The Combinatorial Structure of Spatial STIT Tessellations

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Abstract Spatially homogeneous random tessellations that are *stable* under *iteration* (nesting) in the 3-dimensional Euclidean space are considered, so-called STIT tessellations. They arise as outcome of a space-time process of subsequent cell division and, consequently, they are not facet-to-facet. The intent of this paper is to develop a detailed analysis of the combinatorial structure of such tessellations and to determine a number of new geometric mean values, for example for the neighbourhood of the typical vertex. The heart of the results is a fine classification of tessellation edges based on the type of their endpoints or on the equality relationship with other types of line segments. In the background of the proofs are delicate distributional properties of spatial STIT tessellations.

Keywords Combinatorial topology · Geometric mean values · Iteration/nesting · Random geometry · Random polytopes · Random tessellations · Tilings · Stochastic geometry

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1 Introduction

A random tessellation (or mosaic) of a d -dimensional Euclidean space is a locally finite family of pairwise non-overlapping d -dimensional random convex polytopes—called the cells of the tessellation—that cover the whole space. While non-random tessellations and tilings are a central object in discrete geometry, random tessellations form one of the classical structures considered in stochastic geometry (see [9, 11] for some aspects of the classical theory and [1, 4, 8] for some more recent developments). The most popular models are hyperplane or Voronoi tessellations, where mainly the Poisson case has been studied. All these tessellations share the property of being side-to-side or facet-to-facet in higher dimensions. In recent years there has been a growing interest also in tessellation models that do not fulfill this property. In [15] a first systematic study of the complications has been presented when a tessellation is not side-to-side or facet-to-facet. Tessellations of this kind arise for example by subsequent cell division, see [2], which makes them interesting for particular applications in geology, material sciences or biology. However, such models have a much more complex combinatorial structure, because the spatial arrangement of their cells can be rather complicated. For example it is no more necessarily the case that the number of facets of a cell coincides with the number of its neighbouring cells. It is therefore interesting to calculate mean values describing the ‘mean’ cell-arrangement and allow some insight to the complex tessellation geometry.

Among the models of subsequent cell divisions, the iteration stable or STIT tessellations were of particular interest because of the number of analytically available results, see [3, 6, 7, 10, 12–14] and the references cited therein. A detailed combinatorial analysis of planar STIT tessellations has been carried out in [3, 6]. The intent of this paper is to study the combinatorial structure of STIT tessellations in $\mathbb{R}^3$ and to continue the work initiated in [7, 13, 14]. The structure of spatial STIT tessellations is very rich and considerably more complex compared with the planar case. This is mainly due to the fact that they do not only have T -shaped vertices as in the planar case, but also vertices of so-called X -type, see Fig. 3. This causes that in a detailed analysis we have to take into account the effect generated by the different types of vertices.

To motivate our approach we discuss now one of our main results. As noticed above, a spatial STIT tessellation has vertices of type T and of type X . We will denote by $\varrho_{E[TT]}$ the probability that the typical edge of the tessellation (see Sect. 2.2 for the concept of a typical object) has both of its endpoints of type T . The exact and numerical value for this probability will be discussed later in Theorem 2. Now, we consider the typical vertex V . It has exactly four neighbouring vertices which may be of type T or X and we denote by $\zeta_{V,V[T]}$ the mean number of neighbouring T -vertices and by $\zeta_{V,V[X]}$ the mean number of neighbouring X -vertices of the typical vertex. It will be interesting to compare these means with those where the typical vertex is replaced by the typical T - or X -vertex, i.e. the typical vertex conditioned on being T or X . The corresponding mean values will be denoted by $\zeta_{V[T],V[T]}$, $\zeta_{V[T],V[X]}$, etc.

Theorem 1 *For a spatial STIT tessellation it holds that*

$$\zeta_{V,V[T]} = \frac{8}{3} \quad \text{and} \quad \zeta_{V,V[X]} = \frac{4}{3}.$$

Moreover, for the typical T - and the typical X -vertex the mean values are

$$\begin{aligned} \zeta_{V[T],V[T]} &= 6\varrho_{E[TT]} \approx 2.65727, & \zeta_{V[T],V[X]} &= 4 - 6\varrho_{E[TT]} \approx 1.34273, \\ \zeta_{V[X],V[T]} &= 8 - 12\varrho_{E[TT]} \approx 2.68546, & \zeta_{V[X],V[X]} &= 12\varrho_{E[TT]} - 4 \approx 1.31454. \end{aligned}$$

Theorem 1 formalises for example that in a spatial STIT tessellation the typical X -vertex favours a slightly higher number of T -neighbours compared with the typical T -vertex or the typical tessellation vertex. Up to our best knowledge this is one of the first results in random tessellation theory dealing with the neighbourhood of the typical vertex.

The paper is organised as follows. We recall the construction of STIT tessellations in Sect. 2. We also introduce there the basic geometric objects determined by a spatial STIT tessellation and recall some basic combinatorial mean values for them to keep the paper self-contained. Our main results are the content of Sect. 3. A discussion of our results is presented in Sect. 4, whereas all proofs are in the final Sect. 5.

2 Preliminaries

2.1 STIT Tessellations

Iteration stable tessellations (called STIT tessellations for short) have been introduced in [5]. We confine ourself here to a less formal local description and refer to that paper for more details. To this end, let $W \subset \mathbb{R}^3$ be a compact convex polytope, and let Λ be a translation invariant measure on the space $\mathcal{H}$ of planes in $\mathbb{R}^3$ with the property that $\Lambda([W]) = 1$, where $[W] = \{H \in \mathcal{H} : H \cap W \neq \emptyset\}$ stands for the set of planes that hit the window W .

We assign now to W a random lifetime which is exponentially distributed with parameter $\Lambda([W]) = 1$. When this time has run out, we choose a plane H according to the probability distribution $\Lambda(\cdot \cap [W])$ and divide W into the two sub-cells $W^\pm = W \cap H^\pm$, where $H^\pm$ are the two half-spaces generated by H . These two random polytopes $W^\pm$ are now independently of each other equipped with exponentially distributed random lifetimes that have parameter $\Lambda([W^\pm])$ and upon expiry of their lifetimes, we choose planes according to $\Lambda(\cdot \cap [W^\pm])/\Lambda([W^\pm])$, respectively, and subdivide W^+ and W^- further. This construction continues independently and recursively until a previously fixed deterministic time threshold $t \in (0, \infty)$ is reached, see Figs. 1 and 2 for illustrations.

The outcome of this space-time construction, which is denoted by $Y(t, W)$, is an almost surely finite collection of random convex polytopes that subdivide the convex polytope W and constitute a tessellation of W . It is one of the main results of [5] that the described construction is spatially consistent. As a consequence, there exists a whole space random and spatially homogeneous (stationary) tessellation $Y(t)$ with

Fig. 1 Illustration of the space-time construction of a STIT tessellation in a cube

the property that $Y(t) \cap W$ has the same distribution as the previously constructed $Y(t, W)$. The random tessellation $Y(t)$ is usually called a *STIT tessellation*.

We remark that although the measure Λ controls the direction of the planes inserted into W , the combinatorial mean values and probabilities and also the relative object intensities calculated below are universal in the sense that they do not depend on Λ . Moreover, they are also independent of the time threshold t . The reason for this being the fact that the dilated tessellation $tY(t)$ has the same distribution as $Y(1)$. Therefore, the parameters Λ and t are suppressed in our notation and we speak from now on about a spatial STIT tessellation and think of it as $Y(t)$ for some fixed $t > 0$ and some fixed measure Λ .

2.2 Object Intensities and Typical Objects

The translation invariance of the plane measure Λ (implying the spatial homogeneity or stationarity of the STIT tessellations) allows us to make sense of several *object intensities* and the notion of *typical objects*. Formally, these are introduced by means of Palm distributions for which we refer to [9, 11]. Here we only give an intuitive and informal idea. Let $\mathbf{X}$ be some class of random geometric objects determined by the spatial STIT tessellations, which are assumed to be convex polytopes of a fixed dimension $i \in \{0, 1, 2, 3\}$. For example $\mathbf{X}$ can be the class of cells or the class of cell-separating polygons that are born in the course of the space-time construction. This allows us to regard $\mathbf{X}$ as a stationary point process in the space $\mathcal{P}_i$ of i -dimensional convex polytopes. For any $p \in \mathcal{P}_i$ let $c(p)$ be the midpoint of the smallest circumscribed ball around p . Then the limit

Fig. 2 A spatial STIT tessellation in a cube

$$\lambda_X = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{x \in X} \mathbf{1}\{c(x) \in [n]\} \quad (1)$$

is well defined, where $[n] = [-n/2, n/2]^3$ stands for the centered cube of volume n^3 , and $\mathbf{1}\{\cdot\}$ for the indicator function which is 1 if the statement in brackets is fulfilled and 0 otherwise, cf. [9, 11]. It will be assumed from now on that $0 < \lambda_X < \infty$, which is the case for all classes considered below. In view of (1), λ_X has an interpretation as the mean number of objects of class $\mathbf{X}$ per unit volume.

We turn now to what can be considered as the typical object of a class $\mathbf{X}$. More precisely, we will define the *distribution* of the typical object and refer to [9, 11] for more details. Besides $\mathcal{P}_i$, $0 \leq i \leq 3$, define $\mathcal{P}_i^o = \{p \in \mathcal{P}_i : c(p) = o\}$, where o denotes the origin. The space $\mathcal{P}_i^o$ can be equipped with the Borel σ -field $\mathcal{B}(\mathcal{P}_i^o)$ generated by the usual Hausdorff distance. We now define a distribution $\mathbb{Q}_X$ on $\mathcal{P}_i^o$ by

$$\mathbb{Q}_X(B) = \frac{1}{\lambda_X} \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{x \in X} \mathbf{1}\{c(x) \in [n]\} \mathbf{1}\{x - c(x) \in B\}, \quad (2)$$

where $B \in \mathcal{B}(\mathcal{P}_i^o)$. A random element in $\mathcal{P}_i^o$ with distribution $\mathbb{Q}_X$ is called the *typical object* of class $\mathbf{X}$. Intuitively it can be considered as a uniformly selected object from $\mathbf{X}$ independent of its size and shape.

Also certain adjacency relationships are of interest for us. To define them, let $\mathbf{Y}$ be another class of objects determined by a spatial STIT tessellation. An object $x \in \mathbf{X}$ is said to be *adjacent* to $y \in \mathbf{Y}$ if either $x \subseteq y$ or $y \subseteq x$. For fixed $x \in \mathbf{X}$, we denote by

$m_Y(x)$ the number of objects from $\mathbf{Y}$ adjacent to x and further define

$$\mu_{X,Y} := \int m_Y(x) \mathbb{Q}_X(dx).$$

We interpret $\mu_{X,Y}$ as the mean number of objects from $\mathbf{Y}$ that are adjacent to the typical object from $\mathbf{X}$. Note that in general $\mu_{X,Y}$ and $\mu_{Y,X}$ are entirely different quantities. For example, if $\mathbf{X}$ is the class $\mathbf{Z}$ of cells and $\mathbf{Y}$ is the class $\mathbf{V}$ of vertices of the tessellation, then $\mu_{X,Y}$ denotes the mean number of vertices of the typical cell ($\mu_{Z,V} = 24$ for spatial STIT tessellations), whereas $\mu_{Y,X}$ is the mean number of cells adjacent to the typical vertex ($\mu_{V,Z} = 4$ for the STIT model in $\mathbb{R}^3$). To enhance the readability of our text we use in our notation for intensities and adjacencies the style X for the indices without distinction between object classes and typical objects.

In some cases we will have to consider subclasses of $\mathbf{X}$, which will be denoted by $\mathbf{X}[\cdot]$, where the content of the brackets $[\cdot]$ will be a suitable suggestive symbol that is introduced in an ad hoc manner and whose meaning should be clear from the context. For example, if $\mathbf{X}$ is the class of vertices, $\mathbf{X}[T]$ could be the collection of vertices of a special type T .

To introduce some further notation we recall that we have assumed that the members of $\mathbf{X}$ are convex polytopes having a certain dimension $i \in \{0, 1, 2, 3\}$. Then $\mathbf{X}_j$ with $0 \leq j < i$ will be the class of objects which are the j -dimensional faces of the polytopes from $\mathbf{X}$ and, as above, X_j stands for the typical object of $\mathbf{X}_j$. For example if $\mathbf{X}$ is the class of tessellation cells (here $i = 3$), then $\mathbf{X}_2$ or $\mathbf{X}_1$ are, respectively, the class of cell facets (2-dimensional faces) and ridges (1-dimensional faces). Furthermore, by ∂X and $\overset{\circ}{X}$ we denote, respectively, the topological boundary and the relative interior of the typical object from class $\mathbf{X}$.

2.3 Basic Combinatorial Structure

The primitive objects of a spatial STIT tessellation are its building blocks, see [15]. By $\mathbf{Z}$ (from the German word ‘Zellen’) we denote the class of cells of the tessellation, which are the primitive objects of dimension 3. For a point $x \in \mathbb{R}^3$ let $D(x)$ be the intersection of all cells containing x , i.e. $D(x) = \bigcap_{z \in \mathbf{Z}, x \in z} z$. We observe that $D(x)$ is a finite intersection of random convex polytopes, whence $D(x)$ itself is a random convex polytope. We can now introduce three other classes of primitive objects, namely

$$\mathbf{V} := \{D(x) : \dim D(x) = 0, x \in \mathbb{R}^3\}, \quad \mathbf{E} := \{D(x) : \dim D(x) = 1, x \in \mathbb{R}^3\},$$

the class of *vertices* and *edges*, respectively, and

$$\mathbf{P} := \{D(x) : \dim D(x) = 2, x \in \mathbb{R}^3\},$$

the class of *plates*. Although a primitive object does not contain lower-dimensional primitive elements in its relative interior, the boundary structure of plates or cells can be rather complicated. For example a plate can be a triangle but with additional vertices

Table 1 Relative object intensities

X	E	P	Z	Z_1	Z_2	P_1	I_1	I
$\frac{\lambda_X}{\lambda_V}$	2	$\frac{7}{6}$	$\frac{1}{6}$	2	1	$\frac{14}{3}$	$\frac{2}{3}$	$\frac{1}{6}$

Table 2 Adjacency mean values for primitive objects

$\mu_{X,Y}$	$X = V$	$X = E$	$X = P$	$X = Z$
$Y = V$	1	2	$\frac{36}{7}$	24
$Y = E$	4	1	$\frac{36}{7}$	36
$Y = P$	6	3	1	14
$Y = Z$	4	3	2	1

Table 3 Adjacency mean values for faces of primitive objects, I -segments and I -polygons

$\mu_{X,Y}$	$X = P_1$	$X = Z_1$	$X = Z_2$	$X = I_1$	$X = I$
$Y = V$	$\frac{16}{7}$	3	$\frac{26}{3}$	4	18
$Y = E$	$\frac{9}{7}$	2	10	3	24
$Y = P$	–	–	$\frac{7}{3}$	–	7

on its boundary that are no corners of it. The same also holds for the cells, whereby a cell can have additional vertices and edges nested into its facets.

We now introduce the lower-dimensional faces of primitive objects. Let us start with the faces of cells. Using the notation given in Sect. 2.2 the class of cell *facets* (2-dimensional faces) is denoted by Z_2 and the class of *ridges* (1-dimensional cell faces) is Z_1 . Sometimes we will also refer to the elements from Z_1 as Z_1 -*segments* for simplicity. Note, moreover, that up to multiple counting the class Z_0 is the same as V . We can also consider the class P_1 of 1-dimensional faces of tessellation plates, which are also referred to as P_1 -*segments* or *plate sides*. Finally, we notice that P_0 and E_0 are the same as V up to multiplicities.

The last class of objects we introduce in this paper is the class I of I -polygons. In the language of the space-time construction of STIT tessellations, the I -polygons are the cell separating facets that are introduced/born during the cell dividing process. By I_1 we denote the class of I -segments, which are the 1-dimensional faces of I -polygons. I -polygons are also characterised by saying that they are the maximal unions of connected and coplanar plates. In the same spirit, I -segments are the maximal unions of connected and collinear tessellation edges.

The relative object intensities for the introduced objects for a spatial STIT tessellation are summarised in Table 1. Some basic adjacency mean values are collected in Tables 2 and 3 (items marked by “–” are currently unknown). The values have been taken from the papers [13] and [15].

3 Fine Combinatorial Structure

3.1 Vertex Geometry

We have introduced in Sect. 2.3 the class V of vertices. In sharp contrast to the planar case, which has only vertices of type T (these are vertices in which three edges meet

Fig. 3 Two T -vertices (left) and a vertex of type X (right)

and exactly two of them are collinear), a spatial STIT tessellation has two types of vertices, see Fig. 3. In the course of the space-time construction from Sect. 2.1 a T -vertex (X -vertex) arises if two I -polygons intersect in the *same* (*different*) half-space(s) specified by a given I -polygon that has been born earlier in time.

This way, we split $\mathbf{V}$ into the two subclasses $\mathbf{V} = \mathbf{V}[T] \cup \mathbf{V}[X]$ of T - and X -vertices and denote by

$$q_{V[T]} := \mathbb{P}(V \in \mathbf{V}[T]) \quad \text{and} \quad q_{V[X]} := \mathbb{P}(V \in \mathbf{V}[X])$$

the probability that the typical vertex is of type T or X , respectively. Denoting by $\lambda_{V[T]}$ and $\lambda_{V[X]}$ the intensity of T - and X -vertices we have

$$q_{V[T]} = \frac{\lambda_{V[T]}}{\lambda_V} = \frac{2}{3} \quad \text{and} \quad q_{V[X]} = \frac{\lambda_{V[X]}}{\lambda_V} = \frac{1}{3} \quad (3)$$

in view of the results in [13, 15]. This may also be rephrased by saying that, on average, in a spatial STIT tessellation we have twice as many vertices of T -type compared with the X -vertices.

3.2 Some Distributional Results for the Typical I -Segment with Internal Vertices

Our refined analysis of the combinatorial structure of spatial STIT tessellations rests on distributional properties of the typical I -segment, which have been investigated in [14] and will be further developed in this section.

We consider the subclass $\mathbf{I}_1[\text{int}V]$ of $\mathbf{I}_1$ containing all I -segments with internal vertices and we call the typical object of $\mathbf{I}_1[\text{int}V]$ *the typical I -segment with internal vertices*. The breakdown of the class of vertices from Sect. 3.1 implies that also the vertices in the relative interior of the typical I -segment (if there are any) can either be of type T or X . Thus, we can have two different types of vertices in the relative interior of an I -segment and we calculate the probabilities that a randomly chosen vertex in the relative interior of the typical I -segment with internal vertices is of one of these types.

Proposition 1 *Consider a spatial STIT tessellation. A randomly chosen vertex in the relative interior of the typical I -segment with internal vertices is of T -type with*

probability p_T , where

$$p_T = \frac{4(21 + 56 \ln 2 - 54 \ln 3)}{68 + 208 \ln 2 - 189 \ln 3} \approx 0.433053$$

and of type X with probability p_X , where

$$p_X = \frac{27 \ln 3 - 16 \ln 2 - 16}{68 + 208 \ln 2 - 189 \ln 3} \approx 0.566947.$$

Remark 1 At first sight the probabilities from the previous proposition seem to be rather surprising, because for the mean number of internal vertices of type T or X in the typical I -segment we know from [14] that

$$\mu_{I_1, V[T]}^\circ = \mu_{I_1, V[X]}^\circ = 1.$$

However, the condition ‘having internal vertices’ has a significant impact on the values of p_T and p_X .

Proposition 1 deals with the typical I -segment that has *at least* one internal vertex. Interestingly, if we condition the typical I -segment on having *exactly* $n \geq 1$ internal vertices, the probability for a randomly chosen vertex being of T - or X -type changes with n . This is a special feature of spatial STIT tessellations and is in sharp contrast to the much simpler planar situation considered in [3]. We denote these probabilities by $p_{T|n}$ and $p_{X|n}$, respectively, $n \geq 1$. To calculate $p_{T|n}$ and $p_{X|n}$ we recall at first that the probability $p_n = \mathbb{P}(v = n)$ ($n \in \mathbb{N}$) that the typical I -segment has exactly n vertices in its relative interior equals

$$p_n = 3 \int_0^1 \int_0^1 (1-a)^3 \frac{(3 - (1-a)(3-b))^n}{(3 - (1-a)(2-b))^{n+1}} db da, \quad (4)$$

a formula that has been established in [14]. In the present paper we will always denote by v the number of vertices in the relative interior of the typical I -segment and also introduce the decomposition $v = v_T + v_X$, which takes into account the two different types of vertices. Let us introduce the abbreviation $p_{m,j} = \mathbb{P}(v_T = m, v_X = j)$ for the probability that the typical I -segment has exactly m vertices of type T and j vertices of type X in its relative interior (a precise formula will follow in Sect. 5.1, see (6) there). Using p_n and $p_{m,j}$ we can formally introduce

$$p_{T|n} = \frac{1}{p_n} \sum_{k=0}^n \frac{k}{n} p_{k,n-k} \quad \text{and} \quad p_{X|n} = 1 - p_{T|n}. \quad (5)$$

For example $p_{T|1} \approx 0.3854$, $p_{T|2} \approx 0.4114$ or $p_{T|20} \approx 0.6058$. This indicates that in I -segments with a few number of internal vertices it is much more likely to choose a vertex of type X compared with I -segments having a large number of vertices in their

relative interior. We will see in the next section that the numbers $p_{T|n}$ and p_n will play an important role in our fine combinatorial analysis.

It is of great importance that the probability for choosing a vertex of T - or X -type on the typical I -segment with an fixed number of internal vertices depends on that number of internal vertices. Another crucial observation is that—by construction—the types of two neighbouring internal vertices are independent and identically distributed. Thus, looking along a typical I -segment, the vertex type is determined by independent coin tosses with probabilities $p_{T|n}$ and $p_{X|n} = 1 - p_{T|n}$.

3.3 Classification of Edges

We now refine the analysis initiated in Sect. 2.3 by dividing the class **E** of edges into subclasses, where we explore several possibilities. The first one is based on a classification according to the type of endpoints, whereas the other two classifications concentrate on the equality relationship with other geometric objects. It is worth pointing out that in [3] a similar study has been carried out for the planar STIT tessellation, where one only has one vertex type. The situation with two different types of vertices is much more tricky, produces a lot of new effects and allows us to consider several different classifications.

Classification According to Endpoints The first way is to classify an edge by the type of its two endpoints. Recall that a spatial STIT tessellation has T - and X -type vertices and so an edge can have two T -vertices or two X -vertices or one T - and one X -vertex as its endpoints. This way, the class **E** of edges can be split into the three subclasses **E**[TT], **E**[XX] and **E**[TX], whose elements are referred to as TT -, XX - and TX -edges, respectively (like T - and X -vertices as elements of **V**[T] and **V**[X]). We are interested in the probabilities that the typical edge belongs to one of these classes. More formally, we define

$$Q_{E[\cdot]} := \mathbb{P}(E \in \mathbf{E}[\cdot]) = \frac{\lambda_{E[\cdot]}}{\lambda_E},$$

where **E**[$\cdot$] stands for one of the subclasses **E**[TT], **E**[XX] and **E**[TX].

Theorem 2 *For any spatial STIT tessellation we have that*

$$\begin{aligned} Q_{E[TT]} &= \frac{1}{3}p_0 + \frac{2}{3}p_{T|1}p_1 + \frac{1}{3}\sum_{n=2}^{\infty}(2p_{T|n} + (n-1)p_{T|n}^2)p_n \approx 0.442878, \\ Q_{E[XX]} &= Q_{E[TT]} - \frac{1}{3} \approx 0.109545, \\ Q_{E[TX]} &= \frac{4}{3} - 2Q_{E[TT]} \approx 0.447577, \end{aligned}$$

where p_n is given by (4) and $p_{T|n}$ by (5).

Remark 2 We were not able to compute the exact value of $Q_{E[TT]}$, which is mainly due to the fact that the probabilities $p_{T|n}$ and p_n have rather complicated expressions and, moreover, that the squared value of $p_{T|n}$ and even $1/p_n$ enters the formula. A similar remark also applies to the probabilities in Theorems 3 and 4.

Remark 3 To obtain our numerical values we have used Mathematica and have evaluated the first 10,000 terms of the sums by the implemented numerical integration methods. To truncate the sums for considerably smaller n , $n = 100$ say, does not lead to accurate results, because of the very long tails of the distribution $\{p_n\}_{n \in \mathbb{N}}$.

Classification According to Equality with P_1 -Segments Any edge of a STIT tessellation is adjacent to three plates. Our next edge classification is based on the observation that an edge can be *equal* to either one, two or three plate sides, the P_1 -segments, whereas it is not possible, that an edge is not equal to any plate side, see Fig. 4. In this figure we illustrate the different edge types on an I -segment. We start with an I -segment having no internal vertices, then with an I -segment that has one internal vertex of type T or X , respectively, etc. In the classification according to equality with P_1 -segments we divide the class $\mathbf{E}$ into the subclasses $\mathbf{E}[P_1, 1]$, $\mathbf{E}[P_1, 2]$ and $\mathbf{E}[P_1, 3]$ and define

$$\varepsilon_{E[P_1, i]} := \mathbb{P}(E \in \mathbf{E}[P_1, i]), \quad i = 1, 2, 3$$

as the probability that the typical edge belongs to class $\mathbf{E}[P_1, i]$, where i stands for the number of plate sides equal to the edge. Again we can say that $\varepsilon_{E[P_1, i]}$ denotes the proportion of edges of the STIT tessellation which are equal to i plate sides.

Theorem 3 *A spatial STIT tessellation satisfies*

$$\begin{aligned} \varepsilon_{E[P_1, 1]} &= \frac{2}{3}p_1 + \frac{1}{3} \sum_{n=2}^{\infty} (2 + (n-1)p_{T|n}^2/2 + (n-1)p_{X|n}^2)p_n \approx 0.555046, \\ \varepsilon_{E[P_1, 2]} &= \frac{1}{3} \sum_{n=2}^{\infty} ((n-1)p_{T|n}^2/2 + 2(n-1)p_{T|n}p_{X|n})p_n \approx 0.300657, \\ \varepsilon_{E[P_1, 3]} &= \frac{1}{3}p_0 = \frac{68}{8} \ln 3 - \frac{26}{3} \ln 2 - \frac{5}{2} \approx 0.144296 \end{aligned}$$

with p_n and $p_{T|n}$ given, respectively, by (4) and (5).

Classification According to Equality with Ridges Another breakdown of $\mathbf{E}$ can be based on the equality relationship with ridges (Z_1 -segments) instead of P_1 -segments. In this context we observe that an edge is adjacent to two ridges of the spatial STIT tessellation, so it can be equal to either two, one or no ridge, see Fig. 4. We can this way decompose $\mathbf{E}$ into $\mathbf{E} = \mathbf{E}[Z_1, 0] \cup \mathbf{E}[Z_1, 1] \cup \mathbf{E}[Z_1, 2]$ and introduce

$$\varepsilon_{E[Z_1, i]} := \mathbb{P}(E \in \mathbf{E}[Z_1, i]), \quad i = 0, 1, 2,$$

Fig. 4 Illustration of different edge types on an I-segment of a spatial STIT tessellation. *First row (left)* $E[TT]$, $E[P_1, 3]$, $E[Z_1, 2]$; *first row (right)* $E[TT]$, $E[P_1, 2]$, $E[Z_1, 1]$; *second row (left)* $E[TX]$, $E[P_1, 2]$, $E[Z_1, 0]$; *second row (right)* $E[XX]$, $E[P_1, 2]$, $E[Z_1, 0]$; *third row (left)* $E[TX]$, $E[P_1, 1]$, $E[Z_1, 0]$; *third row (right)* $E[TT]$, $E[P_1, 2]$, $E[Z_1, 1]$; *fourth row* $E[TT]$, $E[P_1, 1]$, $E[Z_1, 0]$

where $\mathbf{E}[Z_1, i]$ is the class of tessellation edges that are equal to exactly i ridges ($i = 0, 1, 2$).

Theorem 4 *The values of $\varepsilon_{E[Z_1, i]}$ for $i = 0, 1, 2$ are given by*

$$\begin{aligned}\varepsilon_{E[Z_1, 0]} &= \frac{2}{3}p_1p_{X|1} + \frac{1}{3}\sum_{n=2}^{\infty}(2p_{X|n} + (n-1)p_{X|n}^2 + 2(n-1)p_{T|n}p_{X|n} \\ &\quad + (n-1)p_{T|n}^2/2)p_n \approx 0.624550, \\ \varepsilon_{E[Z_1, 1]} &= \frac{2}{3}p_1p_{T|1} + \frac{1}{3}\sum_{n=2}^{\infty}(2p_{T|n} + (n-1)p_{T|n}^2/2)p_n \approx 0.231154 \\ \varepsilon_{E[Z_1, 2]} &= \frac{1}{3}p_0 = \frac{68}{8}\ln 3 - \frac{26}{3}\ln 2 - \frac{5}{2} \approx 0.144296,\end{aligned}$$

where again p_n and $p_{T|n}$ are as in (4) and (5).

It is important to note that the presented edge classifications are essentially different from each other and that any of the probabilities calculated above carries important information about the structure of a spatial STIT tessellation. We will in the next section use these ε - and ϱ -values to obtain new geometric mean values that arise in a refined analysis of the combinatorial structure of the model under consideration.

Remark 4 It is worth noticing that with the ε -values from Theorems 3 and 4 and simple mean value relations further ε -values can be calculated, for example the probability that the typical plate-side (P_1 -segment) and the typical ridge (Z_1 -segment), respectively, is equal (not adjacent) to one edge, $\varepsilon_{P_1[E, 1]}$ and $\varepsilon_{Z_1[E, 1]}$. We obtain

$$\varepsilon_{P_1[E, 1]} = \frac{\lambda_E}{\lambda_{P_1}}(\varepsilon_{E[P_1, 1]} + 2\varepsilon_{E[P_1, 2]} + 3\varepsilon_{E[P_1, 3]}) \approx 0.681106$$

and

$$\varepsilon_{Z_1[E, 1]} = \frac{\lambda_E}{\lambda_{Z_1}}(\varepsilon_{E[Z_1, 1]} + 2\varepsilon_{E[Z_1, 2]}) \approx 0.519746.$$

Another interesting interpretation of these values for the typical plate side P_1 and the typical ridge Z_1 is the following:

$$\begin{aligned}\varepsilon_{P_1[E, 1]} &= \mathbb{P}(P_1 \text{ has no internal vertices}), \\ \varepsilon_{Z_1[E, 1]} &= \mathbb{P}(Z_1 \text{ has no internal vertices}).\end{aligned}$$

3.4 New Geometric Mean Values

In this section we calculate a number of new adjacency mean values for geometric parameters of a spatial STIT tessellation and continue the studies initiated in [7, 13]. However, we emphasise that these mean values were not available before as they rely on the new probabilities from Theorems 2–4.

Vertex-Edge Adjacencies In this paragraph we are dealing with certain vertex-edge adjacencies. These adjacencies will be important to derive Theorem 1 stated in the introduction. We start with the typical vertex.

Proposition 2 *For the typical vertex we have*

$$\begin{aligned}\mu_{V,E[TT]} &= 4Q_{E[TT]}, \quad \mu_{V,E[TX]} = \frac{16}{3} - 8Q_{E[TX]}, \quad \mu_{V,E[XX]} = 4Q_{E[TT]} - \frac{4}{3}, \\ \mu_{V,E[P_1,i]} &= 4\varepsilon_{E[P_1,i]} \quad \text{and} \quad \mu_{V,E[Z_1,j]} = 4\varepsilon_{E[Z_1,j]}\end{aligned}$$

for $i = 1, 2, 3$ and $j = 0, 1, 2$.

We refine the results of the previous proposition now and present the vertex-edge adjacencies for the typical T -vertex.

Proposition 3 *For the typical T -vertex it holds that*

$$\begin{aligned}\mu_{V[T],E[TT]} &= 6Q_{E[TT]}, \quad \mu_{V[T],E[TX]} = 4 - 6Q_{E[TT]}, \quad \mu_{V[T],E[XX]} = 0, \\ \mu_{V[T],E[P_1,1]} &= 27 \ln 3 - 28 \ln 2 - \frac{19}{2}, \\ \mu_{V[T],E[P_1,2]} &= \frac{27}{2} - 27 \ln 3 + 28 \ln 2 - 6\varepsilon_{E[P_1,3]}, \quad \mu_{V[T],E[P_1,3]} = 6\varepsilon_{E[P_1,3]}, \\ \mu_{V[T],E[Z_1,0]} &= 4 - 6(\varepsilon_{E[Z_1,0]} - \varepsilon_{E[Z_1,2]}) \quad \text{and} \quad \mu_{V[T],E[Z_1,j]} = 6\varepsilon_{E[Z_1,j]}\end{aligned}$$

for $j = 1, 2$.

Finally, we summarise the mean vertex-edge adjacencies for the typical X -vertex.

Proposition 4 *For the typical X -vertex we have*

$$\begin{aligned}\mu_{V[X],E[TT]} &= 0, \quad \mu_{V[X],E[TX]} = 8 - 12Q_{E[TT]}, \quad \mu_{V[X],E[XX]} = 12Q_{E[TT]} - 4, \\ \mu_{V[X],E[P_1,1]} &= 2 \sum_{n=2}^{\infty} (n-1) p_{T|n} p_{X|n} p_n = 4 - \mu_{V[X],E[P_1,2]}, \\ \mu_{V[X],E[P_1,3]} &= 0, \quad \mu_{V[X],E[Z_1,0]} = 4 \quad \text{and} \quad \mu_{V[X],E[Z_1,j]} = 0\end{aligned}$$

for $j = 1, 2$.

The proof of Propositions 2–4 are postponed to Sect. 5.4 at the end of this paper. The numerical values for the vertex-edge adjacencies are summarised in Tables 4 and 5.

Table 4 Numerical values for the vertex-edge adjacencies (part 1)

	TT	XX	TX
$\mu_{V[T],E[\cdot]}$	2.65727	0	1.34273
$\mu_{V[X],E[\cdot]}$	0	1.31454	2.68546
$\mu_{V,E[\cdot]}$	1.77151	0.43817	1.79031

Table 5 Numerical values for the vertex-edge adjacencies (part 2)

	$E[P_1, 1]$	$E[P_1, 2]$	$E[P_1, 3]$	$E[Z_1, 0]$	$E[Z_1, 1]$	$E[Z_1, 2]$
$\mu_{V, \cdot}$	2.22019	1.20263	0.57718	2.49820	0.92461	0.57718
$\mu_{V[T], \cdot}$	0.75441	2.37981	0.86577	1.74730	1.38692	0.86577
$\mu_{V[X], \cdot}$	0.69968	3.30032	0	4	0	0

Edge Adjacencies Any edge is adjacent to exactly 3 plates and to exactly 3 cells of the STIT tessellation. Furthermore, it is located on the boundary of 4 cell facets and in the relative interior of another one. Then any edge belongs to exactly 2 I -polygons. In one it is located on the boundary and in the relative interior of the other one. Of course this holds for the edges of all the considered subclasses $\mathbf{E}[\cdot]$ introduced in Sect. 3.3. Using these adjacencies and the relation $\lambda_X \mu_{X,Y} = \lambda_Y \mu_{Y,X}$, valid for all object classes $\mathbf{X}$ and $\mathbf{Y}$ considered in this paper, we arrive at the following result.

Proposition 5 *For a spatial STIT tessellation we have, for the edge classification according to equality,*

$$\begin{aligned} \mu_{P,E[\cdot]} &= \frac{36}{7} \varepsilon_{E[\cdot]}, \quad \mu_{Z,E[\cdot]} = 36 \varepsilon_{E[\cdot]}, \\ \mu_{Z_2,E[\cdot]} &= 10 \varepsilon_{E[\cdot]}, \quad \mu_{Z_2^\circ,E[\cdot]} = 2 \varepsilon_{E[\cdot]}, \quad \mu_{\partial Z_2,E[\cdot]} = 8 \varepsilon_{E[\cdot]}, \\ \mu_{I,E[\cdot]} &= 24 \varepsilon_{E[\cdot]}, \quad \mu_{I^\circ,E[\cdot]} = 12 \varepsilon_{E[\cdot]}, \quad \mu_{\partial I,E[\cdot]} = 12 \varepsilon_{E[\cdot]}. \end{aligned}$$

Analog results are valid for the edge classification according to the endpoints ($E[TT]$, $E[TX]$, $E[XX]$) with $q_{E[\cdot]}$ instead of $\varepsilon_{E[\cdot]}$.

4 Discussion

The reader might wonder why our refined combinatorial structures are based only on edges and not on a similar classification of plates, which are the 2-dimensional primitive objects. Clearly, a classification through the type of vertices is not very promising, because a plate can have arbitrary many corners or vertices on its boundary. However, a classification according to the equality relationship for plates with facets Z_2 would be possible and would lead to a breakdown of $\mathbf{P}$ into the classes $\mathbf{P}[Z_2, i]$ and to the related probabilities $\varepsilon_{P[Z_2,i]} = \mathbb{P}(P \in \mathbf{P}[Z_2, i])$ for $i = 0, 1, 2$. To calculate these probabilities, one would have to investigate the geometry of I -polygons (instead of I -segments). From [10] it is known that the distribution of the typical I -polygon is a mixture of typical cell distributions of Poisson line tessellations. Thus, in order to calculate $\varepsilon_{P[Z_2,i]}$ one would have to know the area distribution of a typical cell in a Poisson line tessellation. This, however, is a long standing open problem in stochastic geometry. This way a calculation of the probabilities $\varepsilon_{P[Z_2,i]}$ is currently out of reach.

Let us finally remark that for some of the quantities considered in this paper one can derive the whole distribution. However, these distributions appear to be very complicated in general and therefore we have restricted our attention to mean values.

To give a simple example, let us consider the number $v_{E[XX]}$ of XX -edges that are located on the typical I -segment. It holds that

$$\mathbb{P}(v_{E[XX]} = 0) = p_0 + p_1 + \sum_{n=2}^{\infty} (1 - p_{X|n}^2)^{n-1} p_n$$

and that

$$\mathbb{P}(v_{E[XX]} = N) = \sum_{n=N+1}^{\infty} \binom{n-1}{N} p_{X|n}^{2N} (1 - p_{X|n}^2)^{n-1-N}, \quad N \geq 1,$$

with p_n and $p_{X|n} = 1 - p_{T|n}$ given by (4) and (5), respectively. For small N these expressions can be evaluated numerically:

$$\begin{aligned} \mathbb{P}(v_{E[XX]} = 0) &\approx 0.795161, & \mathbb{P}(v_{E[XX]} = 1) &\approx 0.131115, \\ \mathbb{P}(v_{E[XX]} = 2) &\approx 0.046467, & \mathbb{P}(v_{E[XX]} = 3) &\approx 0.016359. \end{aligned}$$

5 Proofs

5.1 More Distributions for the Typical I -Segment

In this section we continue the distributional analysis of the typical I -segment started in Sect. 3.2 and recall some other results from [14]. They prepare the proofs below. First, formula (4) can be refined by considering the joint distribution of the number of T - and X -vertices in the relative interior of the typical I -segment. Recall that $p_{m,j}$ ($m, j \in \mathbb{N}$) is the probability that in its relative interior the typical I -segment has exactly m T -vertices and j X -vertices. Then

$$p_{m,j} = 3 \cdot 2^m \binom{m+j}{m} \int_0^1 \int_0^1 (1-a)^3 a^m \frac{(1-(1-a)(1-b))^j}{(3-(1-a)(2-b))^{m+j+1}} db da \quad (6)$$

from [14].

For later results we need a further classification of the vertices of type T . For that purpose we use that an I -segment s is the intersection of two I -polygons D_1 and D_2 with birth times β_1 and β_2 , respectively. If $\beta_1 < \beta_2$ then s is located in the relative interior of D_1 . Any T -vertex in the interior of s has exactly 4 emanating edges, two of them are collinear and contained in s whereas a third one is located in the interior of D_2 . The fourth edge lies in D_1 and the classification of T -vertices depends on the location of that edge as follows. The plane H containing D_2 creates two half-spaces H^L and H^R , where we assign the labels L and R by an independent toss of a fair coin. The polygon D_1 gets divided by H into the two sub-polygons $D_1^L = D_1 \cap H^L$ and $D_1^R = D_1 \cap H^R$. Thereby, we have at birth time β_2 two mosaic cells with D_2 as their common facet and with another facet contained in D_1^L or D_1^R , respectively. By

construction of STIT tessellations, the numbers of further I -polygons that intersect s and are born after time β_2 until some fixed time threshold $t > \beta_2$ within each of these two cells are identically distributed. Using that property of the STIT tessellations, the assignment of the labels L and R for the half-spaces is coherent. Now, we attribute the label L (for “left”) to those T -vertices in the relative interior of s for which the fourth emanating edge is contained in D_1^L , and the label R (for “right”) otherwise, i.e., if the fourth emanating edge is contained in D_1^R . Next we observe that any T -vertex is located in the relative interior of exactly one I -segment. This way, each vertex of type T receives one of the labels L and R , which leads to a breakdown of the class $\mathbf{V}[T]$ of T -vertices into $\mathbf{V}[T, L]$ and $\mathbf{V}[T, R]$, respectively. We notice that by construction, $\mathbf{V}[T, L]$ and $\mathbf{V}[T, R]$ are stationary point processes in $\mathbb{R}^3$ (which are additionally isotropic if the underlying tessellation is). Moreover, by construction it holds that

$$\mathbb{P}(V \in \mathbf{V}[T, L]) = \mathbb{P}(V \in \mathbf{V}[T, R]) = \frac{1}{2} \mathbb{P}(V \in \mathbf{V}[T]).$$

Similarly, regarding the I -segments of the tessellation, our breakdown $\mathbf{V}[T] = \mathbf{V}[T, L] \cup \mathbf{V}[T, R]$ can also be used to further classify their internal T -vertices. First, let us consider the typical I -segment with exactly $n \geq 1$ internal vertices. We denote by $p_{T,L|n}$ and $p_{T,R|n}$ the probabilities that a uniformly selected internal vertex of it belongs to $\mathbf{V}[T, L]$ or $\mathbf{V}[T, R]$, respectively. Then, with the same arguments as above

$$p_{T,L|n} = p_{T,R|n} = \frac{1}{2} p_{T|n}. \quad (7)$$

This will be utilized frequently in our proofs below. Next, we drop conditioning on the number of internal vertices and let $p_{l,r}^{L,R}$ be the probability that the typical I -segment contains exactly l vertices of type T with label L and r vertices with label R ($l, r \in \mathbb{N}$). It is given by

$$p_{l,r}^{L,R} = 3 \binom{l+r}{l} \int_0^1 (1-a)^3 \frac{a^{l+r}}{(1+a)^{l+r+1}} da. \quad (8)$$

Although (8) is not used below, we indicate a proof in a restricted special case for reference in future works.

Proof of (8) Let us consider the space-time construction of STIT tessellations in the isotropic case, that is the case where the plane measure Λ is the isometry invariant measure on $\mathcal{H}$, see Sect. 2.1. (This is only for simplicity, the anisotropic case can also be considered, but then also the direction of the segment has to be taken into account. However, it appears that (8) is independent of the choice of the direction and hence independent of the plane measure Λ used in the STIT construction, see Sect. 2.1.) In this construction every I -segment receives a birth-time, which is defined as the birth-time of the I -polygon that creates the segment by intersection of another I -polygon that has been born earlier. The birth-time of the typical I -segment

is denoted by β . Moreover, we denote by ℓ the length of the typical I -segment. From [14] we know that the joint density $f_{\ell,\beta}(x, s)$ of (ℓ, β) for the STIT tessellation $Y(t)$ equals

$$f_{\ell,\beta}(x, s) = \frac{3s^3}{2t^3} e^{-\frac{1}{2}sx} \quad x > 0, \quad 0 < s < t.$$

We condition now on the event that $(\ell, \beta) = (x, s) \in (0, \infty) \times (0, t)$ and conclude from the intersection property of STIT tessellations that under this condition the number v_L of T -vertices with label L and the number v_R of R -labelled T -vertices in the relative interior of the typical I -segment are independent and Poisson distributed with parameter $\frac{1}{2}x(t-s)$, which is to say that

$$\mathbb{P}(v_L = l, v_R = r | (\ell, \beta) = (x, s)) = \frac{\left(\frac{1}{2}x(t-s)\right)^{l+r}}{l!r!} e^{-x(t-s)},$$

see for example [14] or the references cited therein. Averaging with respect to the joint density $f_{\ell,\beta}(x, s)$ yields

$$\begin{aligned} p_{l,r}^{LR} &= \int_0^t \int_0^\infty \mathbb{P}(v_L = l, v_R = r | (\ell, \beta) = (x, s)) f_{\ell,\beta}(x, s) \, dx \, ds \\ &= \int_0^t \int_0^\infty \frac{\left(\frac{1}{2}x(t-s)\right)^{l+r}}{l!r!} e^{-x(t-s)} \cdot \frac{3s^3}{2t^3} e^{-\frac{1}{2}sx} \, dx \, ds \\ &= 3 \binom{l+r}{l} \int_0^1 (1-a)^3 \frac{a^{l+r}}{(1+a)^{l+r+1}} \, da. \end{aligned}$$

This proves our claim. $\square$

5.2 Proof of Proposition 1

Recall that by $p_{m,j}$ ($m, j \in \mathbb{N}$) we denote the probability that the typical I -segment contains exactly m internal vertices from $\mathbf{V}[T]$ and j internal vertices from $\mathbf{V}[X]$. Note, moreover, that $1 - p_0 = 1 - \mathbb{P}(v = 0)$ is the probability that the typical I -segment has internal vertices. Thus, given the typical I -segment contains exactly $n \geq 1$ internal vertices, and given that $m \in \{0, \dots, n\}$ of them are of type T , the probability for choosing a T -vertex is m/n . Averaging over all possible numbers m and n we find that

$$p_T = \frac{1}{1 - p_0} \sum_{n=1}^{\infty} \sum_{m=0}^n \frac{m}{n} p_{m,n-m}, \quad (9)$$

where the factor $1/(1 - p_0)$ comes from conditioning on the event that the typical I -segment has internal vertices as explained above. Combining now (9) with (6) we calculate p_T as follows:

$$\begin{aligned} p_T &= \frac{1}{1 - p_0} \sum_{n=1}^{\infty} \sum_{m=0}^n \frac{m}{n} p_{m,n-m} \\ &= \frac{3}{1 - p_0} \int_0^1 \int_0^1 \sum_{n=1}^{\infty} \sum_{m=0}^n \frac{m}{n} \binom{n}{m} (2a)^m (1-a)^3 \frac{(1 - (1-a)(1-b))^{n-m}}{(3 - (1-a)(2-b))^{n+1}} db da \\ &= \frac{3}{1 - p_0} \int_0^1 \int_0^1 \frac{2a(1-a)^2}{1 + 2a + b - ab} db da \\ &= \frac{3}{1 - p_0} \left(\frac{7}{2} + \frac{28}{3} \ln 2 - 9 \ln 3 \right). \end{aligned}$$

The precise value for p_T is obtained by taking into account that $p_0 = \frac{189}{8} \ln 3 - 26 \ln 2 - \frac{15}{2}$. In addition, the relation $p_T + p_X = 1$ implies the value for p_X , which completes our argument. $\square$

5.3 Proofs of Theorems 2–4

Proof of Theorem 2 We start by noting that any edge is located on exactly one I -segment and that the two endpoints of an I -segment are T -vertices. Now we consider on the typical I -segment the different edge-types induced by the different internal vertices. If $v = 0$ (the typical I -segment has no internal vertices) we have exactly one edge of type TT . If $v = 1$, then the typical I -segment comprises two TT -edges, if the internal vertex is of type T – an event having probability $p_{T|1}$. And we have two TX -edges, if it is an X -vertex (with probability $p_{X|1}$). For $v \geq 2$ at first we are interested in the edges, whose endpoints both are internal vertices of the I -segment. For $v = n$ there are $n - 1$ of these edges on the typical I -segment. If we uniformly select one of them, it is a TT -edge with probability $p_{T|n}^2$, thanks to the independence of the vertex type of two neighboring vertices on an I -segment. And in the same way it is an XX -edge with probability $p_{X|n}^2$ and a TX -edge with probability $2p_{T|n}p_{X|n}$. Furthermore, there are two “boundary” edges (where one endpoint is also an endpoint of the I -segment and therefore a T -vertex), they are of type TT with probability $p_{T|n}$, of type TX with probability $p_{X|n}$ and of type XX with probability 0.

For this reason the mean number $\mu_{I, E[TT]}$ of edges of type TT that are located on the typical I -segment of the tessellation can be calculated by

$$\mu_{I, E[TT]} = p_0 + 2p_{T|1}p_1 + \sum_{n=2}^{\infty} (2p_{T|n} + (n-1)p_{T|n}^2)p_n. \quad (10)$$

We observe next that the intensity $\lambda_{E[TT]}$ of TT -edges equals $\lambda_{E[TT]} = \lambda_{I_1} \mu_{I_1, E[TT]}$. Moreover, we have the relation $\lambda_E = 3\lambda_{I_1}$ and thus the probability $\varrho_{E[TT]}$ is given by

$$\varrho_{E[TT]} = \frac{\lambda_{E[TT]}}{\lambda_E} = \frac{1}{3} \mu_{I_1, E[TT]},$$

which in view of (10) proves our first claim.

To calculate the intensities $\lambda_{E[XX]}$ and $\lambda_{E[TX]}$ of XX - and TX -edges we observe that any vertex (regardless of its type) has exactly 4 outgoing edges, which implies the two intensity relationships

$$2\lambda_{E[XX]} + \lambda_{E[TX]} = 4\lambda_{V[X]} \quad \text{and} \quad 2\lambda_{E[TT]} + \lambda_{E[TX]} = 4\lambda_{V[T]}$$

by counting the X - and T -vertices through the different types of edges. Thus,

$$\varrho_{E[TX]} = \frac{\lambda_{E[TX]}}{\lambda_E} = \frac{4\lambda_{V[T]} - 2\lambda_{E[TT]}}{\lambda_E} = 4\frac{\lambda_{V[T]}}{\lambda_E} - 2\varrho_{E[TT]} = \frac{4}{3} - 2\varrho_{E[TT]}$$

and similarly $\varrho_{E[XX]} = \varrho_{E[TT]} - \frac{1}{3}$, which completes the proof. $\square$

Proof of Theorem 3 This can be shown with the help of the same technique as already used in the proof of Theorem 2. We first find for the mean number of edges from $\mathbf{E}[P_1, i]$ that are located in the relative interior of the typical I -segment

$$\varepsilon_{E[P_1, i]} = \frac{\lambda_{E[P_1, i]}}{\lambda_E} = \frac{1}{3} \mu_{I_1, E[P_1, i]}, \quad i = 1, 2, 3.$$

However, an edge belongs to class $\mathbf{E}[P_1, 3]$ if and only if it is an I -segment at the same time. Hence, $\mu_{I_1, E[P_1, 3]} = \mathbb{P}(v = 0) = p_0$, which can be evaluated using (4). In addition, if $v = 1$ both edges on the I -segment belong to $\mathbf{E}[P_1, 2]$.

For $v \geq 2$ the two “boundary” edges are again in $\mathbf{E}[P_1, 2]$. An edge whose both endpoints are internal vertices of the I -segment is an element of subclass $\mathbf{E}[P_1, 2]$ if its endpoints are X and X , L and L or R and R , see Fig. 4. It belongs to $\mathbf{E}[P_1, 1]$ if its endpoints are of type T and X or L and R , respectively. So, using (7), we find

$$\mu_{I_1, E[P_1, 2]} = 2p_1 + \sum_{n=2}^{\infty} (2 + (n-1)p_{T|n}^2/2 + (n-1)p_{X|n}^2)p_n$$

and similarly

$$\mu_{I_1, E[P_1, 1]} = \sum_{n=2}^{\infty} ((n-1)p_{T|n}^2 + 2(n-1)p_{T|n}p_{X|n})p_n,$$

which completes the argument. $\square$

Proof of Theorem 4 Because this follows once again by similar arguments as above we restrict ourself to a rough sketch. We have again

$$\varepsilon_{E[Z_1, j]} = \frac{\lambda_{E[Z_1, j]}}{\lambda_E} = \frac{1}{3} \mu_{I_1, E[Z_1, j]}, \quad j = 0, 1, 2,$$

where $\mu_{I_1, E[Z_1, j]}$ is the mean number of edges from $\mathbf{E}[Z_1, j]$ in the relative interior of the typical I -segment and hence it remains to determine $\mu_{I_1, E[Z_1, i]}$.

As in the proof of Theorem 3, $\mu_{I_1, E[Z_1, 2]} = \mathbb{P}(v = 0) = p_0$ and further

$$\mu_{I_1, E[Z_1, 1]} = 2p_{T|1}p_1 + \sum_{n=2}^{\infty} (2p_{T|n} + (n-1)p_{T|n}^2)p_n$$

and

$$\begin{aligned} \mu_{I_1, E[Z_1, 0]} = 2p_{X|1}p_1 + \sum_{n=2}^{\infty} (2p_{X|n} + (n-1)p_{X|n}^2 + 2(n-1)p_{T|n}p_{X|n} \\ + (n-1)p_{T|n}^2/2)p_n, \end{aligned}$$

which may be seen by taking into account the different possibilities for subsequent internal vertices in the relative interior of the typical I -segment and the two “boundary” edges, see again Fig. 4. $\square$

5.4 Proof of Propositions 2–4

We start with the mean values $\mu_{V, E[TT]}$, $\mu_{V, E[XX]}$ and $\mu_{V, E[TX]}$ and those for the typical T - and X -vertex.

With $\lambda_{V[T]}\mu_{V[T], E[TT]} = 2\lambda_{E[TT]}$ and $\lambda_{V[T]} = \frac{2}{3}\lambda_V = \frac{1}{3}\lambda_E$, see Table 1, we obtain

$$\mu_{V[T], E[TT]} = 6 \frac{\lambda_{E[TT]}}{\lambda_E} = 6q_{E[TT]}$$

for the mean number of TT -edges emanating from the typical T -vertex. For the other edge types it is easy to see, that

$$\mu_{V[T], E[XX]} = 0 \quad \text{and} \quad \mu_{V[T], E[TX]} = 4 - 6q_{E[TT]},$$

because $\mu_{V[T], E} = 4$. Similar considerations for the typical X -vertex yield

$$\mu_{V[X], E[XX]} = 2 \frac{\lambda_{E[XX]}}{\lambda_E} = 12q_{E[XX]} = 12q_{E[TT]} - 4$$

and

$$\mu_{V[X], E[TT]} = 0, \quad \mu_{V[X], E[TX]} = 8 - 12q_{E[TT]}.$$

For the typical vertex we have

$$\lambda_V \mu_{V,E[TT]} = \lambda_{V[T]} \mu_{V[T],E[TT]} \implies \mu_{V,E[TT]} = 4\varrho_{E[TT]}$$

and

$$\lambda_V \mu_{V,E[XX]} = \lambda_{V[X]} \mu_{V[X],E[XX]} \implies \mu_{V,E[XX]} = 4\varrho_{E[TT]} - \frac{4}{3}.$$

Finally,

$$\mu_{V,E[TX]} = \frac{16}{3} - 8\varrho_{E[TT]}.$$

We deal now with the mean adjacencies $\mu_{V,E[P_1,i]}$ ($i = 1, 2, 3$) and $\mu_{V,E[Z_1,j]}$ ($j = 0, 1, 2$) for the typical vertex. With $\lambda_V \mu_{V,E[P_1,i]} = 2\lambda_{E[P_1,i]}$ and a similar relation for the edges from $\mathbf{E}[Z_1, j]$ we have the exact values

$$\mu_{V,E[P_1,i]} = 4\varepsilon_{E[P_1,i]} \quad (i = 1, 2, 3) \quad \text{and} \quad \mu_{V,E[Z_1,j]} = 4\varepsilon_{E[Z_1,j]} \quad (j = 0, 1, 2).$$

The adjacencies between the typical T -vertex or the typical X -vertex, respectively, and the edges from the classes $\mathbf{E}[P_1, i]$ and $\mathbf{E}[Z_1, j]$ are more complicated. All edges in the subclasses $\mathbf{E}[P_1, 3]$, $\mathbf{E}[Z_1, 1]$ and $\mathbf{E}[Z_1, 2]$ are TT -edges. With the mean value relations $\lambda_{V[T]} \mu_{V[T],E[P_1,3]} = 2\lambda_{E[P_1,3]}$ and $\lambda_{E[P_1,3]} = \lambda_E \varepsilon_{E[P_1,3]}$ and the two intensity relations $\lambda_E = 2\lambda_V$ and $\lambda_{V[T]} = \frac{2}{3}\lambda_V$ (see Table 1) we obtain the result for $\mu_{V[T],E[P_1,3]}$ and analogously for $\mu_{V[T],E[Z_1,j]}$, $j = 1, 2$ and of course $\mu_{V[X],E[P_1,3]} = \mu_{V[X],E[Z_1,1]} = \mu_{V[X],E[Z_1,2]} = 0$. From

$$\mu_{V[T],E[Z_1,0]} + \mu_{V[T],E[Z_1,1]} + \mu_{V[T],E[Z_1,2]} = \mu_{V[T],E} = 4$$

we infer

$$\mu_{V[T],E[Z_1,0]} = 4 - 6(\varepsilon_{E[Z_1,1]} + \varepsilon_{E[Z_1,2]})$$

and analogously for the typical X -vertex $\mu_{V[X],E[Z_1,0]} = 4$. The last result is obvious from the X -vertex geometry, because an X -vertex is an internal vertex of four ridges and therefore all emanating edges can not be equal to a ridge, see also Fig. 3. To find the relations for $\mu_{V[\cdot],E[P_1,1]}$ and $\mu_{V[\cdot],E[P_1,2]}$ we need the same technique as the one used in the proof of Theorems 2–4. To establish the relations for the mean number of edges equal to one P_1 -segment adjacent to the typical T - or X -vertex $\mu_{V[\cdot],E[P_1,1]}$ we first consider the edges in subclass $\mathbf{E}[P_1, 1]$. All of them are edges on an I -segment whose both endpoints are internal vertices of the I -segment. Moreover, these endpoints must be of type L and R or of type T (L or R) and X , see Fig. 4. Using the same method as in the previous proofs, in particular relation (7), we obtain

$$\lambda_{V[T]} \mu_{V[T],E[P_1,1]} = \lambda_{I_1} \sum_{n=2}^{\infty} (n-1) (p_{T|n}^2 + p_{T|n} p_{X|n}) p_n.$$

With $\lambda_{I_1} = \frac{2}{3}\lambda_V$, $\lambda_{V[T]} = \frac{2}{3}\lambda_V$ (see Table 1) and $p_{X|n} = 1 - p_{T|n}$ we thus find

$$\mu_{V[T], E[P_1, 1]} = \sum_{n=2}^{\infty} (n-1) p_{T|n} p_n.$$

The sum can be evaluated explicitly by using (4), (5) and (6). This yields

$$\begin{aligned} \mu_{V[T], E[P_1, 1]} &= \sum_{n=2}^{\infty} (n-1) \sum_{k=0}^n \frac{k}{n} 2^k \binom{n}{k} \int_0^1 \int_0^1 a^k (1-a)^3 \frac{1 - (1-a)(a-b)^{n-k}}{(3 - (1-a)(2-b))^{n+1}} db da \\ &= 27 \ln 3 - 28 \ln 2 - \frac{19}{2}. \end{aligned}$$

Similar considerations for the typical X -vertex imply

$$\lambda_{V[X]} \mu_{V[X], E[P_1, 1]} = \lambda_{I_1} \sum_{n=2}^{\infty} (n-1) p_{T|n} p_{X|n} p_n$$

and with $\lambda_{I_1} = \frac{2}{3}\lambda_V$ and $\lambda_{V[X]} = \frac{1}{3}\lambda_V$ from Table 1 this reduces to

$$\mu_{V[X], E[P_1, 1]} = 2 \sum_{n=2}^{\infty} (n-1) p_{T|n} p_{X|n} p_n.$$

The relations for $\mu_{V[\cdot], E[P_1, 2]}$ for both types of typical vertices follow from $\mu_{V[\cdot], E[P_1, 1]} + \mu_{V[\cdot], E[P_1, 2]} + \mu_{V[\cdot], E[P_1, 3]} = 4$. $\square$

5.5 Proof of Theorem 1

By definition we have that

$$\zeta_{V[T], V[\cdot]} = \mu_{V[T], E[T \cdot]}, \quad \zeta_{V[X], V[\cdot]} = \mu_{V[X], E[X \cdot]}$$

and similarly

$$\zeta_{V, V[\cdot]} = \varrho_{V[T]} \mu_{V[T], E[T \cdot]} + \varrho_{V[X]} \mu_{V[X], E[X \cdot]}.$$

The results from Propositions 2–4 complete the proof. $\square$

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