 depends only on whether the triangular two-cells of Σ are contained in $\mathcal{D}(\Gamma_w)$ or in $\mathcal{M}(\Gamma_w)$ as indicated in Fig. 21; in particular we note that ℓ is tangent to ∇_w^* at the intersection point of $\hat{u}_1$ and $\hat{u}_m$.

Pick now a pseudoline ℓ' such that $L \cup \{\ell'\}$ is a simple arrangement of pseudolines. Assume that ℓ' and ℓ cross three times. Clearly ℓ' avoids the cyclic two-cell of Ψ_0 and consequently—thanks to our previous discussion on the position of the ∇_t in the arrangement Ψ_0^+ — ℓ' is transversal to any ∇_t , $t \in I \setminus (K \setminus K')$, not contained in σ_0 . It follows that $\ell' \notin L \setminus \Psi$ and, consequently, there is no pseudoline of $L \setminus \Psi$ crossing ℓ three times: thus L' is a simple arrangement of pseudolines.

We now prove that L' is a raiponce and a primal representation of Γ' . Given a subfamily S of L we define S' to be the corresponding subfamily of L' . For any index $i \in I$ let M_i be the arrangement of pseudolines composed of the L_α , $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, $j \in I \setminus \{i\}$, and let $N_{ij} = M_i \cap M_j$, $i \neq j \in I$. Observe that N_{ij} contains at most one element of Σ and that $\ell \notin L$: consequently N'_{ij} is an arrangement

Fig. 22 Stability under splitting mutations

of four pseudolines. By construction

$$M'_t = \begin{cases} M_t \setminus \{\widehat{w}, \widehat{t}\} \cup \{\ell\} & \text{if } t \in K \setminus \{w\}; \\ M_t \setminus \Sigma \cup \{\ell\} & \text{if } t = w; \\ M_t & \text{otherwise.} \end{cases} \quad (4)$$

Since for any $i \in K \setminus \{w\}$ the pseudoline ℓ is tangent to ∇_i at the intersection point of $\widehat{w}$ and $\widehat{t}$, and since ℓ is tangent to ∇_w^* at the intersection point of $\widehat{u}_1$ and $\widehat{u}_m$ it follows, according to Lemma 16, that for any i the arrangement M'_i is cyclic, that ∇_i is contained in one of its central two-cells ∇'_i , and that walking along its boundary (oriented according to the orientation of ∇_i) we encounter for any $j \in I \setminus i$ the pseudolines L'_α , $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, in the circular order $L'_{ij}, L'_{\bar{i}j}, L'_{i\bar{j}}, L'_{\bar{i}\bar{j}}$; consequently L' is a raiponce and is (by construction) a primal representation of Γ' . $\square$

We now examine the case of a splitting mutation (Fig. 22).

Let K be the set of indices of the double pseudolines involved in the splitting mutation, let w be the index of the moving double pseudoline, let $\widehat{w}$ be the vertex involved in the mutation, and for any $v \in K \setminus \{w\}$ let $x(v) \in \{wv, \overline{w}v, w\overline{v}, \overline{w}\overline{v}\}$ defined by the condition $L_{x(v)} = \widehat{w}$.

Let w^* be a double pseudoline containing $\widehat{w}$ in its crosscap side such that any pseudoline of L crosses w^* in exactly two points and such that no vertex of the arrangement L belongs to the Möbius strip $\mathcal{M}(w^*)$ bounded by w^* . The pseudolines of L induce a decomposition of $\mathcal{M}(w^*)$ into quadrilateral regions. In particular the trace of the central cell of the raiponce L indexed by w onto $\mathcal{M}(w^*)$ is one of its quadrilateral regions that we shall denote by Q . We denote by S and S^* the sides of Q supported by $\widehat{w}$ and w^* , respectively, and we denote by Q' the second quadrilateral region of $\mathcal{M}(w^*)$ bounded by S .

Let B_1 be a generic point of Q if $\widehat{w} \in \mathcal{D}(\Gamma'_w)$; otherwise let B_1 be a generic point of Q' .

For any $i \in K \setminus \{w\}$ we insert a generic point B_i on the interior of the edge of the central cell of L indexed by i supported by $\widehat{w}$, and we insert in the underlying pseudoline arrangement of L a pseudoline ℓ_i such that

- (1) ℓ_i goes through the points B_1 and B_i , and is contained in $\mathcal{M}(w^*)$;
- (2) the vertices of $L \cup \Psi$ are simple, except B_1 ;

and we perturb the pencil of pseudolines ℓ_i in the vicinity of B_1 into a cyclic arrangement ℓ_i^* with a central cell containing S^* or S depending on whether $\widehat{w} \in \mathcal{D}(\Gamma'_w)$ or not.

Now let L' be the indexed family of pseudolines defined by

$$L'_\tau = \begin{cases} \ell_v^* & \text{if } \tau = x(v) \text{ with } v \in K \setminus \{w\}; \\ L_\tau & \text{otherwise,} \end{cases}$$

where τ ranges over the indexing set of L . A simple case analysis shows that L' is a well-defined raiponce and is a primal representation of Γ' . Details are left to the reader. $\square$

Remark 7 Our proof of the Geometric Representation Theorem is constructive. For an alternative construction see [20].

4 Cycles, Cocycles and Chirotopes

In this section we prove that (1) the isomorphism class of an indexed arrangement of oriented double pseudolines depends only on its family of isomorphism classes of subarrangements of size three, i.e., depends only on what we have called its chirotope; and that (2) the map that assigns to an indexed configuration of oriented convex bodies the isomorphism class of its dual arrangement is compatible with the isomorphism relations on the set of configurations of convex bodies, and it induces a one-to-one correspondence between the set of isomorphism classes of indexed configurations of oriented convex bodies and the set of isomorphism classes of indexed arrangements of oriented double pseudolines. The main ingredients of our proof are

- (1) the coding of the isomorphism class of an indexed arrangement of oriented double pseudolines by its family of side cycles;
- (2) the list of martagons on three and four double pseudolines, established in Sect. 2;

- (3) the injectivity of the map that assigns to each cell of the dual arrangement of an indexed configuration of two oriented convex bodies the cocycle of the configuration at some (hence any) element of the cell; and
- (4) the injectivity of the map that assigns to a bitangent cocycle of an indexed family of at least three oriented convex bodies the sub-cocycles obtained by removing in turn each of the convex bodies.

4.1 Side Cycles

We repeat the definition of side cycles given in the introduction. Let Γ be an indexed arrangement of oriented double pseudolines and recall that Γ is extended to the negative indices by assigning to a negative index the reoriented version of the oriented double pseudoline assigned to its positive version. The *side cycle of disk type* assigned to the (signed) indice i , denoted D_i , is the circular sequence of indices of the double pseudolines crossed by the side wheel of a sidecar rolling on Γ_i , side wheel on the disk side of Γ_i , that are (locally) oriented away from Γ_i . Similarly the *side cycle of crosscap type* assigned to the index i , denoted M_i , is the circular sequence of indices of the double pseudolines crossed by the side wheel of a sidecar rolling on Γ_i , side wheel on the crosscap side of Γ_i , that are (locally) oriented away from Γ_i . Note that the side cycles of disk (crosscap) type assigned to an index and its complement are reverse to one another and that for simple arrangements the side cycle of disk type assigned to an index is the complement of its side cycle of crosscap type and vice versa.

Example 5 The side cycles of disk type of an arrangement Γ on two double pseudolines, say indexed by i, j , are

$$\begin{aligned} i &: \overline{j} \overline{j} j j, \\ j &: \overline{i} \overline{i} i i. \end{aligned}$$

This can be easily read in Fig. 23 where we have displayed the first barycentric subdivision of the one-skeleton of the arrangement and labeled each edge of the subdivision with the index of the supporting double pseudoline of the edge that is, locally on the edge, oriented away from the vertex of the arrangement to which the edge is incident. Observe that each symbol in these cycles corresponds in the natural way to a unique node of the arrangement, namely the linear sequence of symbols $\overline{j} \overline{j} j j$ corresponds to the linear sequence of nodes $\{ij\}\{\overline{i}j\}\{\overline{i}\overline{j}\}\{\overline{i}\overline{j}\}$, as illustrated in Fig. 23. The side cycles of crosscap type of Γ coincide with its side cycles of disk type but now the linear sequence of symbols $\overline{j} \overline{j} j j$ corresponds to the linear sequence of nodes $\{\overline{i}\overline{j}\}\{\overline{i}\overline{j}\}\{ij\}\{ij\}$.

Let Γ be an indexed arrangement of oriented double pseudolines with indexing set I , let D_i be its side cycles of disk type and M_i those of crosscap type. Let S_i be the result of replacing in D_i the linear subsequences $\overline{j} \overline{j} j j$, $j \neq i$, by the linear sequences $\{ij\}\{\overline{i}j\}\{\overline{i}\overline{j}\}\{\overline{i}\overline{j}\}$; similarly let T_i be the result of replacing in M_i the linear subsequences $j j \overline{j} \overline{j}$, $j \neq i$, by the linear sequences $\{ij\}\{\overline{i}j\}\{\overline{i}\overline{j}\}\{\overline{i}\overline{j}\}$. Clearly there

Fig. 23 The first barycentric subdivision of the one-skeleton of an arrangement of two double pseudolines: each edge of the subdivision is labeled with the signed index of its signed supporting curve that is, locally on the edge, oriented away from the vertex of the arrangement to which the edge is incident

Fig. 24 Implicit description of the 4×4 possible nodes involving three curves indexed by i, j and j_* where $i \in I, j, j_* \in I \cup \bar{I}$. The dashed sides stand for the crosscap sides

is a one-to-one correspondence between the vertices of the arrangement lying on the curve indexed by i and the maximal factors $\{i_1 j_1\} \{i_2 j_2\} \{i_3 j_3\} \dots \{i_k j_k\}$ of S_i with $j_l \notin \{j_{l'}, \bar{j}_{l'}\}$ for all $1 \leq l < l' \leq k$ that appear in reverse order $\{i_k j_k\} \dots \{i_3 j_3\} \{i_2 j_2\} \{i_1 j_1\}$ in T_i , *prime factors* for short. More precisely:

- (1) the node of vertex v associated with the prime factor $\{i_1 j_1\} \{i_2 j_2\} \{i_3 j_3\} \dots \{i_k j_k\}$ of S_i is the set of $\{i_l j_l\} \otimes \{i_{l'} j_{l'}\}$, $1 \leq l \leq l' \leq k$, where $\{i_l j_l\} \otimes \{i_{l'} j_{l'}\}$ is the element of the 4-set $\{j_l j_{l'}, \bar{j}_l j_{l'}, j_l \bar{j}_{l'}, \bar{j}_l \bar{j}_{l'}\}$ indexing the intersection point of Γ_{j_l} and $\Gamma_{j_{l'}}$ that coincides with v . As illustrated in Fig. 24 (which depicts implicitly the 4×4 possible nodes involving three curves indexed by i, j and j_* where $i \in I, j, j_* \in I \cup \bar{I}$ and where the dashed sides stand for the crosscap sides) this element depends solely on the information contained in the (ordered) pair $\{i_l j_l\} \{i_{l'} j_{l'}\}$, and the multiplication table of $\otimes$ is the following

$$\begin{cases} \{ij\} \otimes \{ij\} = \{ji\}, \\ \{ij\} \otimes \{ij_*\} = \{j\bar{j}_*\}, \\ \{ij\} \otimes \{\bar{i}j_*\} = \{\bar{j}\bar{j}_*\}, \\ \{\bar{i}j\} \otimes \{ij_*\} = \{jj_*\}, \\ \{\bar{i}j\} \otimes \{\bar{i}j_*\} = \{j\bar{j}_*\}, \end{cases}$$

where $i \in I, j, j_* \in I \cup \bar{I}$ with $j \notin \{j_*, \bar{j}_*\}$;

- (2) Conversely, the prime factor of S_i corresponding to vertex v is the sequence of $\{\eta j\}, \eta \in \{i, \bar{i}\}$, that belong to the node N of v , ordered according to the dominance relation $\{\eta j\} \prec \{\eta' j'\}$ if $\{\eta j\} \otimes \{\eta' j'\} \in N$.

Thus node cycles and side cycles carry exactly the same information about the arrangement. Since, according to Theorem 15, two indexed and oriented arrangements are isomorphic if and only if they have the same family of node cycles we get

Theorem 21 *Two indexed arrangements of oriented double pseudolines are isomorphic if and only if they have the same family of side cycles.*

Let X be an arrangement of double pseudolines, let X_* be an indexed and oriented version of X and recall that we have extended X_* to the complements of the original indices by assigning to a negative index the reoriented version of the double pseudoline assigned to its complement. Let G be the group of permutations of the signed indices which are compatible with the operation of taking the complement, let G_{X_*} be the stabilizer of X_* , i.e., the subgroup of G whose elements are the permutations σ such that $X_*\sigma$ and X_* are isomorphic, and let G_X be the group of automorphisms of X , which we think of as a subgroup of the group of permutations of the signed (or oriented) double pseudolines of the arrangement. Clearly the map that assigns to $\sigma \in G_X$ its conjugate $X_*^{-1}\sigma X_* \in G$ under X_* is a monomorphism of G_X onto G_{X_*} . Thus we can see G_X as a subgroup G_{X_*} of G and the number of distinct indexed and oriented versions of X is the index $[G : G_{X_*}]$ of G_{X_*} in G . In the sequel we use the notation $X(\sigma)$ for the arrangement $X_*\sigma$, $\sigma \in G$; hence $X(1) = X_*$, where 1 is the unit of G .

Example 6 Let Z be the hemi-cube arrangement and let $Z_* = Z(123)$ be one of its indexed and oriented version on the indexing set $\{1, 2, 3\}$; cf. Fig. 25. The group G_Z is, as explained in Sect. 1, S_4 . Thus the number of distinct indexed and oriented versions of Z is $3!2^3/24 = 2$. The group G_{Z_*} is of order 24 generated by the permutations $\bar{1}32$ and $\bar{1}23$ and $2\bar{3}\bar{1}$ (which correspond to the automorphisms τ_1, τ_2 and τ_3 of Sect. 1), respectively. Its two cosets are

Fig. 25 The hemi-cube arrangement and two of its indexed and oriented versions on the indexing set $\{1, 2, 3\}$

Fig. 26 Oriented and indexed versions of the martagons on 4 double pseudolines

$$G_{Z_*} = \begin{pmatrix} 123 & 231 & 312 \\ \overline{123} & \overline{231} & \overline{312} \\ \overline{12\overline{3}} & \overline{23\overline{1}} & \overline{31\overline{2}} \\ \overline{1\overline{2}3} & \overline{2\overline{3}1} & \overline{3\overline{1}2} \\ 21\overline{3} & 32\overline{1} & 13\overline{2} \\ 3\overline{2}1 & 1\overline{3}2 & 2\overline{1}3 \\ \overline{13}2 & \overline{21}3 & \overline{32}1 \\ \overline{213} & \overline{321} & \overline{132} \end{pmatrix}, \quad (213)G_{Z_*} = \begin{pmatrix} 213 & 321 & 132 \\ \overline{213} & \overline{321} & \overline{132} \\ \overline{21\overline{3}} & \overline{32\overline{1}} & \overline{13\overline{2}} \\ \overline{2\overline{1}3} & \overline{3\overline{2}1} & \overline{1\overline{3}2} \\ 12\overline{3} & 23\overline{1} & 31\overline{2} \\ 3\overline{1}2 & 1\overline{2}3 & 2\overline{3}1 \\ \overline{23}1 & \overline{31}2 & \overline{12}3 \\ \overline{123} & \overline{231} & \overline{312} \end{pmatrix}.$$

Example 7 As illustrated in Fig. 26 the two families of cycles on the indexing set $\{1, 2, 3, 4\}$

$$\begin{array}{ll} 1: & 222\overline{2}333\overline{3}444\overline{4}, \\ 2: & 441\overline{1}3344\overline{1}133, \\ 3: & 221\overline{1}4422\overline{1}144, \\ 4: & 331\overline{1}2233\overline{1}122, \end{array} \quad \text{and} \quad \begin{array}{ll} 1: & \overline{22}22\overline{33}33\overline{44}44\overline{4}, \\ 2: & \overline{33}1\overline{1}4433\overline{1}144, \\ 3: & \overline{22}44\overline{11}2244\overline{11}, \\ 4: & \overline{22}1\overline{1}2233\overline{11}33, \end{array}$$

are the side cycles of disk type of indexed and oriented versions of the two martagons M_1 and M_2 on four double pseudolines, depicted in Fig. 14. The automorphism group of M_1 is S_3 (dihedral group D_3) generated by the involutions $\overline{1324}$ and $\overline{1243}$, for example. The automorphism group of M_2 is of order 2 generated by the permutation $\overline{1324}$.

We close this section by introducing, in Figs. 27 and 28, one oriented and indexed version of each of the thirteen simple arrangements of three double pseudolines of Fig. 13. We let the reader check that the four subarrangements of size three of $M_1(1234)$ are $C_{22}(\overline{123})$, $C_{22}(\overline{134})$, $C_{22}(\overline{142})$ and $C_{04}(234)$. Similarly the four subarrangements of size three of the martagon $M_2(1234)$ are $C_{22}(123)$, $C_{22}(42\overline{3})$, $C_{32}(142)$, $C_{32}(143)$.

Fig. 27 Indexed and oriented versions of the arrangements C_{04} , C_{07} , C_{18} , C_{37} , C_{15} , C_{43} . Each diagram is labeled at its *top right* by its number of (distinct) reindexed and reoriented versions and at its *bottom right* by its side cycles of disk type

4.2 Chirotopes

We are now ready to show that the isomorphism class of an arrangement of oriented double pseudolines depends only on its chirotope; cf. first part of Theorem 2.

Theorem 22 *The map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its chirotope is one-to-one.*

Proof Let Γ be an indexed arrangement of oriented double pseudolines. According to Theorem 15 the isomorphism class of Γ depends only on the family of cycles of Γ ; therefore it is sufficient to show that the family of cycles of Γ depends only on the chirotope of Γ .

Clearly the set of nodes of Γ depends only on the chirotope of Γ , and clearly we can restrict our attention to the case where Γ has four elements, say $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 .

We now show that the cycle of Γ indexed by 1 depends only on the chirotope of Γ . We write $\mathcal{N}(k)$, $k \in \{2, 3, 4\}$, for the set of nodes of Γ indexing the intersection points of Γ_1 and Γ_k , $\mathcal{N}(k, k', \dots)$ for $\mathcal{N}(\{k, k', \dots\})$, and for any $A, A' \in \mathcal{N}(2, 3, 4)$, $A \neq A'$, we write $[A, A']$ for the set of $X \in \mathcal{N}(2, 3, 4)$, including A and A' , that appear between A and A' on the cycle of Γ indexed by 1.

Fig. 28 Indexed and oriented versions of the arrangements C_{22} , C_{33} , C_{32} , C_{25_2} , C_{25_1} , C_{36} , C_{64} . Each diagram is labeled at its *top right* by its number of (distinct) reindexed and reoriented versions and at its *bottom right* by its side cycles of disk type

Let $A, A' \in \mathcal{N}(2)$, $A \neq A'$, and let $B, B' \in \mathcal{N}(2, 3, 4)$. We say that the pair (A, A') separates the pair (B, B') if $B \in [A, A']$ and $B' \in [A', A]$. Clearly one can decide, using only the chirotope of Γ ,

- (1) whether B belongs to the interval $[A, A']$ or not; and
- (2) whether the pair (A, A') separates the pair (B, B') or not.

Assume that a pair (A, A') of distinct elements of $\mathcal{N}(2)$ separates a pair (B, B') of elements of $\mathcal{N}(3, 4)$. (In particular this happens if one of the intersection points between Γ_1 and Γ_2 is not an ordinary vertex of the arrangement Γ .) In that case a

pair X, Y of distinct elements of $\mathcal{N}(3, 4)$ lying in the open interval $[A, A'] \setminus \{A, A'\}$ appears in the linear order XY in the interval $[A, A']$ if and only if B', X, Y appear in the cyclic order $B'XY$ on the cycle of the arrangement $\{\Gamma_1, \Gamma_3, \Gamma_4\}$ indexed by 1. Consequently the cycle of Γ indexed by 1 depends only on the chirotope of Γ and we are done. Similarly we are done if a pair of distinct elements of $\mathcal{N}(3)$ separates a pair of elements of $\mathcal{N}(2, 4)$, or if a pair of distinct elements of $\mathcal{N}(4)$ separates a pair of elements of $\mathcal{N}(2, 3)$. Thus it remains to examine the case where for every $k \in \{2, 3, 4\}$ no pair of distinct elements of $\mathcal{N}(k)$ separates a pair of elements of $\mathcal{N}(\{2, 3, 4\} \setminus \{k\})$, i.e., using the terminology introduced in the previous section, the case where the arrangement Γ is a martagon with respect to Γ_1 . According to Lemma 14 and the notations introduced in Example 7, this means that, up to permutation of the indices 1, 2, 3, 4 and their negatives, $\Gamma = M_1(1234)$ or $\Gamma = M_2(1234)$. The theorem follows. Indeed if the chirotope of Γ is the chirotope of $M_1(1234)$ then the family of side cycles (of disk type) of Γ is necessarily either the family

$$\begin{aligned} C_1 : & \quad 22\overline{22}33\overline{33}44\overline{44}, \\ C_2 : & \quad 44\overline{11}33\overline{44}1\overline{1}33, \\ C_3 : & \quad 22\overline{11}44\overline{22}1\overline{1}44, \\ C_4 : & \quad 33\overline{11}22\overline{33}1\overline{1}22 \end{aligned}$$

of side cycles of $M_1(1234)$ or the family

$$\begin{aligned} C_1^* : & \quad 33\overline{33}22\overline{22}44\overline{44}, \\ C_2^* : & \quad 44\overline{11}33\overline{44}1\overline{1}33, \\ C_3^* : & \quad 22\overline{11}44\overline{22}1\overline{1}44, \\ C_4^* : & \quad 33\overline{11}22\overline{33}1\overline{1}22 \end{aligned}$$

obtained from the family $\mathcal{C}$ by switching the blocks $22\overline{22}$ and $33\overline{33}$ in the cycle assigned to 1 and by leaving the other cycles unchanged. To rule out $\mathcal{C}^*$ from the set of families of cycles of double pseudoline arrangements it remains to observe that the permutations that carry C_1 onto C_1^* are exactly the 4 permutations 1324, 1243, 1432, $\overline{1}234$ and that none of these 4 permutations leaves unchanged the triplet C_1, C_2, C_3 . Similarly if the chirotope of Γ is the chirotope of $M_2(12234)$ then the family of side cycles of Γ is either the family

$$\begin{aligned} C_1 : & \quad \overline{22}22\overline{33}33\overline{44}44, \\ C_2 : & \quad \overline{33}1\overline{1}44\overline{33}1\overline{1}44, \\ C_3 : & \quad \overline{22}44\overline{11}22\overline{44}1\overline{1}, \\ C_4 : & \quad \overline{22}1\overline{1}22\overline{33}1\overline{1}33 \end{aligned}$$

of side cycles of $M_2(1234)$ or the family

$$\begin{aligned} C_1^* : & \quad \overline{33}33\overline{22}22\overline{44}44, \\ C_2^* : & \quad \overline{33}1\overline{1}44\overline{33}1\overline{1}44, \\ C_3^* : & \quad \overline{22}44\overline{11}22\overline{44}1\overline{1}, \\ C_4^* : & \quad \overline{22}1\overline{1}22\overline{33}1\overline{1}33 \end{aligned}$$

obtained from the family $\mathcal{C}$ by switching the blocks $\overline{22\overline{22}}$ and $\overline{33\overline{33}}$ in the cycle assigned to 1 and by leaving the other cycles invariant. Again to rule out $\mathcal{C}^*$ from the set of families of cycles of double pseudoline arrangements it remains to observe that the permutations that carry $\mathcal{C}_1$ onto $\mathcal{C}_1^*$ are exactly the two permutations 1324 , $\overline{12\overline{34}}$, and that none of these 2 permutations leaves invariant the cycle $\mathcal{C}_4$. (In Sect. 5 we interpret the $\mathcal{C}^*$ as side cycles of arrangements of double pseudolines living in a triple cross surface.) $\square$

4.3 Cocycles

Let Δ be an indexed configuration of oriented convex bodies of a projective plane $(\mathcal{P}, \mathcal{L})$ and let τ be a line of $(\mathcal{P}, \mathcal{L})$. Recall that we have defined

- (1) the *cocycle of Δ at τ* or the *cocycle of τ with respect to Δ* or the *cocycle of the pair (Δ, τ)* as the homeomorphism class of the image of the pair (Δ, τ) under the quotient map $\omega_\tau : \mathcal{P} \rightarrow \mathcal{P}/\mathcal{R}_\tau$ relative to the equivalence relation $\mathcal{R}_\tau$ on $\mathcal{P}$ generated by the pairs of points lying on a same line segment of $\Delta \cap \tau$;
- (2) a *bitangent cocycle* or *zero-cocycle* as a cocycle at a bitangent;
- (3) the *isomorphism class of Δ* as the set of configurations that have the same set of bitangent cocycles as Δ ; and
- (4) the *chirotope of Δ* as the map that assigns to each 3-subset J of the indexing set of Δ the isomorphism class of the subfamily indexed by J .

To these four definitions we add the following one

- (5) the *cocycle map of Δ* is the map that assigns to each cell σ of the dual arrangement of Δ the cocycle of Δ at some (hence any) element of σ .

Figure 29 depicts, up to reorientation and reindexing of the convex bodies, the cocycles of families of two and three pairwise disjoint convex bodies with respective indexing sets $\{1, 2\}$ and $\{1, 2, 3\}$; in this figure each circular diagram is labeled at its bottom right by its number of reoriented and reindexed versions and at its bottom left by its signature, a natural coding of the cocycle introduced in Sect. 1 and that we will not repeat here.

Figure 30 depicts examples of cocycle maps of families of one, two, and three pairwise disjoint convex bodies with respective indexing sets $\{1\}$, $\{1, 2\}$, and $\{1, 2, 3\}$.

In particular one can easily check that the cocycle map of a family of two bodies is one-to-one.

Lemma 23 *Cocycle maps of families of two disjoint convex bodies are one-to-one.*

We are now ready to prove that the map that assigns to an indexed configuration of oriented convex bodies the isomorphism class of its dual arrangement is compatible with the isomorphism relation on families of convex bodies, and that the quotient map is one-to-one (and onto). This means, for example, that the signatures $\{1\bullet\overline{2}\bullet\overline{3}\bullet\}$, $\{1\bullet\overline{3}\bullet\overline{2}\bullet\}$, $\{1\bullet\overline{2}\bullet\overline{3}\bullet\}$, $\{1\bullet\overline{3}\bullet\overline{2}\bullet\}$ of the bitangent cocycles of the configuration of three convex bodies depicted at the bottom left of Fig. 30 is a coding of the isomorphism class of the dual arrangement of the configuration.

Fig. 29 Cocycles of indexed configurations of two and three oriented convex bodies. Each cocycle is labeled at its bottom left with its signature and at its bottom right by its number of reoriented and reindexed versions

Theorem 24 Let Δ and Δ' be two indexed configurations of oriented convex bodies. Then the following four assertions are equivalent:

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Table 2 The map that assigns to a bitangent cocycle of a family of three convex bodies its sub-cocycles on two bodies is one-to-one

$\mu(123, \tau)$	$\mu(12, \tau)$	$\mu(13, \tau)$	$\mu(23, \tau)$
123●●●	12●●	13●●	23●●
1●2●3●	12●●	31●●	23●●
12●●, 3	12●●	1●, 3	2●, 3
132● $\bar{3}$ ●	12●●	$\bar{3}$ 13●	32 $\bar{3}$ ●
$\bar{3}$ 123●●	12●●	$\bar{3}$ 13●	$\bar{3}$ 23●

We first prove that (2) $\Rightarrow$ (1). Let $\mathcal{V}$ be the (finite) set of signatures $\mu(\Delta, \tau)$ of the pairs (Δ, τ) as Δ ranges over the set of families of $n \geq 3$ convex bodies indexed by $\{1, 2, 3, \dots, n\}$ and where τ ranges over the set of bitangents of Δ . We leave the verification of the following property of the set $\mathcal{V}$ to the reader: the map that assigns to any element $\mu(\Delta, \tau)$ of $\mathcal{V}$ the set of $\mu(\Delta', \tau)$ where Δ' ranges over the set of subfamilies of size $n - 1$ of Δ is one-to-one: see Table 2 for the case $n = 3$; this proves that (2) $\Rightarrow$ (1).

We now prove that (1) $\Rightarrow$ (2). It is sufficient to prove it for families of three bodies. Let I be the indexing set of Δ , let $\mathcal{I}$ be the set of pairs (i, J) where i ranges over I and where J ranges over the set of 3-subsets of I that contains i , and for $(i, J) \in \mathcal{I}$ let $C_{i,J}$ be the circular ordering of the bitangents $v_\alpha, \alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j} \mid j \in I \setminus i\}$, along the (oriented) dual curve Δ_i^* of Δ_i . According to Theorem 15 the isomorphism class of the dual arrangement of Δ depends only on the family of $C_{i,J}, (i, J) \in \mathcal{I}$. Thus proving that (1) $\Rightarrow$ (2) comes down to proving that the $C_{i,J}$ depend only on the chirotope of Δ . This latter statement is a simple consequence of the following two observations:

- (1) for any $j \in I \setminus \{i\}$, the four vertices $v_\alpha, \alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$, appear by definition in the circular order $v_{ij}, v_{\bar{i}j}, v_{i\bar{j}}, v_{\bar{i}\bar{j}}$ along Δ_i^* ;
- (2) for any $j \in I \setminus \{i\}$ and any $\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j}\}$ the position of v_α with respect to the $v_\beta, \beta \in \{ik, \bar{i}k, i\bar{k}, \bar{i}\bar{k}\}, k \in I \setminus \{i, j\}$, depends only on the chirotope of Δ for the cocycle map is one-to-one for families of two bodies. $\square$

Remark 8 It is incorrect to say, as we did in [31], that cocycle maps are one-to-one. However one can show that the space of transversals with given cocycle is connected. This can be used, as explained in the forthcoming paper [23], to extend to the projective setting one of the Wenger's generalizations of the Hadwiger's Transversal Theorem [33, 54]: *Let $\Delta_1, \Delta_2, \dots, \Delta_n$ be a finite indexed family of at least 4 pairwise disjoint oriented convex bodies of a projective plane with the property that for any quadruplet of indices $i < j < k < l$ there is a line whose signature with respect to the subfamily $\Delta_i, \Delta_j, \Delta_k, \Delta_l$ is $ijkl\bar{i}\bar{j}\bar{k}\bar{l}$. Then there is a line whose signature with respect to the family Δ is $123 \dots n\bar{1}\bar{2}\bar{3} \dots \bar{n}$.*

Figures 31, 32 and 33 depict the zero-cocycle labeled versions of the thirteen indexed and oriented simple arrangements on three double pseudolines of Figs. 27 and 28.

Fig. 31 The zero-cocycle labeled version of the arrangement $C_{64}(123)$

5 LR Characterization

In this section we prove the LR characterization of chirotopes of double pseudoline arrangements; cf. Theorem 2. As said in the introduction, the proof goes through the notion of arrangements of double pseudolines living in nonorientable surfaces of any genus. In addition, we apply the same proof technique to give two new proofs of the classical LR characterization of chirotopes of pseudoline arrangements.

5.1 Arrangements of Genus 1, 2, ...

By an arrangement of double pseudolines of genus $g \geq 1$ we mean a finite family Γ of at least two simple closed curves cellularly embedded in a compact nonorientable surface $\mathcal{P}_\Gamma$ of genus g with the property that there exist closed tubular neighborhoods R_i of the Γ_i (ribbons for short) such that

- (1) for any subfamily ζ of Γ the union of its ribbons, denoted R_ζ , is a closed tubular neighborhood of the union of its curves; the compact surface obtained by attaching topological disks to the boundary curves of R_ζ , using homeomorphisms for the attaching maps, is denoted $\mathcal{P}_\zeta$;
- (2) any subfamily ζ of Γ of size 2 considered as embedded not in $\mathcal{P}_\Gamma$ but in $\mathcal{P}_\zeta$ is homeomorphic to the dual arrangement of some (hence any) configuration of two convex bodies;
- (3) for any $\Gamma_i, \Gamma_j \in \Gamma$ the intersection of the ribbon R_i of Γ_i and the disk side of Γ_i in the subarrangement Γ_i, Γ_j is independent of Γ_j .

Fig. 32 Zero-cocycle labeled versions of the arrangements $C_{04}(123)$, $C_{07}(123)$, $C_{18}(123)$, $C_{37}(123)$, $C_{15}(123)$, $C_{43}(123)$

Thus arrangements of double pseudolines of genus 1 are the arrangements of double pseudolines as defined in the previous sections. Figure 34 depicts two embeddings in 3-space of the tubular neighborhood of an arrangement of two double pseudolines (thus a union of two ribbons). A horizontal dashed line segment indicates the presence

Fig. 33 Zero-cocycle labeled versions of the arrangements $C_{22}(123)$, $C_{33}(123)$, $C_{32}(123)$, $C_{25_2}(123)$, $C_{25_1}(123)$ and $C_{36}(123)$

of a half-twist (180°) of the ribbon crossed by the line segment and the numbers, in the right diagram, label the corners of the polygonal boundary curves of the neighborhood (the corners of a polygonal boundary curve being labeled by the same number). We extend in the natural way to the class of arrangements of double pseudolines of arbitrary

Fig. 34 Two embeddings in 3-space of the tubular neighborhood of an indexed arrangement of two oriented double pseudolines

genus the notions of thinness, mutations, isomorphism classes, node cycles (you must *not* forget the binary operation $\otimes$), side cycles of disk and crosscap type together with their prime factors, (Δ) -chirotopes, and so on associated with the class of arrangements of double pseudolines of genus 1. As for arrangements of genus 1, the isomorphism class of an arrangement of any genus depends only on its family of side cycles, with the net benefit that there is now a very simple characterization of cycles that arose as side cycles of simple arrangements: *a family of circular sequences D_i , $i \in I$, is the family of side cycles of disk type of a simple arrangement of oriented double pseudolines indexed by I if and only if the D_i are shuffles of the elementary circular sequences $jj\bar{j}\bar{j}$, $j \neq i$.* The case of any arrangements is hardly more complicated: only the condition that prime factors occur consistently on side cycles has to be taken in account; the exact formulation is postponed to the end of the section. In this broader context the (range part of the) LR characterization of chirotopes of arrangements of double pseudolines of genus 1 is a direct consequence of the two following theorems.

Theorem 25 *The map which assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its 4-chirotope is one-to-one and that which assigns its 5-chirotope is (one-to-one and) onto.*

Theorem 26 *The class of arrangements of double pseudolines of genus 1 is the class of arrangements of double pseudolines whose subarrangements of size at most 5 are of genus 1.*

In a similar way, we introduce the notion of arrangements of pseudolines of arbitrary genus (ribbons are now crosscaps) and we extend the related terminology : mutations, isomorphism classes, side cycles, (Δ) -chirotopes, and so on. Furthermore, exactly as we did for the collection of isomorphism classes of simple arrangements of pseudolines of genus 1, we embed the collection of isomorphism classes of simple arrangements of pseudolines into the collection of isomorphism classes of arrangements of double pseudolines via the support of the isomorphism classes of thin arrangements of double pseudolines. Figure 35 shows an embedding in 3-space of the tubular neighborhood of an indexed arrangement of two oriented pseudolines. Again, in this broader context, the

Fig. 35 Embedding in 3-space of the tubular neighborhood of an indexed arrangement of two oriented pseudolines

Fig. 36 **a** A family of three curves cellularly embedded in a Klein bottle that fulfills condition (2) but not condition (3) of the definition of an arrangement of double pseudolines; **b** an arrangement of three curves living in a double Klein bottle (Color figure online)

classical LR characterization of chirotopes of arrangements of pseudolines of genus 1 is a direct consequence of the two following theorems.

Theorem 27 *The map which assigns to an isomorphism class of indexed arrangements of oriented pseudolines its 3-chirotope is one-to-one and that which assigns its 5-chirotope is (one-to-one and) onto.*

Theorem 28 *The class of arrangements of pseudolines of genus 1 is the class of arrangements pseudolines whose subarrangements of size at most 4 are of genus 1.*

Before proving these theorems (and discuss improved versions of Theorems 26 and 28) we give few examples of arrangements.

Example 8 Figure 36a depicts a family of three curves cellularly embedded in a Klein bottle (decomposed by the curves into 2 digons, 2 trigons, 6 tetragons, 1 hexagon and 1 octagon) that fulfills condition (2) but not condition (3) of the definition of an arrangement of double pseudolines: the disk side of the green curve in the arrangement composed of the green and red curves and the disk side of the green curve in the

Fig. 37 Two arrangements on three curves living in a double Klein bottle (Color figure online)

arrangement composed of the green and black curves intersect the ribbon of the green curve in two distinct cylinders (on the other hand, disk and crosscap sides of the red and black curves are well-defined). Figure 36b depicts an arrangement of three curves, obtained by adding two twists on the green curve of the configuration of Fig. 36a. It is composed of 2 digons, 6 tetragons, 1 octagon and 1 dodecagon (the corners of the octagon are labeled with the numeral 8 and those of the dodecagon by the letter γ). It lives in a double Klein bottle. Its node cycles are the same as those of the previous example, i.e.,

$$\begin{aligned} 1 : & \quad 12, \bar{1}2, \bar{1}\bar{2}, \bar{1}\bar{2}, \bar{1}3, \bar{1}\bar{3}, 13, \bar{1}3, \\ 2 : & \quad 21, \bar{2}1, \bar{2}\bar{1}, \bar{2}\bar{1}, 23, \bar{2}3, 23, \bar{2}3, \\ 3 : & \quad 31, \bar{3}2, \bar{3}\bar{2}, \bar{3}\bar{1}, 31, 32, \bar{3}2, \bar{3}1. \end{aligned}$$

Example 9 Figure 37 depicts embeddings in 3-space of tubular neighborhoods of two arrangements on three curves. Again the horizontal dashed line segments indicate the presence of half-twists (180°) of the ribbons of the tubular neighborhood.

Both live in a sphere with 4 crosscaps (a double Klein bottle) decomposed by the curves into 1 trigon, 7 tetragons, 1 octagon and 1 nonagon. (In the left diagram the corners of the octagon are labeled by the numeral 8 and those of the nonagon by the numeral 9.) If we orient clockwise the curves and use, respectively, the indices 1, 2 and 3 for the green, blue and red curves, then the side cycles of disk type of the arrangements are, respectively,

$$\begin{aligned} 1 : & \quad \bar{2}\bar{2}\bar{2}\bar{2}\bar{3}\bar{3}\bar{3}\bar{3}, & 1 : & \quad \bar{2}\bar{2}\bar{3}\bar{3}\bar{2}\bar{2}\bar{3}\bar{3}, \\ 2 : & \quad \bar{3}\bar{3}\bar{1}\bar{1}\bar{3}\bar{3}\bar{1}\bar{1}, & \text{and} & \quad 2 : & \quad \bar{3}\bar{3}\bar{1}\bar{1}\bar{3}\bar{3}\bar{1}\bar{1}, \\ 3 : & \quad \bar{2}\bar{2}\bar{1}\bar{1}\bar{2}\bar{2}\bar{1}\bar{1}, & 3 : & \quad \bar{2}\bar{2}\bar{1}\bar{1}\bar{2}\bar{2}\bar{1}\bar{1}. \end{aligned}$$

Observe that the first arrangement is a martagon (with respect to and only to the green curve) but the second one is not and that these two arrangements are connected by a sequence of four mutations.

Fig. 38 Two thin arrangements of three double pseudolines

Fig. 39 Two arrangements of three pseudolines

Example 10 Figure 38 depicts embeddings in 3-space of tubular neighborhoods of two thin arrangements on three curves. The first arrangement lives in a cross surface decomposed by the curves into 4 trigons and 9 tetragons, and the second one in a surface with 3 crosscaps decomposed by the curves into 2 hexagons and 9 tetragons. Their side cycles of disk type are, respectively,

$$\begin{array}{ll}
 1 : 2233\overline{2233}, & 1 : 2233\overline{2233}, \\
 2 : 3311\overline{3311}, & \text{and} \quad 2 : 3311\overline{3311}, \\
 3 : 1122\overline{1122}, & 3 : 1122\overline{1122}.
 \end{array}$$

These two arrangements are doubles of those of Fig. 39. Note that a family of circular sequences D_i , $i \in I$, is the family of side cycles of disk type of a thin arrangement of oriented double pseudolines indexed by I if and only if the D_i are the images under the morphism $\varphi(x) = xx$ of the side cycles of a simple arrangement of oriented pseudolines indexed by I .

Example 11 Figure 39 depicts embeddings in 3-space of tubular neighborhoods of two arrangements of three pseudolines. The first arrangement lives in a cross surface decomposed by the curves into 4 trigons, and the second one in a surface with 3 crosscaps decomposed by the curves into 2 hexagons. Their side cycles are, respectively,

Fig. 40 An arrangement of three double pseudolines living in a triple cross surface. The double pseudolines are drawn red, green and purple in colored pdf (Color figure online)

$$\begin{array}{ll}
 1 : 23\overline{23}, & 1 : 23\overline{23}, \\
 2 : 31\overline{31}, & \text{and} \quad 2 : 31\overline{31}, \\
 3 : 12\overline{12}, & 3 : 12\overline{12}.
 \end{array}$$

These two arrangements are core arrangements of those of Fig. 38. Note that a family of circular sequences D_i , $i \in I$, is the family of side cycles of a simple arrangement of oriented pseudolines indexed by I if and only if the D_i are *antipodal* shuffles of the elementary circular sequences $j\bar{j}$, $j \neq i$. (Here antipodal means that j and $\bar{j}$ occur at positions that differ by the maximum amount, i.e., the cardinality of $I - 1$.)

Example 12 Figure 40 depicts the cell complex of a simple arrangement of three octagonal curves (colored red, green and purple in colored pdf)

Fig. 41 An arrangement of three double pseudolines living in a triple cross surface

$$\begin{aligned}\tau_1 &= 1a2b3c4d5e6f7g8h, \\ \tau_2 &= \hat{1}\hat{a}\hat{2}\hat{b}\hat{3}\hat{c}\hat{4}\hat{d}\hat{5}\hat{e}\hat{6}\hat{f}\hat{7}\hat{g}\hat{8}\hat{h}, \\ \tau_3 &= \tilde{1}\tilde{a}\tilde{2}\tilde{b}\tilde{3}\tilde{c}\tilde{4}\tilde{d}\tilde{5}\tilde{e}\tilde{6}\tilde{f}\tilde{7}\tilde{g}\tilde{8}\tilde{h},\end{aligned}$$

living in a triple cross surface as one can check by calculating the Euler characteristic of the surface. In the figure the cell complex is augmented with its dual graph (oriented arbitrarily at our convenience). Using the symbol of an edge of the cell complex to denote its dual we get a dual presentation composed of a system of 12 equations in 24 symbols

$$\begin{aligned}a\hat{f} &= \hat{e}b, & a\hat{h} &= \hat{g}h, & c\hat{f} &= \hat{g}b, & c\hat{d} &= \hat{e}d, \\ \tilde{a}f &= \tilde{e}\tilde{b}, & \tilde{a}h &= \tilde{g}\tilde{h}, & \tilde{c}f &= \tilde{g}b, & \tilde{c}d &= \tilde{e}d, \\ \hat{a}\tilde{f} &= \tilde{e}\hat{b}, & \hat{a}\tilde{h} &= \tilde{g}\hat{h}, & \hat{c}\tilde{f} &= \tilde{g}\hat{b}, & \hat{c}\tilde{d} &= \tilde{e}\hat{d},\end{aligned}$$

providing evidence that this system of curves is a well-defined arrangement, as illustrated in Fig. 41 where the shaded regions denote the crosscap sides of the curves.

We built it as the simple arrangement with side cycles (of disk type)

$$\begin{aligned}1 &: 22\overline{22}33\overline{33}, \\ 2 &: 33\overline{33}11\overline{11}, \\ 3 &: 11\overline{11}22\overline{22}\end{aligned}$$

Fig. 42 A 4-chirotope on the indexing set $\{1, 2, 3, 4, 5\}$ that is not a five-chirotope

(this can be read easily on the dual presentation). Observe that it is a martagon with respect to each of its three curves.

Example 13 The thin 3-chirotope χ on the indexing set $\{1, 2, 3, 4, 5\}$ with entries

$$\begin{array}{cccccc} C_{04}(123), & C_{04}(124), & C_{04}(125), & C_{04}(134), & C_{04}(145), \\ C_{04}(234), & C_{04}(245), & C_{04}(345), & C_{04}(153), & C_{04}(253), \end{array}$$

admits a 4-extension (i.e., χ is the restriction of a 4-chirotope), depicted in Fig. 42, but no 5-extension because there is no cycle involving the indices 2, 3, 4, 5 and their negatives exactly twice in which the side cycles of disk type $\overline{22332233}$, $\overline{22442244}$, $\overline{22552255}$, $\overline{33443344}$, $\overline{44554455}$, $\overline{55335533}$ assigned to the index 1 of the entries $C_{04}(123)$, $C_{04}(124)$, $C_{04}(125)$, $C_{04}(134)$, $C_{04}(145)$ and $C_{04}(153)$ of χ are subcycles. The same conclusion holds if we interpret the entries as entries of a 3-chirotope of pseudoline arrangements.

Example 14 The simple 3-chirotope on the indexing set $\{1, 2, 3, 4\}$ with entries $C_{64}(123)$, $C_{64}(124)$, $C_{64}(134)$, $C_{64}(234)$ is the chirotope of a unique simple arrangement Υ on four curves whose side cycles of disk type are

$$\begin{array}{ll} 1 : & \overline{223344223344}, \\ 2 : & \overline{334411334411}, \\ 3 : & \overline{441122441122}, \\ 4 : & \overline{112233112233}. \end{array}$$

The surface is a sphere with 7 crosscaps decomposed by the curves into 19 two-cells (12 digons, 3 octagons, and 4 dodecagons) put together according to the following presentation:

$$\begin{array}{ll}
 1 : & a\hat{a}^{-1} = 1, \\
 3 : & \hat{i}\tilde{f}^{-1} = 1, \\
 5 : & \tilde{b}\check{f}^{-1} = 1, \\
 7 : & e\check{b}^{-1} = 1, \\
 9 : & c\check{d}^{-1} = 1, \\
 11 : & \hat{k}\check{d}^{-1} = 1, \\
 13 : & \hat{b}^{-1}b\check{c}^{-1}\check{g}\hat{l}\hat{l}^{-1}\check{e}\hat{a}^{-1} = 1, \\
 14 : & \check{a}\check{g}\hat{j}^{-1}\check{c}f^{-1}\hat{h}^{-1}\check{e}\hat{d}^{-1} = 1, \\
 16 : & a\hat{b}\hat{l}^{-1}\hat{d}\hat{j}^{-1}\hat{f}\hat{g}^{-1}\hat{h}\hat{f}\hat{j}\hat{d}\hat{l} = 1, \\
 18 : & \check{e}\hat{i}\check{g}\hat{l}^{-1}\check{i}\hat{i}^{-1}\check{k}\hat{c}^{-1}\check{a}\check{f}\check{c}\hat{c} = 1, \\
 2 : & g\hat{g} = 1, \\
 4 : & \hat{c}\hat{l}^{-1} = 1, \\
 6 : & \hat{h}\check{l} = 1, \\
 8 : & k\check{h} = 1, \\
 10 : & i\check{j} = 1, \\
 12 : & \hat{e}\check{j} = 1, \\
 15 : & f\hat{h}\hat{k}\hat{d}\hat{k}\hat{i}\hat{j}\hat{i} = 1, \\
 17 : & \hat{a}\hat{b}\hat{d}\hat{d}\hat{b}\hat{f}\hat{g}^{-1}\hat{h}\hat{j}^{-1}\hat{j}\hat{h}^{-1}\hat{l} = 1, \\
 19 : & \hat{c}\hat{k}\check{e}\check{b}\check{g}\check{k}^{-1}\check{i}\hat{e}^{-1}\check{k}\hat{h}^{-1}\hat{a}\hat{e} = 1,
 \end{array}$$

where

$$\begin{aligned}
 \Upsilon_1 &= abcdefghijkl, \\
 \Upsilon_2 &= \hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}\hat{g}\hat{h}\hat{i}\hat{j}\hat{k}\hat{l}, \\
 \Upsilon_3 &= \check{a}\check{b}\check{c}\check{d}\check{e}\check{f}\check{g}\check{h}\check{i}\check{j}\check{k}\check{l}, \\
 \Upsilon_4 &= \check{a}\check{b}\check{c}\check{d}\check{e}\check{f}\check{g}\check{h}\check{i}\check{j}\check{k}\check{l}.
 \end{aligned}$$

A dual presentation is given by the following system of 24 equations in 48 symbols

$$\begin{array}{llll}
 a\hat{b} = \hat{a}b, & \hat{g}f = g\hat{h}, & g\hat{f} = \hat{g}f, & a\hat{l} = \hat{a}l, \\
 c\check{e} = \check{d}d, & \hat{j}h = i\check{k}, & i\check{i} = \check{j}j, & \check{d}b = c\check{c}, \\
 e\check{c} = \check{b}f, & \check{h}j = k\check{i}, & k\check{g} = \check{h}l, & \check{b}d = e\check{a}, \\
 \hat{i}\hat{g} = \check{f}\hat{j}, & \hat{c}\hat{a} = \check{l}\hat{b}, & \hat{c}\hat{k} = \check{l}\hat{d}, & \hat{f}\hat{h} = \hat{i}\hat{e}, \\
 \hat{k}\hat{e} = \check{d}\hat{l}, & \hat{j}\hat{d} = \hat{e}\hat{k}, & \hat{e}\hat{i} = \check{j}\hat{f}, & \hat{d}\hat{j} = \hat{k}\hat{c}, \\
 \check{b}\check{g} = \check{f}\check{c}, & \check{l}\check{g} = \check{h}\check{a}, & \check{h}\check{k} = \check{l}\check{i}, & \check{f}\check{a} = \check{b}\check{e},
 \end{array}$$

where we use the same symbol to denote an edge and its dual; the dual presentation is also depicted in Fig. 43.

Example 15 The chirotope of the martagon $M_1(1234)$ is the chirotope of a second martagon $M_1^*(1234)$ defined by the (side) cycles (of disk type)

$$\begin{array}{l}
 1 : \overline{2222}\overline{3333}\overline{4444}, \\
 2 : \overline{3344}\overline{113344}\overline{11}, \\
 3 : \overline{4422}\overline{114422}\overline{11}, \\
 4 : \overline{2233}\overline{112233}\overline{11},
 \end{array}$$

which are obtained from the cycles of $M_1(1234)$ by simply changing the order of the blocks $\overline{2222}$, $\overline{3333}$, and $\overline{4444}$ in the cycle indexed by 1. This arrangement lives in a triple cross surface that is decomposed by the curves into 3 digons, 15 tetragons, 3

Fig. 43 The dual presentation of the unique arrangement on four curves whose chirotope is the one with entries $C_{64}(123)$, $C_{64}(124)$, $C_{64}(134)$, $C_{64}(234)$. The underlying surface of this arrangement is a sphere with seven crosscaps

Fig. 44 The four martagons on four double pseudolines. Two live in a cross surface and two in a triple cross surface

pentagons, 1 hexagon and 1 nonagon. Note that this arrangement has no triangular faces. Similarly the chirotope of the martagon $M_2(1234)$ is the chirotope of a second martagon $M_2^*(1234)$ defined by the cycles

$$\begin{aligned} 1 : & \overline{222244443333}, \\ 2 : & \overline{331144331144}, \\ 3 : & \overline{224411224411}, \\ 4 : & \overline{221122331133}. \end{aligned}$$

This arrangement lives in a triple cross surface, decomposed by the curves into 4 digons, 14 tetragons, 3 pentagons, 1 octagon and 1 nonagon. Graphical representations of (tubular neighborhoods of) these arrangements are given in Fig. 44.

Example 16 The 3-chirotope on the indexing set $\{1, 2, 3, 4\}$ with entries $C_{32}(123)$, $C_{32}(124)$, $C_{32}(134)$, $C_{32}(234)$ is not the chirotope of an arrangement because the 3 side cycles indexed by 3 of the entries $C_{32}(123)$, $C_{32}(134)$, $C_{32}(234)$, namely $1\overline{1}22\overline{1}122$, $4\overline{1}144\overline{1}14$, and $\overline{4}2244\overline{2}24$, understood as partial circular orders on the indices 1, 2, 4 and their negatives, are incompatible. Similarly for the 3-chirotope on $\{1, 2, 3, 4\}$ with entries $C_{22}(123)$, $C_{22}(423)$, $C_{32}(124)$, $C_{32}(134)$.

Example 17 Example 14 generalizes to any number of indices, i.e., the $C_{64}(ijk)$, where $1 \leq i < j < k \leq n$, are the entries of the chirotope of an arrangement on n curves. For $n = 5, 6, 7, 8, 9$ we get surfaces of genus 14, 21, 33, 43, 58 decomposed

by the curves into twenty 2-gons, one 5-gon, one 10-gon, five 16-gons, one 25-gon for $n = 5$; thirty 2-gons, five 12-gons and six 20-gons for $n = 6$; forty-two 2-gons, one 7-gon, two 14-gons, seven 24-gons, one 49-gon for $n = 7$; fifty-six 2-gons, seven 16-gons, eight 28-gons, for $n = 8$; and seventy-two 2-gons, one 9-gon, three 18-gons, three 27-gons, nine 32-gons for $n = 9$.

Back to the proof of Theorems 25, 26, 27, and 28.

The proof needs some preparations. Given a sequence $B = \{i_1 j_1\} \{i_2 j_2\} \{i_3 j_3\} \dots \{i_k j_k\}$, $i_l \in \{i, \bar{i}\}$, we denote by $B_{\otimes j_p}$ the sequence $\alpha_{p+1} \alpha_{p+2} \dots \alpha_k \alpha_p \alpha_1 \alpha_2 \dots \alpha_{p-1}$ or its reverse $\alpha_{p-1} \alpha_{p-2} \dots \alpha_2 \alpha_1 \alpha_p \alpha_{k_m} \dots \alpha_{p+2} \alpha_{p+1}$, depending on whether $i_p = \bar{i}$ or $i_p = i$, where

$$\alpha_q = \begin{cases} \{i_p j_p\} \otimes \{i_q j_q\} & \text{if } p < q \leq k, \\ \{i_p j_p\} & \text{if } q = p, \\ \{i_q j_q\} \otimes \{i_p j_p\} & \text{if } 1 \leq q < p. \end{cases}$$

We leave the verification of the following property to the reader: if B is a prime factor of the side cycle of disk type indexed by i of an arrangement of double pseudolines then $B_{\otimes j_p}$ is the corresponding prime factor of the side cycle of disk type indexed by j_p . Routine considerations that we leave to the reader yield the following characterizations of families of cycles that arose as families of side cycles of arrangements of double pseudolines and those that arose as families of side cycles of arrangements of pseudolines.

Lemma 29 *Let I be a finite set of (at least 2) indices, let D_i and M_i , $i \in I$, be two families of circular words on the signed version $\hat{I}$ of I , with the property that D_i and M_i are shuffles of the elementary cycles $j \bar{j} j \bar{j}$, $j \neq i$, let S_i be the result of replacing in D_i the linear subsequences $\bar{j} j j \bar{j}$, $j \neq i$, by the linear sequences $\{i j\} \{i \bar{j}\} \{i \bar{j}\} \{i j\}$, let T_i be the result of replacing in M_i the linear subsequences $\bar{j} j j \bar{j}$, $j \neq i$, by the linear sequences $\{i j\} \{i \bar{j}\} \{i \bar{j}\} \{i j\}$, and let $S_{\bar{i}}$ and $T_{\bar{i}}$ be the reversal of S_i and T_i , respectively. Then the D_i and M_i are the side cycles of disk type and crosscap type of an arrangement Γ of oriented double pseudolines indexed by I if and only if there exist block decompositions*

$$B_{i1} B_{i2} \dots B_{i n_i}$$

of the S_i , $i \in I$, where $B_{im} = \{i_1 j_{m1}\} \{i_2 j_{m2}\} \{i_3 j_{m3}\} \dots \{i_k j_{mk_m}\}$, $1 \leq m \leq n_i$, with $j_{ml} \notin \{j_{m'l'}, \bar{j}_{m'l'}\}$ for all $1 \leq l < l' \leq k_m$, such that

- (1) $T_i = B_{i1}^* B_{i2}^* \dots B_{i n_i}^*$ where B_{im}^* is the reversal of B_{im} (note that the pairs S_i, T_i determine their block decompositions);
- (2) $(B_{im})_{\otimes j_{mp}}$ is one of the blocks of the block decomposition of $S_{j_{mp}}$.

Lemma 30 *Let I be a finite set of (at least 2) indices, let C_i , $i \in I$, be a family of circular words on the signed version $\hat{I}$ of I , with the property that the C_i are shuffles of the elementary cycles $j \bar{j}$, $j \neq i$, and let $C_{\bar{i}}$ be the reversal of C_i . Then the C_i are*

the side cycles of an arrangement Γ of oriented pseudolines indexed by I if and only if there exist block decompositions

$$B_{i1}B_{i2}\dots B_{in_i}B'_{i1}B'_{i2}\dots B'_{in_i}$$

of the C_i , $i \in I$, where $B_{im} = j_{m1}j_{m2}j_{m3}\dots j_{mk_m}$, $1 \leq m \leq n_i$, with $j_{m_l} \notin \{j_{m_{l'}}, \bar{j}_{m_{l'}}\}$ for all $1 \leq l < l' \leq k_m$, such that

- (1) B'_{im} is the complement of the reversal of B_{im} (note that this condition determines the block decomposition of C_i);
- (2) $j_{m2}j_{m3}\dots j_{mk_m}\bar{i}$ is one of the blocks of the block decomposition of $C_{j_{m1}}$.

Remark 9 According to Lemma 29, the number b_n of simple indexed arrangements of oriented double pseudolines on a given set of n indices is the n -th power of the number of shuffles of the $n - 1$ circular sequences $j\bar{j}\bar{j}\bar{j}$, $1 \leq j \leq n$, $j \neq 1$, or, equivalently, the n -th power of the product of the number of permutations of a multiset of $4n - 5$ elements of multiplicities $3, 4, 4, \dots, 4$ and the number of cyclic shifts of the $n - 2$ linear sequences $j\bar{j}\bar{j}\bar{j}$, $2 < j \leq n$. Hence, using the standard notation for multinomial coefficients,

$$b_n = \left\{ 4^{n-2} \binom{4n-5}{3, 4, 4, \dots, 4} \right\}^n.$$

The first values are : $b_2 = 1^2$, $b_3 = 140^3$, $b_4 = 184800^4$ and $b_5 = 10090080005^5$. Similarly, according to Lemma 30, the number c_n of simple indexed arrangements of oriented pseudolines on a given set of n indices is the n -th power of the number of antipodal shuffles of the $n - 1$ circular sequences $\bar{j}j$, $1 \leq j \leq n$, $j \neq 1$, or, equivalently, the n -th power of the number of signed permutations on a set of $n - 2$ elements. Hence

$$c_n = \{2^{n-2}(n-2)!\}^n.$$

The first values are: $c_2 = 1^2$, $c_3 = 2^3$, $c_4 = 8^4 = 4096$ and $c_5 = 48^5 = 254803968$. It will be interesting to have closed formulae also for nonsimple arrangements.

Proof of Theorem 25 Let I be finite indexing set, let Δ_5 be the complex of subsets of size at most 5 of I , let χ be a 5-chirotope on the indexing set I , and for $J \in \Delta_5$, let $D_i(J)$ and $M_i(J)$ be the families of side cycles of the entry $\chi(J)$ of χ . Proving the theorem boils down to prove that for any index $i \in I$ there exists

- (1) a unique shuffle D_i of the elementary cycles $j\bar{j}\bar{j}\bar{j}$, $j \neq i$, of which the $D_i(J)$, $i \in J \in \Delta_5$, are subcycles;
- (2) a unique shuffle M_i of the elementary cycles $j\bar{j}\bar{j}\bar{j}$, $j \neq i$, of which the $M_i(J)$, $i \in J \in \Delta_5$, are subcycles; and that
- (3) the two families of cycles D_i and M_i are the families of side cycles of an arrangement of oriented double pseudolines indexed by I whose 5-chirotope is χ .

For $X \in \{D, M\}$ and $i \in I$, let $\mathcal{R}_i^X$ be the ternary relation defined on distinct elements $\alpha = i_\alpha j_\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j} \mid j \in I \setminus i\}$, by $(\alpha, \alpha', \alpha'') \in \mathcal{R}_i^X$ if α, α' and α'' appear in this order on the cycle $X_i(\{i, j_\alpha, j_{\alpha'}, j_{\alpha''}\})$ and let $\mathcal{B}_i^X$, $i \in I$, be the binary relation defined on distinct elements $\alpha = i_\alpha j_\alpha \in \{ij, \bar{i}j, i\bar{j}, \bar{i}\bar{j} \mid j \in I \setminus i\}$, by $(\alpha, \alpha') \in \mathcal{B}_i^X$ if α and α' appear in this order in the same prime factor of the prime factor decomposition of $X_i(\{i, j_\alpha, j_{\alpha'}\})$. Clearly

- (1) $\mathcal{R}_i^X$ is well-defined;
- (2) for every triple $(\alpha, \alpha', \alpha'')$ one has $(\alpha, \alpha', \alpha'') \in \mathcal{R}_i^X$ or (exclusive) $(\alpha, \alpha'', \alpha') \in \mathcal{R}_i^X$;
- (3) if $(\alpha, \alpha', \alpha'') \in \mathcal{R}_i^X$ then $(\alpha', \alpha'', \alpha) \in \mathcal{R}_i^X$;
- (4) $\mathcal{R}_i^X$ is transitive, i.e., if $(\alpha, \alpha', \alpha'') \in \mathcal{R}_i^X$ and $(\alpha, \alpha'', \alpha''') \in \mathcal{R}_i^X$ then $(\alpha, \alpha', \alpha''') \in \mathcal{R}_i^X$ (because $X_i(i, j_\alpha, j_{\alpha'}, j_{\alpha''})$, $X_i(i, j_\alpha, j_{\alpha''}, j_{\alpha'''})$, and $X_i(i, j_\alpha, j_{\alpha'}, j_{\alpha'''})$ are subcycles of $X_i(i, j_\alpha, j_{\alpha'}, j_{\alpha''}, j_{\alpha'''})$);
- (5) $\mathcal{B}_i^X$ is well-defined;
- (6) if $(\alpha, \alpha') \in \mathcal{B}_i^X$ and $(\alpha', \alpha'') \in \mathcal{B}_i^X$ then $(\alpha, \alpha'') \in \mathcal{B}_i^X$;
- (7) if $(\alpha, \alpha'') \in \mathcal{B}_i^X$ and $(\alpha, \alpha', \alpha'') \in \mathcal{R}_i^X$ then $(\alpha, \alpha') \in \mathcal{B}_i^X$ and $(\alpha', \alpha'') \in \mathcal{B}_i^X$;
- (8) if $(\alpha, \alpha') \in \mathcal{B}_i^X$ then $(\alpha', \alpha) \in \mathcal{B}_i^{\bar{X}}$.

This proves that the shuffle X_i of the elementary cycles $j\bar{j}\bar{j}\bar{j}$, $j \neq i$, given by the ternary relation $\mathcal{R}_i^X$, is the unique shuffle of the elementary cycles $j\bar{j}\bar{j}\bar{j}$, $j \neq i$, of which the $X_i(J)$, $i \in J \in \Delta_5$, are subcycles and that there is a unique block decomposition $B_{i1}B_{i2} \dots B_{in_i}$ of D_i , where $B_{im} = \{i_1j_{m1}\}\{i_2j_{m2}\}\{i_3j_{m3}\} \dots \{i_kj_{mk_m}\}$, $1 \leq m \leq n_i$, with $j_{m_l} \notin \{j_{m_{l'}}, \bar{j}_{m_{l'}}\}$ for all $1 \leq l < l' \leq k_m$ (the one given by the binary relation $\mathcal{B}_i^X$) such that $T_i = B_{i1}^*B_{i2}^* \dots B_{in_i}^*$ where B_{im}^* is the reversal of B_{im} . Since by construction $B_{\otimes j_p}$ is one of the blocks of the block decomposition of $S_{j_{mp}}$ we are done, thanks to Lemma 29. $\square$

Proof of Theorem 27 Similar to the proof of Theorem 25. $\square$

We come now to the proof of Theorems 26 and 28.

As said in the introduction, to prove Theorem 26, it can be argued that the mutation graph on the space of arrangements of double pseudolines of given size whose subarrangements of size at most 5 are of genus 1 is connected, or that for any pair of distinct faces of an arrangement of double pseudolines of genus 1 there exists a subarrangement of size at most 3 whose corresponding faces are distinct. Similarly, to prove Theorem 28, it can be argued that the mutation graph on the space of arrangements of pseudolines of given size whose subarrangements of size at most 4 are of genus 1 is connected, or that for any pair of distinct faces of an arrangement of pseudolines of genus 1 there exists a subarrangement of size at most 2 whose corresponding faces are distinct. We set out independently these two arguments in the two next sections.

5.2 Pumping Lemma and Mutations

We prove the connectedness of the mutation graph on the space of arrangements of double pseudolines of given size whose subarrangements of size at most 5 are of

genus 1 and the connectedness of the mutation graph on the space of arrangements of pseudolines of given size whose subarrangements of size at most 4 are of genus 1. The proof is based on the following abstractions of the pumping lemmas of Sect. 2.

Lemma 31 *Let Γ be a simple arrangement of double pseudolines whose subarrangements of size at most 5 are of genus 1 and let $\gamma \in \Gamma$. Assume that there exists a vertex v of the arrangement Γ contained in the crosscap side of γ in the subarrangement of size three composed of γ and the two double pseudolines crossing at v . Then there exists a triangular 2-cell of the arrangement Γ with a side supported by γ and a vertex w supported by the crosscap side of γ in the subarrangement composed of γ and the two double pseudolines crossing at w .*

Lemma 32 *Let Γ be a simple arrangement of pseudolines whose subarrangements of size at most 4 are of genus 1, let $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$, and let $M(\gamma, \gamma')$ be one of two 2-cells of size 2 of the subarrangement $\{\gamma, \gamma'\}$. Assume that there exists a vertex v of the arrangement Γ contained in $M(\gamma, \gamma')$ in the subarrangement of size four composed of γ, γ' and the two pseudolines crossing at v . Then there exists a triangular 2-cell of the arrangement Γ contained in $M(\gamma, \gamma')$ with a side supported by γ in the subarrangement composed of γ, γ' and the two pseudolines crossing at the vertex w opposite the side supported by γ .*

Proof of Lemma 31 Let $p_\Gamma : \tilde{\mathcal{P}}_\Gamma \rightarrow \mathcal{P}_\Gamma$ be a 2-sheeted unbranched covering of $\mathcal{P}_\Gamma$ which is closed and orientable. For example the two relations

$$\begin{cases} c_1 c'_1 c_2 c'_2 \dots c_g c'_g = 1, \\ c'_1 c_1 c'_2 c_2 \dots c'_g c_g = 1 \end{cases}$$

define a closed and orientable 2-sheeted unbranched covering of the nonorientable surface of genus g defined by the relation $c_1 c_1 c_2 c_2 \dots c_g c_g = 1$. For any subarrangement ζ of Γ of size at least 2 the restriction of p_Γ to the pair $p_\Gamma^{-1}(R_\zeta)$, R_ζ extends naturally to a closed and orientable 2-sheeted unbranched covering $p_\zeta : \tilde{\mathcal{P}}_\zeta \rightarrow \mathcal{P}_\zeta$ of the nonorientable surface $\mathcal{P}_\zeta$. Without loss of generality we assume that the surfaces $\mathcal{P}_\zeta$ intersect pairwise only along their common ribbons, i.e., $\mathcal{P}_\zeta \cap \mathcal{P}_{\zeta'} = R_{\zeta''}$ where $\zeta'' = \zeta \cap \zeta'$. Similarly we assume that the surfaces $\tilde{\mathcal{P}}_\zeta$ intersect pairwise only along their common ribbons. The two lifts under p_Γ of a curve τ of Γ are denoted τ_+ and τ_- , and the set of lifts of the curves of Γ is denoted $\tilde{\Gamma}$. Figure 45a shows a subarrangement of two double pseudolines and Fig. 45b shows its 2-sheeted unbranched covering. We note that two curves of $\tilde{\Gamma}$ have exactly 0 or 2 intersection points depending on whether they are the lifts of the same curve in Γ , or not. By convention if B is one of the two intersection points of two crossing curves of $\tilde{\Gamma}$ then the other one is denoted B_* , as illustrated in Fig. 45b. For ζ subarrangement of Γ of size 2,3,4 or 5 containing γ we denote by C_ζ the cylinder of the sphere $\tilde{\mathcal{P}}_\zeta$ bounded by γ_+ and γ_- . We introduce the following terminology.

- (1) A γ -curve supported by $\gamma' \in \Gamma$, $\gamma' \neq \gamma$, is a maximal subcurve of γ'_+ or γ'_- contained in the cylinder C_ζ where $\zeta = \{\gamma, \gamma'\}$. Observe that there are four γ -curves supported by γ' (two per lift of γ') and that a γ -curve has an endpoint

Fig. 45 **a** A subarrangement of two double pseudolines; **b** its 2-sheeted unbranched covering

Fig. 46 The three possible arrangements of two γ -curves

on γ_+ and the other one on γ_- . The γ -curve with endpoint B on γ_+ is denoted $\text{curve}_\gamma(B)$.

- (2) An *arrangement of γ -curves* is a set of at most four γ -curves embedded in the cylinder C_ζ where ζ is the set of supporting curves of the at most four γ -curves augmented with γ . The cell complex of an arrangement of two γ -curves depends only on the number of intersection points, as depicted in Fig. 46.
- (3) A γ -*triangle* is a triangular face of the arrangement of two crossing γ -curves with a side supported by γ_+ ; the vertex of a γ -triangle not on γ_+ is called its *apex* and the side of a γ -triangle supported by γ_+ is called its *base side*. The interior and the exterior of the base side of a γ -triangle T , considered as a subset of γ_+ , are denoted $\text{Int}_\gamma(T)$ and $\text{Ext}_\gamma(T)$, respectively.
- (4) A γ -triangle is *admissible* if one of its two sides with the apex as an endpoint is an edge of $\tilde{\Gamma}$.
- (5) An admissible γ -triangle $\Delta = XYY'$ with apex X and edge side XY is said to enclose an admissible γ -triangle $T = ABB'$ with apex A and edge side AB if T is included in Δ and walking along the base side of Δ from Y to Y' we encounter B' before B . Thus the arrangement of the four γ -curves $\text{curve}_\gamma(Y)$, $\text{curve}_\gamma(Y')$, $\text{curve}_\gamma(B)$, $\text{curve}_\gamma(B')$ is, up to homeomorphism, one of those implicitly depicted in Fig. 47a ($B \neq Y'$) or Fig. 47b ($B = Y'$); and, consequently, one of those implicitly depicted in Fig. 47c or d since one can easily prove that $\text{curve}_\gamma(B')$ crosses the side XY' only once.

Fig. 47 The admissible γ -triangle Δ encloses the admissible γ -triangle T

Fig. 48 Relative positions of an admissible γ -triangle T and its derived admissible γ -triangle T' : **a** $A = A'$, $T = T'$; **b** $B'' \in \text{Int}_\gamma(T)$; **c** $B'' \in \text{Ext}_\gamma(T)$

Lemma 33 *There is at least one admissible γ -triangle.*

Proof Since by assumption there is a vertex of Γ in the crosscap side of the double pseudoline γ in the subarrangement composed of γ and the two double pseudolines meeting at the vertex, there is a γ -triangle, say $T = ABB'$ with apex A . Let A' be the vertex of $\tilde{\Gamma}$ that follows B' on the side $B'A$ of T . Then A' is the apex of an admissible γ -triangle $T' = A'B'B''$ with edge side $A'B'$. This proves that there is at least one admissible γ -triangle. $\square$

Let $T = ABB'$ be an admissible γ -triangle with apex A and edge side AB , and let A' be the vertex of $\tilde{\Gamma}$ that follows B' on the side $B'A$ of T . Then A' is the apex of an admissible γ -triangle $T' = A'B'B''$ with edge side $A'B'$. A simple use of the Jordan curve theorem leads to the following four lemmas that control the relative positions of the base sides of T and T' , possibly in the presence of a third admissible γ -triangle $\Delta = XYY'$ with apex X and edge side XY enclosing T . (Figures 48a–c and 49).

Lemma 34 *Assume that $T = T'$. Then T is a triangular two-cell of $\tilde{\Gamma}$.*

Lemma 35 *Assume that $T \neq T'$ and that $B'' \in \text{Int}_\gamma(T)$. Then: (1) $\text{curve}_\gamma(B'')$ crosses the side $B'A$ of T exactly once (at A') and (2) $\text{Int}_\gamma(T')$ is contained in $\text{Int}_\gamma(T)$.*

Lemma 36 *Assume that $T \neq T'$ and that $B'' \in \text{Ext}_\gamma(T)$. Then:*

- (1) $\text{curve}_\gamma(B')$ and $\text{curve}_\gamma(B'')$ cross twice (at A' and A'_*) on the side $B'A$ of T ,
- (2) $\text{Int}_\gamma(T)$ and $\text{Int}_\gamma(T')$ are interior disjoint,

Fig. 49 **a** Δ encloses T ; **b** $B'' \in \text{Ext}_\gamma(T)$; **c** Δ encloses T'

- (3) B'_* and $B''_* \in \text{Ext}_\gamma(T) \cap \text{Ext}_\gamma(T')$, and
- (4) walking along $\text{Ext}_\gamma(T) \cap \text{Ext}_\gamma(T')$ from B'' to B we encounter successively the points B''_* and B'_* .

Furthermore if Δ encloses T , then Δ encloses T' .

Proof We only comment the furthermore part. Let Υ be the arrangement of the three γ -curves $\text{curve}_\gamma(Y)$, $\text{curve}_\gamma(Y')$, $\text{curve}_\gamma(B')$ and let Υ' be the arrangement of the two γ -curves $\text{curve}_\gamma(B')$ and $\text{curve}_\gamma(B'')$, both arrangements being augmented with the points A , B , A' and A''_* . According to the previous discussion Υ is, up to homeomorphism, one of those implicitly depicted in Fig. 49a (we omit the case $Y' = B$). Similarly Υ' is, up to homeomorphism, one of those implicitly depicted in Fig. 49b. Again using the Jordan curve theorem we see easily that the only compatible superpositions of these two arrangements are, up to homeomorphism, those implicitly depicted in Fig. 49c. The lemma follows. $\square$

Consider now the sequence of admissible γ -triangles $T_0, T_1, T_2, \dots$ defined inductively by $T_0 = T$ and $T_{k+1} = T'_k$ for $k \geq 0$. A simple combination of Lemmas 36 and 35 leads to the conclusion that the sequence T_k is stationary. According to Lemma 34 the lemma follows.

Proof of Lemma 32 As the proof uses similar ideas to the proof of the previous lemma with a much simpler case analysis, we omit it. $\square$

Theorem 37 *The mutation graph on the space of pseudoline arrangements of given size whose subarrangements of size at most 4 are of genus 1 is connected.*

Proof A good arrangement is an arrangement of pseudolines whose subarrangements of size at most 4 are of genus 1. Clearly any good arrangement is connected, via a finite sequence of splitting mutations, to a good simple arrangement. Then by a repeated application of Lemma 32 we see that any good simple arrangement of size $n + 1$ is connected, via a finite sequence of mutations, to a good simple arrangement of pseudolines obtained from a good simple arrangement of size n by adding a copy of one of its pseudolines as indicated in Fig. 50. The result follows by induction. $\square$

Fig. 50 Adding a copy of a pseudoline in a pseudoline arrangement is carried out in the vicinity of the pseudoline. The choice of the position of intersection point between the pseudoline and its copy is arbitrary

Theorem 38 *The mutation graph on the space of double pseudoline arrangements of given size whose subarrangements of size at most 5 are of genus 1 is connected.*

Proof A good arrangement is an arrangement of double pseudolines whose subarrangements of size at most 5 are of genus 1. Clearly any good arrangement is connected, via a finite sequence of splitting mutations, to a good simple arrangement. By a repeated application of Lemma 31 we see that any good simple arrangement is connected, via a finite sequence of mutations, to a good thin arrangement. The results follows thanks to Theorem 37 and the one-to-one correspondence between isomorphism classes of simple pseudoline arrangements and isomorphism classes of thin double pseudoline arrangements. $\square$

Proof of Theorem 26 According to Theorem 38 the mutation graph on the space of double pseudoline arrangements of given size whose subarrangements of size at most 5 are of genus 1 is connected. Since a mutation does not change the Euler characteristic of the underlying surface of an arrangement this proves that a double pseudoline arrangement whose subarrangements of size at most 5 are of genus 1 is of genus 1. $\square$

Proof of Theorem 28 According to Theorem 37 the mutation graph on the space of pseudoline arrangements of given size whose subarrangements of size at most 4 are of genus 1 is connected. Since a mutation does not change the Euler characteristic of the underlying surface of an arrangement this proves that a pseudoline arrangement whose subarrangements of size at most 4 are of genus 1 is of genus 1. $\square$

5.3 Separation Lemma

We come now to the announced alternative proofs of Theorems 26 and 28 before discussing improved versions of both. In particular we offer results in strong support of the conjecture that an arrangement of double pseudolines whose subarrangements of size at most 4 are of genus 1 is of genus 1. The alternative proofs are based on the following related two observations

Lemma 39 (Separation Lemma) *Let F and F' be two distinct faces of an arrangement of pseudolines Γ of genus 1. Then there exists a subarrangement of Γ of size 2 whose faces A and A' containing F and F' are distinct.*

Proof Let $[F]$ and $[F']$ be the subarrangements of Γ composed of the supporting curves of the sides of F and F' , respectively. Clearly the faces of the subarrangement $[F]$ containing F and F' are distinct. Therefore one can assume that $\Gamma = [F] = [F']$

Fig. 51 **a** $NN' \in \mathcal{A}(\Gamma, \gamma)$; **b** $NN'MM' \in \mathcal{B}(\Gamma, \gamma)$

is a cyclic arrangement with at least two central cells. Then the result follows from the classification of cyclic pseudoline arrangements of genus 1 that was recalled in Sect. 3.2. $\square$

Lemma 40 (Separation Lemma) *Let F and F' be two distinct faces of an arrangement of double pseudolines Γ of genus 1. Then there exists a subarrangement of Γ of size at most 3 whose faces A and A' containing F and F' are distinct.*

Proof For the restricted class of thin arrangements the result follows from Lemma 39 and the one-to-one correspondence between the class of isomorphism classes of simple pseudoline arrangements and the class of isomorphism classes of thin double pseudoline arrangements. The general case follows easily from the connectedness of mutation graphs. $\square$

We are now ready for the alternative proofs. Let Γ be an arrangement, let $\gamma \in \Gamma$, and set $\Gamma' = \Gamma \setminus \{\gamma\}$. Endow γ with an orientation and introduce the set $\mathcal{A}(\Gamma, \gamma)$ of pairs of consecutive nodes NN' of the node cycle of γ in the arrangement Γ such the face F of Γ' that γ enters at N and the face of Γ' that γ leaves at N' are distinct, as illustrated in Fig. 51a, and the set $\mathcal{B}(\Gamma, \gamma)$ of pairs of pairs of consecutive nodes NN' and MM' of the node cycle of γ in the arrangement Γ such that (1) γ enters at N and M and leaves at N' and M' the same face F of Γ' , (2) the pair NN' separates the pair MM' on the boundary of F , as illustrated in Fig. 51b. Clearly the genus of Γ' is less than the genus of Γ with equality if and only if $\mathcal{A}(\Gamma, \gamma)$ and $\mathcal{B}(\Gamma, \gamma)$ are both empty.

Second proof of Theorem 26 Let Γ be an arrangement of double pseudolines whose subarrangements of size at most 5 are of genus 1. Our goal is to prove that Γ is of genus 1. We proceed by induction on the size n of Γ . The base case $n \leq 5$ is clear. Assume $n \geq 6$, let $\gamma \in \Gamma$ and assume that $\Gamma \setminus \{\gamma\}$ is of genus 1. We show that $\mathcal{A}(\Gamma, \gamma)$ and $\mathcal{B}(\Gamma, \gamma)$ are both empty. Without loss of generality we can assume that Γ is a simple arrangement. We first show that $\mathcal{A}(\Gamma, \gamma)$ is empty. Let NN' be a pair of consecutive nodes of the node cycle of γ in the arrangement Γ . Let Γ'' be a subarrangement of Γ'

of size at most 3 and let A be a face of Γ'' . We let the reader check that the following four claims are equivalent

- (1) N is contained in A or γ enters A at N ;
- (2) N' is contained in A or γ leaves A at N' ;
- (3) F is contained in A ;
- (4) F' is contained in A .

According to Lemma 40 it follows that $F = F'$. Hence $\mathcal{A}(\Gamma, \gamma)$ is empty. It remains to observe that if $NN'MM' \in \mathcal{B}(\Gamma, \gamma)$ then $NN'MM' \in \mathcal{B}(\Gamma'', \gamma)$ where Γ'' is the subarrangement of Γ composed of γ and the (at most 4) curves of Γ that γ crossed at N, N', M and M' to complete the proof. $\square$

Second proof of Theorem 26 Let Γ be an arrangement of pseudolines whose subarrangements of size at most 4 are of genus 1. Our goal is to prove that Γ is of genus 1. We proceed by induction on the size n of Γ . The base case $n \leq 4$ is clear. Assume $n \geq 5$, let $\gamma \in \Gamma$ and assume that $\Gamma \setminus \{\gamma\}$ is of genus 1. We show that $\mathcal{A}(\Gamma, \gamma)$ and $\mathcal{B}(\Gamma, \gamma)$ are both empty. Without loss of generality we can assume that Γ is a simple arrangement. We first show that $\mathcal{A}(\Gamma, \gamma)$ is empty. Let NN' be a pair of consecutive nodes of the node cycle of γ in the arrangement Γ . Let Γ'' be a subarrangement of Γ' of size at most 2 and let A be a face of Γ'' . We let the reader check that the following four claims are equivalent

- (1) N is contained in A or γ enters A at N ;
- (2) N' is contained in A or γ leaves A at N' ;
- (3) F is contained in A ;
- (4) F' is contained in A .

According to Lemma 39 it follows that $F = F'$. Hence $\mathcal{A}(\Gamma, \gamma)$ is empty. It remains to observe that if $NN'MM' \in \mathcal{B}(\Gamma, \gamma)$ then M is the initial node of a pair of nodes in $\mathcal{A}(\Gamma'', \gamma)$ where Γ'' is the subarrangement of Γ composed of γ and the 3 curves of Γ that γ crossed at N, N' and M to complete the proof. $\square$

We now discuss the improved versions of Theorems 26 and 28. They are consequences of improved versions of the separation lemmas, obtained by looking at two-coverings of arrangements. The case of pseudoline arrangements is particularly simple.

Lemma 41 (Separation Lemma) *Let GG' be a pair of distinct faces of an unbranched 2-covering $\tilde{\Gamma}$ of an arrangement of pseudolines Γ of size 2 and genus 1. Then there exists a subarrangement Γ' of Γ of size 1 whose faces A and A' in $\tilde{\Gamma}'$ containing G and G' are distinct.*

Proof Obvious; cf. Fig. 52. $\square$

Theorem 42 *Let Γ be an arrangement pseudolines whose subarrangements of size 3 are of genus 1. Then Γ is of genus 1.*

Proof We argue exactly as in the proof of Theorem 26 except that we work in a 2-covering of Γ in order to use Lemma 41 instead of Lemma 39. $\square$

Fig. 52 **a** An arrangement of two pseudolines; **b** its 2-sheeted unbranched covering

The case of arrangements of double pseudolines requires a little more work.

Let Γ be an arrangement of double pseudolines of genus 1 and let FF' be a pair of distinct faces of Γ . A *S-witness* of FF' is a subarrangement of Γ whose faces A and A' containing F and F' are distinct. The *S-number* of FF' is the minimum of the sizes of its *S-witnesses*. Thus our Separation Lemma asserts that the *S-number* of FF' is 1, 2 or 3. In case the *S-number* of FF' is 3 we say that the pair FF' is a *critical pair* of Γ . In case FF' is a critical pair and Γ is of size 3 we define the *critical graph* of FF' as the graph with two vertices and three edges embedded in the cross surface of the arrangement whose vertices are two arbitrary points a and a' chosen in F and F' and whose edges are three paths joining a to a' , each path avoiding two of the three curves (and crossing necessarily the third one) of the arrangement. It is no hard to see that the critical graph is unique up to ambient isotopy. Figures 53 and 54 show the critical pairs of the arrangements of size 3 together with their associated critical graphs. (The arrangements C_{04} and C_{07} have no critical pairs.) A crucial observation is that critical graphs contain pseudolines with two exceptions: one of the critical graphs the three critical pairs of C_{22} and one of the critical graphs of the four critical pairs of C_{32} are free of pseudolines. Let us call *T-critical* a critical pair whose critical graph is free of pseudolines and *F-critical* a critical pair whose critical graph contains pseudolines (T is for truly and F is for falsely). Hence our Separation Lemma can be completed as follows

Lemma 43 (Separation Lemma) *Let GG' be a pair of distinct faces of an unbranched 2-covering $\tilde{\Gamma}$ of an arrangement of double pseudolines Γ of size 3 and genus 1. Then there exists a subarrangement Γ' of Γ of size at most 2 whose faces A and A' in $\tilde{\Gamma}'$ containing G and G' are distinct unless the pair GG' is one of the two lifts of a *T-critical* pair FF' of Γ with the property that their faces are connected by the lift of the corresponding critical graph.*

Lemma 44 *Let Γ be an arrangement of size 5 whose subarrangements of size 4 are of genus 1, let $\gamma \in \Gamma$ and assume that $\mathcal{B}(\Gamma, \gamma)$ is nonempty. Then $\mathcal{A}(\Gamma, \gamma)$ is nonempty.*

Proof Let $(NN', MM') \in \mathcal{B}(\Gamma, \gamma)$ with the property that there is no $(NN', XX') \in \mathcal{B}(\Gamma, \gamma)$ such that NN', MM', XX' appear in this order on the node cycle of γ in the arrangement Γ . Let v' be the curve crossed by γ at N . Let M'' be successor of M' in

Fig. 53 Critical pairs of the arrangements C_{18} , C_{37} , C_{15} , C_{43} , and C_{22} together with their associated critical graphs. The arrangements C_{04} and C_{07} have no critical pairs

the node cycle of γ . Let L' be the first node of the node cycle of γ , not supported by v' , that follows M' , and let L be its predecessor. A simple case analysis shows that either $LL' \in \mathcal{A}(\Gamma, \gamma)$ or $M'M'' \in \mathcal{A}(\Gamma, \gamma)$. Hence $\mathcal{A}(\Gamma, \gamma)$ is nonempty and we are done. $\square$

Theorem 45 *Let Γ be an arrangement of size 5 whose subarrangements of size 4 are of genus 1. Then Γ is of genus 1 or Γ contains at least one of the two subarrangements C_{22} and C_{32} .*

Proof We argue exactly as in the proof of Theorem 26 except that we work in a 2-covering of Γ in order to use Lemma 43 instead of Lemma 40. $\square$

Fig. 54 Critical pairs of the arrangements C_{33} , C_{32} , C_{252} , C_{251} , C_{32} , C_{22} , C_{25} , C_{36} , and C_{64} together with their associated critical graphs

We arrive at the result announced in the introduction. A *marked critical arrangement* is an arrangement of size 4 and genus 1 together with a pair FF' of distinct faces (the mark) such that

Fig. 55 Three marked critical arrangements. In these diagrams the double pseudolines whose crosscap sides are free of vertices are simply represented by one of their core pseudolines

- (1) the S-number of FF' is 3;
- (2) for any S-witness of FF' of size 3 the critical pair AA' containing FF' is T-critical;
- (3) the S-witnesses of FF' of size 3 are two in number.

Figure 55 shows three marked critical arrangements : the two S-witnesses of size 3 of the mark are obtained by removing the curves labeled τ and τ' . Observe that the two last have the same underlying critical arrangement. It is no hard to see that any marked critical arrangement is connected to one of these three by a sequence of mutations respecting the mark. It follows that marked critical arrangements are few dozens : this number should be compared to the number (6570) of simple arrangements of size 4 and genus 1; cf. [20].

Theorem 46 *Let Γ be an arrangement of double pseudolines of size 5 whose subarrangements of size 4 are of genus 1. Then Γ is of genus 1 or its subarrangements of size 4 are critical arrangements.*

Proof Let $\gamma \in \Gamma$ and assume that $\mathcal{A}(\Gamma, \gamma)$ is nonempty. Let $NN' \in \mathcal{A}(\Gamma, \gamma)$. The curves crossing γ at N and N' are denoted τ and τ' , the face that γ enters at N is denoted F , and the face that γ leaves at N' is denoted F' . We let the reader check the following four claims:

- (1) The S-number of FF' is 3;
- (2) for any S-witness of size 3 the critical pair AA' containing FF' is T-critical;
- (3) $\tau \neq \tau'$;
- (4) the S-witnesses of FF' are $\Gamma \setminus \{\gamma, \tau\}$ and $\Gamma \setminus \{\gamma, \tau'\}$;

from which it follows that Γ' marked at FF' is a marked critical arrangement. $\square$

As said in the introduction, Theorem 46 shows that a computer check of the conjecture that the arrangements of double pseudolines living in cross surfaces are those whose subarrangements of size at most 4 live in cross surfaces, is doable with modest computing resources. This computer check will be the subject of another paper.

6 An Extension and a Refinement

In this sixth and penultimate section we discuss *arrangements of pseudocircles*, *crosscap* or *Möbius arrangements* and the *fibrations* of the latter. The material on fibrations

Fig. 56 Arrangements on two pseudocircles

was partially motivated by the question raised by Goodman and Pollack in [27] about the realizability of their so-called double permutation sequences by families of pairwise disjoint convex bodies of affine topological planes.

Define an *oval* as the boundary of a convex body of a projective plane and the *dual of an oval* as the set of lines touching the oval but not its disk side.

An *arrangement of pseudocircles* is a finite family of pseudocircles embedded in a cross surface, with the property that its subfamilies of size two are homeomorphic to the dual arrangement of two points, two disjoint ovals, or one point and one oval which are not incident. Observe that arrangements of pseudocircles extend both arrangements of pseudolines and arrangements of double pseudolines. Figure 56 depicts representatives of the isomorphism classes of arrangements of two pseudocircles.

The *order* of an arrangement of pseudocircles is defined as the isomorphism class of the poset of the bicolored curves of the arrangement ordered by reverse inclusion of their disk sides where by convention a pseudoline is colored in blue ($\bullet$) and a double pseudoline is colored in red ($\circ$). For example the orders of the arrangements depicted in the above figure are $\{\circ \rightarrow \circ\}$, $\{\circ, \circ\}$, $\{\bullet \rightarrow \circ\}$, $\{\bullet, \bullet\}$, and $\{\bullet, \circ\}$. According to Theorem 1 examples of arrangements of pseudocircles are given by the dual arrangements of finite families of pairwise disjoint ovals and points of projective planes.

Minor adaptations in our proof of the pumping lemma for arrangements of double pseudolines yield the following pumping lemma for arrangements of pseudocircles. We denote by $\mathcal{D}(\gamma)$ the disk side of a double pseudoline γ .

Lemma 47 (Pumping Lemma for Arrangements of Pseudocircles) *Let Γ be a simple arrangement of pseudocircles, let $\gamma \in \Gamma$ be a double pseudoline of Γ , let Γ' be the set*

$\gamma' \in \Gamma$ such that $\mathcal{D}(\gamma') \supset \mathcal{D}(\gamma)$, and let M be the trace on the crosscap side of γ of the 2-cell of the arrangement Γ' that contains $\mathcal{D}(\gamma)$. Assume that there is a vertex of the arrangement Γ lying in M . Then there is a triangular two-cell of the arrangement Γ contained in M with a side supported by γ .

Minor variations in our proofs of Theorems 3, 4, and 2 based on the above pumping lemma lead to direct extensions of these theorems modulo the following elementary dictionary:

arrangements of double pseudolines	$\longleftrightarrow$	arrangements of pseudocircles
convex bodies	$\longleftrightarrow$	ovals and points
size	$\longleftrightarrow$	order

In particular

Theorem 48 Any arrangement of pseudocircles is isomorphic to the dual family of a finite family of pairwise disjoint ovals and points of a projective plane.

For the numbers of isomorphism classes of simple arrangements with trivial order $(\{\bullet, \dots, \bullet, \circ, \dots, \circ\})$ on at most five curves we refer to [20], where these arrangements are called mixed arrangements.

We now discuss a refinement of Theorem 48 in the context of *crosscap or Möbius arrangements* and their *fibrations*. Let $\mathcal{M}$ be a Möbius strip, let $\overline{\mathcal{M}} = \mathcal{M} \cup \{\infty\}$ be its one-point compactification, and recall that $\overline{\mathcal{M}}$ is a cross surface. Define an *arrangement of pseudocircles in $\mathcal{M}$* as an arrangement of pseudocircles in $\overline{\mathcal{M}}$ with the property that the intersection of the disk sides of its pseudolines and double pseudolines is nonempty and contains the point at infinity; define a *fibration* of an arrangement of pseudocircles Γ in $\mathcal{M}$ as a sup-arrangement Γ' of Γ in $\overline{\mathcal{M}}$ composed of the pseudocircles of Γ and of the pseudolines of a pencil of pseudolines through the point at infinity with the property that any pseudoline of the pencil goes through a vertex of Γ and any vertex of Γ is incident to a pseudoline of the pencil. Figure 57 shows an arrangement of two double pseudolines in a Möbius strip and representatives of its three isomorphism classes of fibrations. According to Theorem 48 we see easily that any arrangement of pseudocircles Γ in $\mathcal{M}$ is the dual arrangement of a family of pairwise disjoint ovals and points of a projective plane $\mathcal{G}$ with line space $\overline{\mathcal{M}}$, with the property that the line at infinity ∞ avoids the ovals and the points of the family. Consequently the family

Fig. 57 An arrangement of two double pseudolines in a Möbius strip with representatives of its three isomorphism classes of fibrations

Γ' composed of the pseudocircles of Γ and of the dual pseudolines of the intersection points of the line at infinity with the vertices of Γ is a fibration of Γ . This fibration is denoted $\Gamma^{\mathcal{G}}$. Conversely let Γ' be a fibration of Γ . Does there exist a projective plane $\mathcal{G}$ such that $\Gamma' = \Gamma^{\mathcal{G}}$? Applying Theorem 48 to Γ' we obtain a positive answer to that question.

Theorem 49 *Let $\mathcal{M}$ be a Möbius strip and let Γ' be a fibration of an arrangement of pseudocircles Γ in $\mathcal{M}$. Then Γ is the dual family of a finite family of pairwise disjoint ovals and points of a projective plane $\mathcal{G}$ with line space $\overline{\mathcal{M}}$ such that $\Gamma' = \Gamma^{\mathcal{G}}$.*

In particular the above theorem answers positively the question raised by Goodman and Pollack in [27] and [13, Problem 20] about the realizability of their so-called double permutation sequences and, more generally, allowable interval sequences by families of pairwise disjoint convex bodies of real two-dimensional affine topological planes. Indeed double permutation sequences and allowable interval sequences are simply a coding of isomorphism classes of our fibrations. For example the allowable interval sequences coding the three isomorphism classes of fibrations of an arrangement of two double pseudolines indexed by $\{1, 2\}$ are the following

$$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \end{bmatrix}. \quad (5)$$

We refer to [13, 27] for the precise definition of double permutation sequences and that of allowable interval sequences. For recent applications of these notions see [44, 43].

We conclude with the statements of the counterparts of Theorems 3 and 2 in the context of arrangements in Möbius strips.

Theorem 50 *Let $\mathcal{M}$ be a Möbius strip. Any two arrangements of pseudocircles in $\mathcal{M}$ of the same order are homotopic in $\mathcal{M}$ via a finite sequence of mutations followed by an isotopy.*

An arrangement of oriented pseudocircles is termed *acyclic* if the orientations of the pseudocircles are coherent, in the sense that the pseudocircles are oriented according to the choice of a generator of the (infinite cyclic) fundamental group of the underlying Möbius strip.

Theorem 51 *The map that assigns to an isomorphism class of Möbius arrangements of pseudocircles its chirotope is one-to-one and its range is the set of maps χ defined on the set of 3-subsets of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of 3-subsets of J is a chirotope of Möbius arrangements of pseudocircles. Furthermore the same result holds for the class of acyclic Möbius arrangements of pseudocircles.*

Note that our homotopy theorem for Möbius arrangements provides an algorithm to enumerate the isomorphism classes of Möbius arrangements by traversing the associated mutation graphs. We have implemented this algorithm to enumerate the isomorphism classes of the simple Möbius arrangements of double pseudolines. Preliminary

counting results (confirmed by two independent implementations) are reported in the following table:

n	1	2	3	4
a_n	1	1	118	541820
b_n	1	1	22	22620
c_n	1	1	16	11502
d_n	1	1	12	5955

Here the index n refers to the number of double pseudolines, a_n is the number of isotopy classes of simple indexed arrangements of double pseudolines, b_n is the number of isotopy classes of simple arrangements of double pseudolines, c_n is the number of isomorphism classes of simple arrangements of double pseudolines, and d_n is the number of isomorphism classes of simple arrangements of double pseudolines considered as projective arrangements. Figure 58 depicts representatives of the 22 isomorphism classes of nonindexed arrangements of three double pseudolines: Each diagram is labeled at its bottom left with a symbol to name it (of type M_α where α is the 2-sequence of its numbers of 2-cells of size 2 and 3 possibly followed, between brackets, with the size of the unbounded 2-cell of the arrangement in the case where there are several arrangements with the same 2-sequence; M_α and M_α^* are mirror images of one another) and is labeled at its bottom right with the size of its automorphism group; thus the number (118) of simple chirotopes of families of three pairwise disjoint convex bodies on a given indexing set of size 3 can be computed as the sum

$$\sum_{k \geq 1} \frac{3!}{k} g_k = \frac{6}{1} \times 18 + \frac{6}{2} \times 2 + \frac{6}{3} \times 2,$$

where g_k is the number of arrangements of Fig. 58 with group of automorphisms of order k . Finally we mention that there are 531 (simple and non simple) chirotopes on a given indexing set of size 3; cf. [20].

It is interesting to mention that the canonical embedding of Lemma 6 can be extended, in the case of Möbius arrangements, to the whole class of simple and non simple arrangements as explained below.

Define a *pencil* of double pseudolines as a Möbius arrangement of double pseudolines with the property that any of its subarrangements has only two *external* vertices, i.e., only two vertices in the boundary of the two-cell that contains the point at infinity of the one-point compactification of the underlying Möbius strip. Figure 59 shows pencils of two, three, four and five double pseudolines; it is not hard to see that the isotopy class of a pencil of double pseudolines depends only on its number of double pseudolines. Now a Möbius arrangement of double pseudolines is termed *thin* if any of its subarrangements whose double pseudolines have associated Möbius strips with nonempty intersection is a pencil of double pseudolines—to put it differently, a Möbius arrangement of double pseudolines is thin if the crosscap sides of its double pseudolines are free of external vertices—and a Möbius arrangement of double pseudolines Γ^* is termed a *double* of a Möbius pseudoline arrangement Γ if there exists a one-to-one correspondence between Γ and Γ^* such that

Fig. 58 Representatives of the 22 isomorphism classes of simple nonindexed arrangements on three double pseudolines

Fig. 59 Pencils of 2, 3, 4, and 5 double pseudolines

- (1) any pseudoline of Γ is contained in the crosscap side of its corresponding double pseudoline in Γ^* , and
- (2) any subarrangement of Γ^* is a pencil of double pseudolines if and only if the corresponding subarrangement of Γ is a pencil of pseudolines.

In this Möbius setting Lemma 6 can be read as follows.

Lemma 52 *Let $\mathcal{M}$ be a Möbius strip. The map that assigns to an arrangement of pseudolines in $\mathcal{M}$ its set of double versions induces a one-to-one and onto correspondence between the set of isotopy classes of pseudoline arrangements in $\mathcal{M}$ and the set of isotopy classes of thin double pseudoline arrangements in $\mathcal{M}$. $\square$*

7 Conclusion and Open Problems

We have introduced the notion of arrangements of double pseudolines as a combinatorial abstraction of families of pairwise disjoint convex bodies of projective planes and we have extended to that setting well-known fundamental properties of arrangements of pseudolines.

Several open questions are raised by our work. We mention five of them below.

The cell poset of a simple arrangement of pseudolines presented by its chirotope is computable in optimal quadratic time and linear working space; cf. [18]. Can we achieve similar bounds for arrangements of double pseudolines presented by their chirotopes? Progress in this direction using the notion of pseudotriangulation is reported in our companion paper [32].

There is a closed formula, due to R. Stanley, counting the number r_n of wiring representations of simple Möbius arrangements of n pseudolines, namely

$$r_n = \frac{\binom{n}{2}!}{(2n-3)(2n-5)^2(2n-7)^3 \dots 5^{n-3}3^{n-2}}.$$

Most of the proofs of this formula, if not all, are based on connections between standard Young Tableaux, reduced words and arrangements of pseudolines; cf. [17, 19, 34, 41, 51]. Are there similar connections for Möbius arrangements of double pseudolines? Is there any similar formula counting the number of wiring representations of simple Möbius arrangements of double pseudolines?

Say that an arrangement of n double pseudolines is realizable if it is the dual of a family of n disjoint disks of the standard two-dimensional projective plane $\mathbb{RP}^2$.

In that case one can think of the arrangement as the trace on the unit sphere of $\mathbb{R}^3$ of a centrally symmetric affine arrangement of $2n$ planes with the property that the distance to the origin of any line defined as the intersection of d of these planes is less than 1. The open question is then the following: Is any arrangement of double pseudolines the trace on the unit sphere of a centrally symmetric affine arrangement of pseudoplanes in $\mathbb{R}^3$?

Arrangements of double pseudolines are dual families of families of pairwise disjoint convex bodies of projective planes. What is the smallest example which is not realizable as the dual of a family of pairwise disjoint disks of the standard projective plane $\mathbb{RP}^2$? (By a disk we mean a convex body whose boundary is a circle, i.e., the intersection of the unit sphere of $\mathbb{R}^3$ with an affine plane.)

Arrangements of double pseudolines generalize arrangements of pseudolines. What are the similar generalizations for arrangements of pseudohyperplanes of dimensions 4, 5, etc.? A generalization that comes naturally to mind defines (1) a double pseudohyperplane as the image of the hypersurface $x_1 = \pm 1/\sqrt{2}$ of the projective space $\mathbb{RP}^d$ (defined as the quotient of the unit sphere of $\mathbb{R}^{d+1}$ under the antipodal map) under a self-homeomorphism of $\mathbb{RP}^d$, and (2) an arrangement of double pseudohyperplanes as a finite family of double pseudohyperplanes with the property that its subfamilies of size d are the images of the arrangement composed of the d hypersurfaces $x_i = \pm 1/\sqrt{2d}$, $i = 1, 2, \dots, d$, under a self-homeomorphism of $\mathbb{RP}^d$. Two questions arise naturally: (1) Does the isomorphism class of a (indexed and oriented) double pseudohyperplane arrangement depend only on its chirotope, i.e., the family of isomorphism classes of its subarrangements of size $d + 1$? (2) Does the class of chirotopes of double pseudohyperplane arrangements coincide with the class of maps that assigns to each $(d + 1)$ -subset of indices an isomorphism class of arrangements of double pseudohyperplanes indexed by that $(d + 1)$ -subset and whose restrictions to the sets of $(d + 1)$ -subsets of $(d + 3)$ -subsets of indices are chirotopes of double pseudohyperplane arrangements?

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Appendix A. Arrangements of Pseudolines

We review the basics of arrangements of pseudolines that fall within the general scope of the paper.

A.1. LR Characterization

An *arrangement of pseudolines* is a finite set of pseudolines living in the same cross surface with the property that any two pseudolines intersect in exactly one point. Two

Fig. 60 Representatives of the isomorphism classes of arrangements of one, two, three, four and five pseudolines. Each diagram is labeled at its *bottom right* by the size of its automorphism group and at its *bottom left* by a symbol to name it

arrangements of pseudolines are *isomorphic* if one is the image of the other by an homeomorphism of their underlying cross surfaces. Figure 60 depicts representatives of the isomorphism classes of arrangements of at most five pseudolines. The question of understanding the isomorphism relation between arrangements of pseudolines was addressed and solved by Ringel [46, 47] for simple arrangements and by Folkman and Lawrence [21] for any arrangements—moreover, in the broader context of arrangements of pseudohyperplanes—essentially as indicated in the following theorem where the term *chirotope* applied to an indexed arrangement of oriented pseudolines means the map that assigns to each 3-subset of indices of the arrangement the isomorphism class of the subarrangement indexed by this 3-subset.

Theorem 53 ([46, 47, 21]) *The map that assigns to an isomorphism class of indexed arrangements of oriented pseudolines its chirotope is one-to-one and its range is the set of maps χ defined on the set of 3-subsets of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of 3-subsets of J is a chirotope of indexed arrangements of oriented pseudolines.* $\square$

A.2. Chirotopes of Small Size

The above theorem can be complemented by a comprehensive description of the indexed and oriented arrangements on 3, 4 and 5 pseudolines as we now

Fig. 61 An arrangement on 5 pseudolines, its first barycentric subdivision, and one of its indexed and oriented versions

explain. We use the idea of signed indices of an indexing set, namely the original indices $i_1, i_2, \dots, i_n$ and their complements $\bar{i}_1, \dots, \bar{i}_n$. The original indices are said to be positive, their complements are said to be negative, and we define the complement $\bar{\eta}$ of a negative index η as its positive version. Let X be an arrangement of pseudolines, let X_* be an indexed and oriented version of X and extend X_* to the complements of the original indices by assigning to a negative index the reoriented version of the pseudoline assigned to its complement. Let G be the group of permutations of the signed indices which are compatible with the complement operation and let G_X be the group of automorphisms of X . Clearly the map that assigns to $\sigma \in G_X$ its conjugate $X_*^{-1}\sigma X_* \in G$ under X_* is a monomorphism of G_X into G . Thus we can see G_X as a subgroup G_{X_*} of G and the number of distinct indexed (on a given set of indices) and oriented versions of X is the index $[G : G_{X_*}]$ of G_{X_*} in G . In the sequel we use the notation $X(\sigma)$ for the arrangement $X_*\sigma$, $\sigma \in G$; hence $X(1) = X_*$, where 1 is the unit of G .

Example 18 Figure 61 depicts an arrangement H on 5 pseudolines, its first barycentric subdivision, and one of its indexed and oriented versions H_* on the indexing set $\{1, 2, 3, 4, 5\}$. The group G_H is D_4 generated, for example, by the automorphism σ_{12} that exchanges the flags numbered 1 and 2 in the figure and the automorphism σ_{18} that exchanges the flags numbered 1 and 8 in the figure. Thus the number of distinct indexed (on a given set of indices) and oriented versions of H is $5!2^5/8 = 480$. The group G_{H_*} is generated by the permutations $\overline{15423}$ and $\overline{13245}$ which correspond to the automorphisms σ_{12} and σ_{18} , respectively.

Figure 62 depicts one indexed and oriented version of each of the isomorphism classes of arrangements of three and four pseudolines; each diagram is labeled at its bottom right by its number of distinct reindexings (on a given set of indices) and reorientations.

Thus the arrangement A has 2 distinct indexed and oriented versions. The group G_{A_*} is S_4 generated, for example, by the permutations $\overline{132}$, $\overline{123}$ and $\overline{231}$ and its 2 cosets are

Fig. 62 Isomorphism classes of arrangements of three and four oriented pseudolines indexed by 1, 2, 3, 4

$$G_{A_*} = \left\{ \begin{array}{ccc} 123 & 231 & 312 \\ \overline{123} & \overline{231} & \overline{312} \\ \overline{123} & \overline{231} & \overline{312} \\ \overline{123} & \overline{231} & \overline{312} \\ 21\overline{3} & 32\overline{1} & 13\overline{2} \\ 3\overline{2}1 & 1\overline{3}2 & 2\overline{1}3 \\ \overline{1}32 & \overline{2}13 & \overline{3}21 \\ \overline{2}13 & \overline{3}21 & \overline{1}32 \end{array} \right\}, \quad (213)G_{A_*} = \left\{ \begin{array}{ccc} 213 & 321 & 132 \\ \overline{2}13 & \overline{3}21 & \overline{1}32 \\ \overline{2}13 & \overline{3}21 & \overline{1}32 \\ \overline{2}13 & \overline{3}21 & \overline{1}32 \\ 12\overline{3} & 23\overline{1} & 31\overline{2} \\ 3\overline{1}2 & 1\overline{2}3 & 2\overline{3}1 \\ \overline{2}31 & \overline{3}12 & \overline{1}23 \\ \overline{1}23 & \overline{2}31 & \overline{3}12 \end{array} \right\}.$$

Similarly the number of distinct indexed and oriented versions of the arrangement B is 4. The group G_{B_*} is $S_3 \times \mathbb{Z}_2$, generated for example by the permutations 231 , 213 , $\overline{123}$ and its 4 cosets are

$$G_{B_*} = \left\{ \begin{array}{ccc} 123 & 231 & 312 \\ 213 & 321 & 132 \\ \overline{123} & \overline{231} & \overline{312} \\ \overline{213} & \overline{321} & \overline{132} \end{array} \right\}, \quad (\overline{123})G_{B_*} = \left\{ \begin{array}{ccc} \overline{123} & \overline{231} & \overline{312} \\ \overline{213} & \overline{321} & \overline{132} \\ \overline{123} & \overline{231} & \overline{312} \\ \overline{213} & \overline{321} & \overline{132} \end{array} \right\},$$

$$(1\overline{2}3)G_{B_*} = \left\{ \begin{array}{ccc} \overline{1}23 & \overline{2}31 & \overline{3}12 \\ \overline{2}13 & \overline{3}21 & \overline{1}32 \\ \overline{123} & \overline{231} & \overline{312} \\ \overline{213} & \overline{321} & \overline{132} \end{array} \right\}, \quad (12\overline{3})G_{B_*} = \left\{ \begin{array}{ccc} \overline{123} & \overline{231} & \overline{312} \\ \overline{213} & \overline{321} & \overline{132} \\ \overline{123} & \overline{231} & \overline{312} \\ \overline{213} & \overline{321} & \overline{132} \end{array} \right\}.$$

Figure 63 depicts these 2 + 4 distinct indexed and oriented versions of A and B .

Fig. 63 The possible entries of a chirotope of arrangement of pseudolines (of genus 1)

Using these notations one can describe the set of chirotopes on the indexing set $\{1, 2, 3, 4\}$ as the set of

$$\chi = \{\chi(123), \chi(124), \chi(134), \chi(234)\}$$

such that, up to a signed permutation of the indices,

(A₁): if $A(213), A(314), A(412) \in \chi$ then $A(234) \in \chi$ or $A(243) \in \chi$ or $B(234) \in \chi$;

(A₂): if $B(12\bar{3}), B(12\bar{4}), B(13\bar{4}) \in \chi$ then $B(2\bar{3}4) \in \chi$.

To describe the set of chirotopes on 5 indices we use the natural coding of an indexed arrangement of oriented pseudolines by its *side cycles*: there is exactly one side cycle per pseudoline γ of the arrangement and the latter is defined as the circular sequence of signed indices obtained by writing down the indices of the pseudolines encountered when walking along the side of the pseudoline γ , each index being signed positively or negatively depending on whether the encountered pseudoline is (locally) oriented away from or towards the pseudoline γ . For example the side cycles of the arrangement $H(12345)$ of Fig. 61 are

$$\begin{aligned} 1 : & \quad 2345\bar{3}2\bar{5}4, \\ 2 : & \quad 1345\bar{3}145, \\ 3 : & \quad \bar{1}245\bar{2}14\bar{5}, \\ 4 : & \quad 32\bar{5}1\bar{3}215, \\ 5 : & \quad 32\bar{1}4\bar{3}2\bar{4}1. \end{aligned}$$

Similarly the three cycles of $A(123)$ are $23\bar{2}\bar{3}$, $31\bar{3}\bar{1}$ and $12\bar{1}\bar{2}$ and those of $B(123)$ are $\bar{2}2\bar{3}\bar{3}$, $\bar{3}3\bar{1}\bar{1}$ and $\bar{1}1\bar{2}\bar{2}$. The set of chirotopes on a set of 5 indices, say $\{1, 2, 3, 4, 5\}$, can then be described as the set of

$$\begin{aligned}\chi &= \left\{ \chi(J) : J \in \binom{\{1, 2, 3, 4, 5\}}{3} \right\} \\ &= \{ \chi(123), \chi(124), \chi(125), \chi(134), \chi(135), \\ &\quad \chi(145), \chi(234), \chi(235), \chi(245), \chi(345) \}\end{aligned}$$

such that the restrictions of χ to the sets of 3-subsets of the 5 subsets of 4 indices of $\{1, 2, 3, 4, 5\}$ are chirotopes on 4 indices, i.e., satisfy the axioms (A_1) and (A_2) mentioned above, which satisfy a single additional axiom (A_3) saying that for any index i the 4 cycles indexed by i of these 5 chirotopes on 4 indices are mergeable.

A.3. Enlargement Theorem

The *enlargement theorem for pseudoline arrangements*, due to Goodman et al. [28], proving a conjecture of Grünbaum [29, Conjecture 4.10, p. 90], is the following.

Theorem 54 ([28,38]) *Any arrangement of pseudolines is an arrangement of lines of a projective plane.*

Combining the enlargement theorem for pseudoline arrangements with the duality principle for projective planes we get the following theorem.

Theorem 55 *Any arrangement of pseudolines is isomorphic to the dual arrangement of a finite set of points of a projective plane.*

Proof Indeed any pseudoline arrangement $\mathcal{A}$ is isomorphic to the dual of the point set $\mathcal{A}$ of the dual projective plane of any projective plane extension of $\mathcal{A}$ —here we implicitly use the fact that a projective plane is isomorphic to its bidual. $\square$

Appendix B. Chirotopes of Finite Planar Families of Points

We now review the “classical” characterization of chirotopes of finite planar indexed families of oriented points mentioned in the abstract. (An oriented point in a projective plane is a point together with an orientation of its neighborhood, indicated in our drawings by an oriented circle surrounding the point.) Our account takes advantage of the relatively recent positive answer of Goodman et al. [28] to the question of Grünbaum [29, Conjecture 4.10, p. 90] about the embeddability of any arrangement of pseudolines in the line space of a projective plane.

Let Δ be a finite indexed family of oriented points of a projective plane $(\mathcal{P}, \mathcal{L})$, and let τ be a line of $(\mathcal{P}, \mathcal{L})$. We define

- (1) the *cocycle of Δ at τ* or the *cocycle of τ* with respect to Δ or the *cocycle of the pair (Δ, τ)* as the homeomorphism class of the pair (Δ, τ) , i.e., the set of $(\varphi\Delta, \varphi\tau)$ as φ ranges over the set of homeomorphisms of surfaces with domain

Fig. 64 The cocycles, up to reindexing and reorientation, of configurations of three points with indexing set $\{1, 2, 3\}$

- $\mathcal{P}$; in other words two pairs (Δ, τ) and (Δ', τ') define the same cocycle if there exists a homeomorphism φ of $\mathcal{P}$ onto $\mathcal{P}'$ such that $\Delta' = \varphi\Delta$ and $\tau' = \varphi\tau$;
- (2) the *cocycle-map* as the map that assigns to each cell σ of the dual arrangement of Δ the cocycle of Δ at an element (hence any) of σ ;
  - (3) a 0-, 1-, 2-cocycle of Δ as a cocycle of Δ at a 0-, 1-, 2-cell of its dual arrangement;
  - (4) the *isomorphism class* of Δ as the set of configurations Δ' that have the same set of 0-cocycles as Δ (hence, using a simple perturbation argument, the same set of cocycles as Δ); and
  - (5) the *chirotope* of Δ as the map that assigns to each 3-subset of its indexing set the isomorphism class of the subconfiguration indexed by this 3-subset.

Figure 64 depicts the cocycles of configurations of three points: each circular diagram is labeled at its bottom right with its number of reindexings and reorientations and at its bottom left with its *signature* $\nu(\Delta, \tau)$ which is defined as follows. Let $\mathcal{D}_\tau$ be the closed 2-cell obtained by cutting the cross surface $\mathcal{P}$ along the line τ , let $\nu_\tau : \mathcal{D}_\tau \rightarrow \mathcal{P}$ be the canonical projection, and let Σ_τ be the pre-image of the set of Δ_i s under ν_τ ; each element of Σ_τ is endowed with the orientation of the corresponding point of Δ and is labeled with the signed index of the corresponding signed point of Δ . Choose an orientation ε of τ , orient $\mathcal{D}_\tau$ accordingly, and define the *signature* of the triple $(\Delta, \tau, \varepsilon)$ as the set of labels of the elements of Σ whose corresponding signed points are contained in the interior of $\mathcal{D}_\tau$ and oriented consistently with the orientation of $\mathcal{D}_\tau$ plus the circular sequence of labels of the elements of Σ_τ oriented consistently with the orientation of $\mathcal{D}_\tau$ encountered when walking along the boundary of $\mathcal{D}_\tau$ according to its orientation. The *signature of the pair* (Δ, τ) or the *signature of Δ at τ* is then defined as the unordered pair of signatures of the triples $(\Delta, \tau, \varepsilon)$ and $(\Delta, \tau, -\varepsilon)$; it can be represented by either of its two elements since the signature of the triple $(\Delta, \tau, -\varepsilon)$ is obtained from the signature of the triple $(\Delta, \tau, \varepsilon)$ by replacing each of its elements with the reversal of its complement. Clearly the cocycle of a pair (Δ, τ) depends only on its signature and vice versa.

A simple case analysis shows that the map that assigns to the isomorphism class of an indexed configuration of oriented points its chirotope is well-defined and one-to-one, that there are exactly six isomorphism classes of indexed configurations of three oriented points on the indexing set $\{1, 2, 3\}$, namely, in signature terms,

$$\begin{aligned}
&\{123\overline{123}\}, \\
&\{132\overline{132}\}, \\
&\{2132\overline{13}\}, \\
&\{132\overline{132}\}, \\
&\{12\overline{12}, 3\}, \{23\overline{23}, 1\}, \{31\overline{31}, 2\}, \\
&\{12\overline{12}, \overline{3}\}, \{23\overline{23}, \overline{1}\}, \{31\overline{31}, \overline{2}\},
\end{aligned}$$

and, finally, that the map that assigns to an indexed configuration of three oriented points the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of three oriented points and that the induced (one-to-one and onto) quotient map is the following:

$$\begin{aligned}
\{123\overline{123}\} &\longrightarrow B(\overline{123}), \\
\{132\overline{132}\} &\longrightarrow B(12\overline{3}), \\
\{2132\overline{13}\} &\longrightarrow B(\overline{123}), \\
\{123\overline{123}\} &\longrightarrow B(123), \\
\{12\overline{12}, 3\}, \{23\overline{23}, 1\}, \{31\overline{31}, 2\} &\longrightarrow A(123), \\
\{12\overline{12}, \overline{3}\}, \{23\overline{23}, \overline{1}\}, \{31\overline{31}, \overline{2}\} &\longrightarrow A(\overline{123}).
\end{aligned}$$

Since the map that assigns to an isomorphism class of indexed arrangement of oriented pseudolines its chirotope is one-to-one and since any arrangement of pseudolines is isomorphic to the dual arrangement of a family of points it follows that the above considerations concerning indexed configurations of three oriented points and indexed arrangements of three oriented pseudolines extend to configurations of any number of points and arrangements of any number of pseudolines. We summarize:

Theorem 56 *The map that assigns to an indexed configuration of oriented points the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of oriented points; furthermore the induced quotient map is one-to-one and onto.*

Therefore there are also six isomorphism classes of cocycle-maps on the indexing set $\{1, 2, 3\}$; they are depicted in Fig. 65.

The well-informed reader will have recognized here a reformulation, taking advantage of the embeddability of any arrangement of pseudolines in the line space of a projective plane, of the existence of an adjoint—or Type II representation—for every oriented matroid of rank

Combining Theorems 55, 56 and 53 we get the characterization of chirotopes of planar families of points mentioned in the abstract.

Theorem 57 *Let χ be a map on the set of 3-subsets of a finite set I . Then χ is a chirotope of finite planar families of points if and only if for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of 3-subsets of J is a chirotope of finite planar families of points.*

Fig. 65 The cocycle maps for families of three points indexed by 1, 2 and 3

Appendix C. Basics of Convexity in Projective Planes

In this section we establish the basics of convexity in projective planes that we have taken for granted in the paper, namely

Theorem 1 *A convex body of a projective plane is a closed topological disk, its polar is a convex body of the dual projective plane, and its dual is the boundary of its polar (hence a double pseudoline). Furthermore, up to homeomorphism, the dual arrangement of a pair of disjoint convex bodies of a projective plane is the unique arrangement of two double pseudolines that intersect in four transversal intersection points and induce a cellular decomposition of their underlying cross surface.*

We proceed in two steps, establishing the basics first for neutral (hence affine) planes, which we shall briefly recall, and second for projective planes by reduction to the first step.

C.1. Background Material on Neutral and Affine Planes

A *neutral plane* is a topological point-line incidence geometry $(\mathcal{P}, \mathcal{L})$ whose point space $\mathcal{P}$ is homeomorphic to $\mathbb{R}^2$, whose line space $\mathcal{L}$ is a subspace of the space of

pseudolines of the point space,³ and whose axiom system is reduced to the following single axiom: any two distinct points belong to exactly one line, called their *joining line*, which depends continuously on the two points.

The *join map*, denoted $\vee$, assigns to any ordered pair of distinct points of $\mathcal{P}$ their joining line in $\mathcal{L}$; the *intersection map*, denoted $\wedge$, assigns to any ordered pair of distinct intersecting lines of $\mathcal{L}$ their common intersection point in $\mathcal{P}$. The join and intersection maps are continuous and open.

Theorem 58 ([6, p. 220]) *Let $\mathcal{L}$ be the line space of a neutral plane and let $\tilde{\mathcal{L}}$ be the space of oriented versions of the lines of $\mathcal{L}$. Then $\mathcal{L}$ is an open crosscap, the natural projection $\tilde{\mathcal{L}} \rightarrow \mathcal{L}$ that assigns to an oriented line its unoriented version is a two-covering map, and the pencil of lines through a point is a pseudoline in $\mathcal{L}$, i.e., a nonseparating simple closed curve embedded in $\mathcal{L}$.*

An *affine plane* is a neutral plane $(\mathcal{P}, \mathcal{L})$ with the property that for every point-line pair (P, L) there exists a unique line K incident to P such that either $K = L$ or K and L are disjoint; the line K is called the *parallel* to L through P , and the lines L and K are said to be *parallel*. The parallelism relation is an equivalence relation, the parallel class of L is denoted $[L]$, and the set of parallel classes is denoted $[\mathcal{L}]$. The *projective completion* of an affine plane $(\mathcal{P}, \mathcal{L})$ is the topological point-line incidence geometry whose line space $\hat{\mathcal{L}}$ and point space $\hat{\mathcal{P}}$ are, respectively,

- (1) the set $\{L \cup \{[L]\} \mid L \in \mathcal{L}\} \cup \{[\mathcal{L}]\}$, endowed with the topology of the one-point compactification $\mathcal{L} \cup \{\infty\}$ of $\mathcal{L}$ via the map that assigns $L \in \mathcal{L}$ to $L \cup \{[L]\} \in \hat{\mathcal{L}}$ and ∞ to $[\mathcal{L}]$; and
- (2) the set $\mathcal{P} \cup [\mathcal{L}]$ endowed with the topology with subbase the $J_P \wedge J_Q$ where P and Q are two points of $\hat{\mathcal{P}}$ and where J_P and J_Q are disjoint open intervals of the pencils of lines through P and Q , respectively.

The *affine parts* of a projective plane $(\mathcal{P}, \mathcal{L})$ are the topological point-line incidence geometries $(\mathcal{P} \setminus L, \mathcal{L} \setminus \{L\})$ where L ranges over $\mathcal{L}$.

Theorem 59 *The projective completion of an affine plane is a projective plane and the affine parts of a projective plane are affine planes with the property that their projective completions are isomorphic to the initial projective plane.*

We refer to the monograph of Salzmann et al. [48, Chap. 3] for supplementary background material on neutral planes, where they are called $\mathbb{R}^2$ -planes.

C.2. Convexity and Duality in Neutral Planes

We work in a neutral plane $(\mathcal{P}, \mathcal{L})$. As in the Euclidean plane, a subset of points is called *convex* if it includes the line segments joining its points. A *convex body* is a

³ The space of pseudolines of $\mathbb{R}^2$ is the quotient of the space of embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ with closed images (i.e., the set of continuous one-to-one maps $\varphi : \mathbb{R} \rightarrow \mathbb{R}^2$, with the property that $\varphi(\mathbb{R})$ is closed in $\mathbb{R}^2$, endowed with the compact-open topology) under the natural action of the group of homeomorphisms of $\mathbb{R}$. As usual we identify a pseudoline φ with its image $\varphi(\mathbb{R})$. Similarly the space of oriented pseudolines of $\mathbb{R}^2$ is the quotient of the space of embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ with closed images under the natural action of the group of direct homeomorphisms of $\mathbb{R}$.

compact convex subset of points with nonempty interior, its *polar* is the set of lines missing its interior, and its *dual* is its set of tangent lines or supporting lines (i.e., the set of lines that intersect the body but not its interior or, equivalently, the set of lines that intersect the convex body and that include the body in one of their two closed sides). A *double pseudoline* of an open crosscap is a double pseudoline of the one-point compactification of the open crosscap with the property that the point at infinity belongs to the disc side of the double pseudoline.

In this section we establish the following transcription of Theorem 1 for neutral planes.

Theorem 60 *A convex body of a neutral plane is a closed topological disk, its polar is a closed topological disk with an interior point deleted, which is closed in the line space and whose intersection with the pencil of lines through any point is a closed line segment, and its dual is the boundary of its polar; hence a double pseudoline of the line space. Furthermore, up to homeomorphism, the dual arrangement of a pair of disjoint convex bodies of a neutral plane is the unique arrangement of two double pseudolines in an open crosscap that intersect in four transversal intersection points and induce a cellular decomposition of the one-point compactification of the open crosscap.*

The proof proceeds by a sequence of auxiliary results.

C.2.1. Boundary of a Convex Body and Tangents

The proofs of the two following lemmas are adapted from [1, Chap. 11.3].

Lemma 61 *The boundary of a convex body is a simple closed curve.*

Proof Let U be a convex body, let A be one of its interior points, and let $L_A \approx \mathbb{S}^1$ be the pencil of oriented lines through A . Consider the application $\varphi : L_A \rightarrow \partial U$ that assigns to $L \in L_A$ the endpoint of the trace of U on L beyond A . Clearly φ is a well-defined one-to-one and onto correspondence whose inverse is continuous. Therefore it is sufficient to show that φ is continuous. Let $L \in L_A$ and let B and C be two points of the interior of U with A contained in the interior of the line segment joining B to C . As illustrated in Fig. 66 the rays with origins A , B and C through $\varphi(L)$ leave U at $\varphi(L)$. Let $L_1, L_2, \dots, L_n, \dots$, be a sequence of oriented lines of L_A converging to L with $L_n \neq L$ for all $n \geq 1$. For n large enough the intersection points of the line L_n with the lines $B \vee \varphi(L)$ and $C \vee \varphi(L)$ are well-defined and converge to $\varphi(L)$. Consequently, for n large enough, the line L_n intersects the rays through $\varphi(L)$ with origins B and C . One of these two intersection points is beyond $\varphi(L)$ and the other one is before $\varphi(L)$. Therefore $\varphi(L_n)$ belongs to the line segment joining these two points. When n goes to infinity, this line segment retracts onto $\varphi(L)$. Therefore φ is continuous. $\square$

Lemma 62 *Let U be a convex body, let A be a point not in the interior of U , let X be the set of lines through A intersecting the interior of U , and let Y be the set of lines through A intersecting U but not its interior. Then X is a nonempty open interval whose endpoints belong to Y . Furthermore Y is a pair if and only if $A \notin \partial U$.*

Fig. 66 Illustration for the proof of Lemma 61

Proof The set X is

- (1) nonempty, because the interior of U is nonempty;
- (2) an open subset of the pencil of lines through A , because the join map is open;
- (3) connected, because the interior of U is connected.

Consequently, X is a nonempty open interval of the pencil of lines through A or X is the pencil of lines through A .

We now show that X is not the pencil of lines through A .

For $L \in X$, let L^+ and L^- be the two connected components of $L \setminus \{A\}$ with the convention that L^+ is the one that intersects the interior of U and, consequently, L^- is the one that misses U . We set $R^+ = \bigcup_{L \in X} L^+$ and $R^- = \bigcup_{L \in X} L^-$. Let $\mathcal{S}$ be the set of closed line segments contained in the interior of U whose supporting line is not incident to A . For $I \in \mathcal{S}$, let X_I be the set of lines of X intersecting the interior of I and let $Q^+(I) = \bigcup_{L \in X_I} L^+$ and $Q^-(I) = \bigcup_{L \in X_I} L^-$. We leave the verification of the following properties to the reader

- (1) $\mathcal{S}$ is nonempty;
- (2) $X = \bigcup_{I \in \mathcal{S}} X_I$;
- (3) $Q^+(I)$ and $Q^-(I)$ are open quadrants;
- (4) $R^+ = \bigcup_{I \in \mathcal{S}} Q^+(I)$ is open and nonempty;
- (5) $R^- = \bigcup_{I \in \mathcal{S}} Q^-(I)$ is open and nonempty.

The sets R^+ and R^- are disjoint nonempty open subsets of the point space minus A . Since this last set is connected, there exists a point E neither in R^+ nor in R^- . The line joining A and E misses the interior of U . Consequently X is not the pencil of lines through A .

Finally the endpoints of X belongs to Y since U is compact. The furthermore part follows easily. $\square$

C.2.2. Duality

The dual of a convex body U is denoted U^* .

Lemma 63 *Let U be a convex body. Then*

- (1) U^* is a simple closed curve in $\mathcal{L}$;

- (2) U^* is a double pseudoline in $\mathcal{L}$;
- (3) the set of lines intersecting the interior of U is the open crosscap bounded by U^* ;
- (4) the set of lines missing U is the one-punctured topological disk bounded by U^* .

Proof We endow the plane with an orientation. Let Δ be the map that assigns to A not in U the tangent to U through A with the property that walking along the tangent from A to U we see the convex body U on our right, let $I = [A, B]$ be a closed line segment missing U with the property that $\Delta(A) \neq \Delta(B)$, and let Γ be a simple closed curve surrounding U . We leave the verification of the following properties to the reader

- (1) Δ is continuous and onto;
- (2) the restriction of Δ to the domain Γ is onto;
- (3) the restriction of Δ to the domain I and codomain $\Delta(I)$ is a homeomorphism;
- (4) Δ is open;

from which it follows that U^* is compact and locally homeomorphic to $\mathbb{R}$, hence a simple closed curve.

We now prove claims (2), (3), and (4). Let A be an interior point of U and let $\mathcal{L} \cup \{\infty\}$ be the one-point compactification of $\mathcal{L}$. Since two pseudolines intersect in at least one point and since the pencil of lines through A is a pseudoline that does not intersect U^* , it follows that U^* is a double pseudoline of $\mathcal{L} \cup \{\infty\}$. Let X be the set of lines intersecting the interior of U . The set X is connected, open, closed in $\mathcal{L} \setminus U^*$ and contains pseudolines. Therefore X is the trace on $\mathcal{L}$ of the open crosscap bounded by U^* in $\mathcal{L} \cup \{\infty\}$. It remains to show that the open crosscap bounded by U^* in $\mathcal{L} \cup \{\infty\}$ does not contain ∞ . This follows from [48, Lemma 31.24] which asserts that the set of lines intersecting a compact set of points is compact. $\square$

Lemma 64 *Let U and V be two disjoint convex bodies. Then the double pseudolines U^* and V^* intersect in exactly four points, where they cross.*

Proof We endow the plane with an orientation. Let Δ be the map that assigns to $A \notin U$ the tangent to U through A with the property that walking along the tangent from A to U we see the convex body U on our right, let $I = [A, B]$ be a closed line segment missing U with the property that $\Delta(A) \neq \Delta(B)$, and let Γ be a simple closed curve surrounding U . We leave the verification of the following properties to the reader:

- (1) Δ is continuous and onto;
- (2) the restriction of Δ to the domain Γ is onto;
- (3) the restriction of Δ to the domain I and codomain $\Delta(I)$ is a homeomorphism;
- (4) $\Delta(V)$ is a closed interval $[T, T']$, $T \neq T'$, of U^* ;
- (5) $\Delta(\text{Int}(V))$ is the open interval $]T, T'[$;

from which it follows that T and T' are the sole tangents to both U and V such that walking along the tangents from V to U we see U on our right (and walking along the tangents from U to V we see V on our left or on our right depending on whether we walk on T or on T'), and that U^* and V^* cross at T and T' . Switching the roles of U and V we get a second pair of common tangents to U and V . This proves the lemma. $\square$

Fig. 67 Illustration for the proof of Lemma 65

Lemma 65 *Let U and V be two disjoint convex bodies. Then the double pseudolines U^* , V^* induce a cellular decomposition of the (one-point compactification of) $\mathcal{L}$.*

Proof Let $u \in \text{Int}(U)$ and let $v \in \text{Int}(V)$. We have seen that

- (1) U^* and V^* are double pseudolines intersecting in exactly four points—where they cross;
- (2) u^* and v^* are pseudolines intersecting in exactly one point—where they cross;
- (3) u^* is contained in the open crosscap bounded by U^* ;
- (4) u^* and V^* intersect in exactly two points—where they cross;
- (5) v^* is contained in the open crosscap bounded by V^* ;
- (6) v^* and U^* intersect in exactly two points—where they cross.

Consequently the arrangements $\{u^*, v^*, U^*\}$ and $\{u^*, v^*, V^*\}$ are (up to homeomorphism) those depicted in the two leftmost diagrams of Fig. 67. Now it is not hard to see that the only overlays of these two diagrams fulfilling condition (1) above are the two rightmost diagrams of Fig. 67. It remains to observe that the set of lines intersecting both U and V is connected to rule out the rightmost diagram from our considerations and to conclude that the double pseudolines U^* , V^* induce a cellular decomposition of the one-point compactification of $\mathcal{L}$. $\square$

Observe that this proves that two disjoint convex bodies have a strictly separating line.

C.3. Convexity and Duality in Projective Planes

We now work in a projective plane $(\mathcal{P}, \mathcal{L})$. Recall that a convex body is a closed subset of points with nonempty interior whose intersection with any line is a (necessarily closed) line segment, that its polar is the set of lines that miss it, and that its dual is its set of tangent lines or supporting lines (i.e., the set of lines intersecting the body but not its interior). According to Theorems 59 and 60 proving Theorem 1 boils down to prove that the set of lines missing two disjoint convex bodies is nonempty. The proof proceeds by a sequence of auxiliary results.

Lemma 66 *Assume that two of the three sides of a triangular face of a simple arrangement of three lines are contained in a convex body. Then the triangular face is contained in the convex body.*

Fig. 68 Illustration for the proof of Lemma 66

Proof Let U be a compact subset of points with nonempty interior whose intersection with any line is a line segment or a line, and let T be a triangular face of a simple arrangement of three lines. Let A, B, C be the three vertices of the triangular face T , as illustrated in the left diagram of Fig. 68 where the triangular face is marked with a little square, let $[AB]$, $[BC]$ and $[CA]$ be the three sides of T , and assume that $[AB]$ and $[AC]$ are contained in U . Proving our lemma comes down to proving that U contains a line or that T is contained in U . Let D be a point of the line (BC) outside the line segment $[BC]$, as illustrated in the right diagram of Fig. 68. For any $X \in [AB]$, we denote by X' the intersection of the line (DX) with the line (AC) (note that X' ranges over the line segment $[AC]$ and that $B' = C$), by $[XX']$ the line segment supported by the line (DX) contained into T , and, for $X \neq A$, by $[X'X]$ the line segment of (DX) contained in the complement of the interior of T . Let I be the set of $X \in [AB]$, $X \neq A$, such that $[XX']$ is contained in U , and let J be the set of $X \in [AB]$, $X \neq A$, such that $[X'X]$ is contained in U . One can easily check that

- (1) $I \cap J = \emptyset$ unless U contains a line (DX) with $X \in [AB]$, $X \neq A$;
- (2) $I \cup J = [AB] \setminus \{A\}$;
- (3) I and J are both closed in $[AB] \setminus \{A\}$.

Assume now that $I \cap J = \emptyset$, otherwise U contains a line (DX) with $X \in [AB]$, $X \neq A$, and we are done. Since $[AB] \setminus \{A\}$ is connected, it follows that I or J is empty. In the first case the line (DA) is contained in U since U is compact and in the second case $T = \bigcup_{X \in [AB]} [XX'] \subseteq U$. In both cases we are done. $\square$

Lemma 67 *The trace of a line on the interior of a convex body is empty or is the interior of the trace of the line on the body.*

Proof Let U be a convex body, let $[AB]$ be the trace of a line on U and assume that $[AB]$ intersects the interior of U at point C . Let $[DE]$ be a line segment through C , contained in the interior of U , and not contained in the line (AB) . Clearly the arrangement composed of the six lines joining two of the four points A, B, D, E is, up to homeomorphism, the one shown in the rightmost diagram of Fig. 69. According to Lemma 66 the four triangles ACD , ACE , BCD , and BCE are included in U . The lemma follows. $\square$

Fig. 69 Illustration for the proof of Lemma 67

Lemma 68 Let U be a convex body, let L be a line intersecting U along a nonempty line segment I , let $\mathcal{D}$ be the closed topological disk obtained by cutting $\mathcal{P}$ along L , let $\mathcal{D} \rightarrow \mathcal{P}$ be the induced canonical map, and let $\tilde{U}$, $\tilde{L}$ and $\tilde{I}$ be the pre-images of U , L and I under $\mathcal{D} \rightarrow \mathcal{P}$. Then

- (1) $\tilde{I}$ has two connected components, denoted $\tilde{I}_+$ and $\tilde{I}_-$ thereafter;
- (2) $\tilde{U}$ has two connected components and the traces of these two connected components on $\tilde{L}$ are the two connected components of $\tilde{I}$; we denote by $\tilde{U}_+$ and $\tilde{U}_-$ the connected components of $\tilde{U}$ that contain $\tilde{I}_+$ and $\tilde{I}_-$, respectively, and we set $U_+ = \tilde{U}_+ \setminus \tilde{I}_+$ and $U_- = \tilde{U}_- \setminus \tilde{I}_-$;
- (3) U_+ and U_- are closed convex subsets of the affine part of $(\mathcal{P}, \mathcal{L})$ obtained by removing the line L ;
- (4) U_+ or U_- is nonempty and U_+ and U_- are both nonempty if and only if L intersects the interior of U ;
- (5) the topological closure of U_+ in $\mathcal{D}$ is $U_+ \cup I_+$ under the assumption that U_+ is nonempty, and a similar result holds for U_- .

Proof Claim (1) is clear since the restriction of $\mathcal{D} \rightarrow \mathcal{P}$ to the domain $\tilde{L}$ and codomain L is a two-covering. For all $X, Y \in U$, $X \neq Y$, we denote by $[X, Y]$ the line segment joining X and Y contained in U . Let $A \in I$. For all $B \in U \setminus I$ the pre-image under $\mathcal{D} \rightarrow \mathcal{P}$ of $[A, B]$ has two connected components: a first line segment reduced to a single point A_B^* and a second line segment $([A, B] \setminus \{A\}) \cup A_B$ where $\{A_B^*, A_B\}$ is the pre-image of A under $\mathcal{D} \rightarrow \mathcal{P}$. See Fig. 70 for an illustration. Let V_+ be the set of $B \in U \setminus I$ such that $A_B \in \tilde{I}_+$ and, similarly, let V_- be the set of $B \in U \setminus I$ such that $A_B \in \tilde{I}_-$. Clearly, by definition, $V_+ \cup V_- = U \setminus I$ and $V_+ \cap V_- = \emptyset$. We claim that

- (1) V_+ and V_- are independent of the choice of $A \in I$;
- (2) V_+ and V_- are closed convex subsets of the affine part of $(\mathcal{P}, \mathcal{L})$ obtained by removing the line L ;
- (3) V_+ or V_- is nonempty and V_+ and V_- are both nonempty if and only if I intersects the interior of U ;
- (4) the topological closure of V_+ in $\mathcal{D}$ is $V_+ \cup \tilde{I}_+$ under the assumption that V_+ is nonempty, and a similar result holds for V_- ;

from which it follows that $\tilde{U}$ has two connected components: $V_+ \cup \tilde{I}_+$ and $V_- \cup \tilde{I}_-$. Claims (1), (2) and (3) are simple applications of Lemmas 66 and 67. Assume now

Fig. 70 Illustration for the proof of Lemma 68

Fig. 71 Illustration for the proof of Lemma 69

that V_+ is nonempty. Clearly, the topological closure of V_+ contains $V_+ \cup \tilde{I}_+$ and is contained in $V_+ \cup \tilde{I}_+ \cup \tilde{I}_-$. Thus we have to prove that the topological closure of $V_+ \cup \tilde{I}_+$ avoids $\tilde{I}_-$. Assume the contrary. Then there exists a convergent sequence of points $A_n \in V_+$ with limit $A' \in \tilde{I}_-$. Let $A \in I$, let $J_n = [A, A_n] \setminus \{A\}$ and let L_n be the supporting line of J_n . Without loss of generality one can assume that the sequence L_n has a limit L' . Let $B \in L'$, $B \notin I$. There exists a convergent sequence of points $B_n \in L_n$ with limit B . For n large enough B_n belongs to J_n . Since V_+ is closed it follows that $B \in V_+$ and, consequently, L' is a subset of U . This contradicts the assumption that U is a convex body. The lemma follows with $U_+ = V_+$ and $U_- = V_-$. $\square$

Lemma 69 Assume that there is a line missing the interior of a convex body. Then there is a line missing the body.

Proof Let U be a convex body, let L be a line missing the interior of U , let I be the trace of L on U and assume that I is nonempty (otherwise we are done). Let L_∞ be a line missing the line segment I and let Q and Q' be the two connected components of the complement of the lines L and L_∞ in $\mathcal{P}$, as indicated in the leftmost diagram of Fig. 71. Let R be a neighborhood of the intersection point of L and L_∞ disjoint from U . Let V_n be a decreasing sequence of open neighborhoods of L with $\bigcap_n V_n = L$ and let W_n and W'_n be the traces of V_n on $Q \cup R$ and $Q' \cup R$, respectively. According to

Fig. 72 Illustration for the proof of Lemma 70

the previous lemma there exists an n_0 such that for all $n \geq n_0$ the trace of W_n on U is empty or for all $n \geq n_0$ the trace of W'_n on U is empty. Without loss of generality one can assume that the trace of W'_n on U is empty for $n \geq n_0$. Using standard compactness arguments we see that there is a line L'' of the pencil of lines through the intersection point of L and L_∞ contained in W'_n and we are done. $\square$

Lemma 70 *Assume that there is a line missing the interiors of two disjoint convex bodies. Then there is a line missing the two bodies.*

Proof Let U and U' be two disjoint convex bodies, let L be a line missing the interiors of U and U' , let I and J be the traces of L on U and U' and assume that I and J are nonempty (otherwise we are done, thanks to the previous lemma), as indicated in the leftmost diagram of Fig. 72. Let $\mathcal{D}$ be the closed topological disk obtained by cutting $\mathcal{P}$ along L , let $\mathcal{D} \rightarrow \mathcal{P}$ be the induced canonical map, let I_+ and I_- be the two connected components of the pre-image of I under $\mathcal{D} \rightarrow \mathcal{P}$ with the convention that I_- is also a connected component of the pre-image of U under $\mathcal{D} \rightarrow \mathcal{P}$, and similarly let J_+ and J_- be the two connected components of the pre-image of J under $\mathcal{D} \rightarrow \mathcal{P}$ with the convention that J_- is also a connected component of the pre-image of U under $\mathcal{D} \rightarrow \mathcal{P}$. The lemma follows from the simple observation that there is a line missing I and J whose pre-image under $\mathcal{D} \rightarrow \mathcal{P}$ separates $I_+ \cup J_+$ from $I_- \cup J_-$, as indicated in the right diagram of Fig. 72. $\square$

Lemma 71 *Any boundary point of a convex body is incident to a line missing the interior of the body.*

Proof Let U be a convex body and let A be a boundary point of U . The color of an oriented line L through A is defined to be

- (1) *blue* if the line L intersects the interior of U and if A is the initial point of the trace on the interior of U of the oriented line L ;
- (2) *white* if the line L does not intersect the interior of U ;
- (3) *red* if the line intersects the interior of U and if A is the terminal point of the trace on the interior of U of the oriented line L .

According to Lemma 67 any oriented line through A has a color, and these colors are mutually exclusive. The sets of blue and red oriented lines are open subsets of the pencil of oriented lines through A . Since none of these two sets is empty and since the pencil of oriented lines through A is connected it follows that the set of white oriented lines is nonempty. This proves the lemma. $\square$

Fig. 73 Illustration for the proof of Lemma 73

Lemma 72 *The set of lines missing a convex body is nonempty.*

Proof Simple consequence of Lemmas 69 and 71. $\square$

Lemma 73 *The set of lines missing two disjoint convex bodies is nonempty.*

Proof According to Lemma 70 it is sufficient to prove that the set of lines missing the interiors of two disjoint convex bodies is nonempty. Let U and V be two disjoint convex bodies and let L be a line missing U . If L avoids the interior of V we are done. Otherwise L intersects V along a closed line segment, say $[RS]$, $R \neq S$, and L intersects the interior of V along the interior of $[RS]$.

Let K be a tangent to V at S , let K' be a tangent to V at R and let A be the intersection point of K and K' . If K or K' misses the interior of U we are done. Otherwise we proceed as follows (Fig. 73).

Let G be a line through A that avoids the interior of V , let B be the intersection point of L and G , let $H \neq L$ be a line through B that avoids U but intersects the interior of V , and let W be the intersection of V with the strip delimited by G and H in the affine plane $(\mathcal{P} \setminus L, \mathcal{L} \setminus \{L\})$. Clearly U and W are disjoint convex bodies of the affine plane $(\mathcal{P} \setminus L, \mathcal{L} \setminus \{L\})$: Let D be the intersection point of their interior bitangents. We let the reader check that D belongs to the triangle in $\mathcal{P} \setminus L$ delimited by the lines K , K' and H and that the line through D of the pencil of lines through B avoids the interiors of U and V . $\square$

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