Maximal Equilateral Sets

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Abstract A subset of a normed space X is called equilateral if the distance between any two points is the same. Let $m(X)$ be the smallest possible size of an equilateral subset of X maximal with respect to inclusion. We first observe that Petty's construction of a d -dimensional X of any finite dimension $d \geq 4$ with $m(X) = 4$ can be generalised to give $m(X \oplus_1 \mathbb{R}) = 4$ for any X of dimension at least 2 which has a smooth point on its unit sphere. By a construction involving Hadamard matrices we then show that for any set Γ , $m(\ell_p(\Gamma))$ is finite and bounded above by a function of p , for all $1 \leq p < 2$. Also, for all $p \in [1, \infty)$ and $d \in \mathbb{N}$ there exists $c = c(p, d) > 1$ such that $m(X) \leq d + 1$ for all d -dimensional X with Banach–Mazur distance less than c from ℓ_p^d . Using Brouwer's fixed-point theorem we show that $m(X) \leq d + 1$ for all d -dimensional X with Banach–Mazur distance less than $3/2$ from ℓ_∞^d . A graph-theoretical argument furthermore shows that $m(\ell_\infty^d) = d + 1$. The above results lead us to conjecture that $m(X) \leq 1 + \dim X$ for all finite-dimensional normed spaces X .

Keywords Equilateral set · Equilateral simplex · Equidistant points · Brouwer's fixed point theorem

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1 Introduction

Vector spaces in this paper are over the field $\mathbb{R}$ of real numbers. Write $[d] := \{1, 2, \dots, d\}$ for any $d \in \mathbb{N}$ and $\binom{V}{k} := \{A \subseteq V : |A| = k\}$ for any set V and $k \in \mathbb{N}$. Consider d -dimensional vectors to be functions $\mathbf{x} : [d] \rightarrow \mathbb{R}$ denoted using the superscript notation $\mathbf{x} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(d)})$. Similarly, write $\mathbf{x} = (\mathbf{x}^{(n)})_{n \in \Gamma}$ for any function $\mathbf{x} : \Gamma \rightarrow \mathbb{R}$. Write $\mathbf{o}$ for zero vectors and zero functions. For any $\gamma \in \Gamma$, let $\mathbf{e}_\gamma$ denote the indicator function of $\{\gamma\}$, i.e., $\mathbf{e}_\gamma(\gamma) = 1$ and $\mathbf{e}_\gamma(\delta) = 0$ for all $\delta \in \Gamma \setminus \{\gamma\}$. Given $\mathbf{a} = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(d)}) \in \mathbb{R}^d$ and $\mathbf{b} \in X$ with X any vector space, define the *Kronecker product* $\mathbf{a} \otimes \mathbf{b}$ by $(\mathbf{a}^{(1)}\mathbf{b}, \dots, \mathbf{a}^{(d)}\mathbf{b}) \in X^d$.

Let X denote a real normed vector space with norm $\|\cdot\| = \|\cdot\|_X$. We also use *space* or *normed space* to refer to such spaces. We will use the multiplicative Banach–Mazur distance between two normed spaces X and Y of the same finite dimension, denoted by $d_{\text{BM}}(X, Y)$ and defined to be the infimum of all $c \geq 1$ such that

$$\|\mathbf{x}\|_X \leq \|T\mathbf{x}\|_Y \leq c\|\mathbf{x}\|_X \quad \text{for all } \mathbf{x} \in X$$

for some invertible linear transformation $T : X \rightarrow Y$.

Let Γ be any set. For $p \in [1, \infty)$ let $\ell_p(\Gamma)$ denote the Banach space of all functions $\mathbf{x} : \Gamma \rightarrow \mathbb{R}$ such that $\sum_{n \in \Gamma} |\mathbf{x}^{(n)}|^p < \infty$ with norm $\|\mathbf{x}\|_p = (\sum_{n \in \Gamma} |\mathbf{x}^{(n)}|^p)^{1/p}$. Let $\ell_\infty(\Gamma)$ denote the Banach space of all bounded scalar-valued functions on Γ with norm $\|\mathbf{x}\|_\infty := \sup_{n \in \Gamma} |\mathbf{x}^{(n)}|$. As usual, for any $p \in [1, \infty]$ we write ℓ_p for the sequence spaces $\ell_p(\mathbb{N})$ and ℓ_p^d for $\ell_p([d])$. If X and Y are two normed spaces, their ℓ_p -sum $X \oplus_p Y$ is defined to be the direct sum $X \oplus Y$ with norm $\|(\mathbf{x}, \mathbf{y})\|_p := \|(\|\mathbf{x}\|_X, \|\mathbf{y}\|_Y)\|_p$. Denote the *ball* and *sphere* in X with centre $\mathbf{c} \in X$ and radius $r > 0$ by

$$S(\mathbf{c}, r) = S_X(\mathbf{c}, r) := \{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{c}\| = r\}$$

and

$$B(\mathbf{c}, r) = B_X(\mathbf{c}, r) := \{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{c}\| \leq r\},$$

respectively. See [6] for further background on the geometry of Banach spaces.

Definition 1 A subset $A \subseteq X$ is λ -*equilateral* if $\|\mathbf{x} - \mathbf{y}\| = \lambda$ for all $\{\mathbf{x}, \mathbf{y}\} \in \binom{A}{2}$. A set $A \subseteq X$ is *equilateral* if A is λ -equilateral for some $\lambda > 0$. An equilateral set $A \subseteq X$ is *maximal* if there does not exist an equilateral set $A' \subseteq X$ with $A \subsetneq A'$.

It is clear that a λ -equilateral set is a maximal equilateral set if and only if it does not lie on a sphere of radius λ . Also, A is λ -equilateral if and only if the balls $B(\mathbf{c}, \lambda/2)$, $\mathbf{c} \in A$, are pairwise touching. It follows (as observed by Petty [10] and Soltan [14]) by a result of Danzer and Grünbaum [3] that an equilateral set in a d -dimensional

normed space has cardinality at most 2^d with equality only if the unit ball is an affine cube, that is, only if the space is isometric to ℓ_∞^d . For a survey on equilateral sets, see [16]. See also [17] for recent results on the existence of large equilateral sets in finite-dimensional spaces. This paper will be exclusively concerned with maximal equilateral sets.

Definition 2 Let $m(X)$ denote the minimum cardinality of a maximal equilateral set in the normed space X .

It follows from the above-mentioned result of Danzer and Grünbaum that $m(X) \leq 2^d$ if $\dim X = d$.

We first dispose of the 2-dimensional case. By a simple continuity argument, any set of two points in a normed space of dimension at least 2 can be extended to an equilateral set of size 3. It is also possible to find a maximal equilateral set of size 3 in any 2-dimensional X . In fact, if X is not isometric to ℓ_∞^2 then the result of Danzer and Grünbaum gives that any equilateral set has cardinality < 4 , which implies that any equilateral set of size 3 is already maximal. Also, in ℓ_∞^2 it is easy to find a maximal equilateral set of size 3. It follows that $m(X) = 3$ for all 2-dimensional X .

Now suppose that the dimension of X is at least 3. Using a topological result, Petty [10] showed that any equilateral set of size 3 in X can be extended to one of size 4. He also constructed, for each dimension $d \geq 3$, a d -dimensional normed space with a maximal equilateral set of size 4. Below (Proposition 9) we modify his example to show for instance that ℓ_1^d and ℓ_1 also have this property.

A simple linear algebra argument shows that $m(\ell_2^d) = d + 1$. Brass [2] and Dekster [4] independently showed that if $d_{\text{BM}}(X, \ell_2^d) \leq 1 + 1/(d + 1)$, then $m(X) \geq d + 1$ (see also [16, Theorem 8]). By a theorem of Schütte [12] (as pointed out by Smyth [13]) if $d_{\text{BM}}(X, \ell_2^d) \leq 1 + 1/(d + 1)$ then $m(X) \leq d + 1$. In particular, using $d_{\text{BM}}(\ell_p^d, \ell_2^d) = d^{|1/p - 1/2|}$ (see for instance [6]) it follows that

$$m(\ell_p^d) = d + 1 \quad \text{if} \quad \left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1 + o(1)}{d \ln d}. \quad (1)$$

Even though ℓ_∞^d has an equilateral set of size 2^d , it turns out to have a maximal equilateral set of size $d + 1$. More generally, we show the following:

Theorem 3 If $d_{\text{BM}}(X, \ell_\infty^d) < 3/2$, then $m(X) \leq d + 1$. In addition, $m(\ell_\infty^d) = d + 1$.

Theorem 3 will follow from Proposition 10 in Sect. 3 and Proposition 12 in Sect. 4. A similar result holds for the ℓ_p^d spaces.

Theorem 4 For all $p \geq 1$ and all $d \in \mathbb{N}$, $m(\ell_p^d) \leq d + 1$.

Theorem 5 For each $p \in (1, \infty)$ and $d \geq 3$ there exists $c = c(p, d) > 1$ such that $m(X) \leq d + 1$ for any d -dimensional X with $d_{\text{BM}}(X, \ell_p^d) < c$.

Theorems 4 and 5 will be proved in Sect. 5. Our main result is the following property of ℓ_p where $1 \leq p < 2$. It gives many examples of finite and infinite dimensional

Table 1 Values of $C(p)$ and $d_0(p)$ in Theorem 6

Range of p	$C(p)$	$d_0(p)$	Proof
$1 \leq p < \frac{\log(5/2)}{\log 2} \approx 1.32$	5	4	Proposition 19
$p = \frac{\log(5/2)}{\log 2}$	6	4	Proposition 19
$\frac{\log(5/2)}{\log 2} < p < \frac{\log 3}{\log 2} \approx 1.58$	8	6	Prop. 23 with $(k_1, k_2) = (2, 2)$
$\frac{\log 3}{\log 2} \leq p \leq \frac{\log(13/4)}{\log 2} \approx 1.70$	12	10	Prop. 23 with $(k_1, k_2) = (2, 4)$
$\frac{\log(13/4)}{\log 2} < p < \frac{\log(7/2)}{\log 2} \approx 1.81$	16	14	Prop. 23 with $(k_1, k_2) = (4, 4)$
$\frac{\log(7/2)}{\log 2} \leq p \leq \frac{\log(29/8)}{\log 2} \approx 1.86$	24	22	Prop. 23 with $(k_1, k_2) = (4, 8)$
$\frac{\log(29/8)}{\log 2} < p < \frac{\log(15/4)}{\log 2} \approx 1.907$	32	30	Prop. 23 with $(k_1, k_2) = (8, 8)$
$\frac{\log(15/4)}{\log 2} \leq p \leq \frac{\log(91/24)}{\log 2} \approx 1.923$	40	38	Prop. 23 with $(k_1, k_2) = (8, 12)$
$\frac{\log(91/24)}{\log 2} < p < \frac{\log(23/6)}{\log 2} \approx 1.939$	48	46	Prop. 23 with $(k_1, k_2) = (12, 12)$

spaces with finite maximal equilateral sets. These examples are essentially different from Petty's example alluded to above (which we also generalise in Proposition 9 below).

Theorem 6 *For each $p \in [1, 2)$ there exist $C = C(p) \in \mathbb{N}$ and $d_0 = d_0(p) \in \mathbb{N}$ such that $m(\ell_p^d) \leq C$ for any $d \geq d_0$, and in fact, for any (finite or infinite dimensional) normed space X and any $q \in [1, \infty)$, also $m(\ell_p^d \oplus_q X) \leq C$.*

When $p \rightarrow 2$, $C(p) = O(1/(2-p))$ and $d_0(p) = O(1/(2-p))$. Upper bounds are given in Table 1 for all $p \in [1, \frac{\log 23/6}{\log 2})$.

In particular, we obtain the following surprising corollary.

Corollary 7 *For any set Γ and any $p \in [1, 2)$, $m(\ell_p(\Gamma))$ is bounded above by a constant depending only on p .*

The asymptotic bounds on $C(p)$ and $d_0(p)$ for $p \rightarrow 2$ in the above theorem are close to optimal, as (1) implies that

$$C(p) = \Omega\left(\frac{1}{(2-p) \ln(2-p)^{-1}}\right) \quad \text{and} \quad d_0(p) = \Omega\left(\frac{1}{(2-p) \ln(2-p)^{-1}}\right).$$

Theorem 6 and Corollary 7 will be proved in Sect. 6 below.

We do not know of any d -dimensional space X for which $m(X) > d + 1$. The above theorems give some evidence for the following conjecture:

Conjecture 8 *For any d -dimensional normed space X , $m(X) \leq d + 1$.*

2 A Generalisation of Petty's Example

Petty [10] showed that $m(\ell_2^d \oplus_1 \mathbb{R}) = 4$ for all $d \geq 2$. In his argument ℓ_2^d can in fact be replaced by any, not necessarily finite-dimensional, normed space which has a *smooth point* on its unit sphere, that is, a point where the norm is Gâteaux differentiable, or equivalently, a point on the unit sphere which has only one supporting hyperplane [6, 11]. By a classical theorem of Mazur [9] any separable normed space enjoys this property [11, Theorem 10].

Proposition 9 *Let X be a normed space of dimension at least 2 with a norm which has a smooth point on its unit sphere. Then $m(X \oplus_1 \mathbb{R}) = 4$.*

Proof Since $X \oplus_1 \mathbb{R}$ is at least 3-dimensional, $m(X \oplus_1 \mathbb{R}) \geq 4$ by Petty's theorem mentioned in Sect. 1. For the upper bound, let $\mathbf{u} \in X$ be a smooth point on the unit sphere of X . Let $A := \{(\mathbf{o}, 1), (\mathbf{o}, -1), (\mathbf{u}, 0), (-\mathbf{u}, 0)\}$. Then A is a 2-equilateral set in $X \oplus_1 \mathbb{R}$. If there exists a point $(\mathbf{x}, r) \in X \oplus_1 \mathbb{R}$ at distance 2 to each point in A , then it easily follows that $r = 0$, $\|\mathbf{x}\| = 1$ and $\|\mathbf{x} \pm \mathbf{u}\| = 2$. Then $\pm \mathbf{x}$, $\pm \mathbf{u}$ and $\pm \frac{1}{2}\mathbf{x} \pm \frac{1}{2}\mathbf{u}$ are all unit vectors in X and by convexity the unit ball of the subspace Y of X generated by $\mathbf{u}$ and $\mathbf{x}$ is the parallelogram with vertices $\pm \mathbf{u}$ and $\pm \mathbf{x}$. In particular, the unit ball of Y has more than one supporting line at $\mathbf{u}$, and so by the Hahn–Banach theorem, the unit ball of X has more than one supporting hyperplane at $\mathbf{u}$. $\square$

As special cases, $m(\ell_1) = m(\ell_1^d) = 4$ for $d \geq 3$. However, if Γ is an uncountable set, then it is well known that no point on the unit sphere of $\ell_1(\Gamma)$ is smooth. (This can be seen as follows: Let $\mathbf{u} \in \ell_1(\Gamma)$ have norm 1. Then $\mathbf{u}: \Gamma \rightarrow \mathbb{R}$ has countable support $U \subset \Gamma$, say. Choose any $i \in \Gamma \setminus U$. Then $\|\mathbf{u} \pm \mathbf{e}_i\|_1 = 2$ and the intersection of the unit ball of $\ell_1(\Gamma)$ with the subspace generated by $\mathbf{u}$ and $\mathbf{e}_i$ is the parallelogram with vertices $\pm \mathbf{u}$ and $\pm \mathbf{e}_i$, as in the proof of Proposition 9.) Nevertheless, Theorem 6 gives the upper bound $m(\ell_1(\Gamma)) \leq 5$ for any set Γ .

3 Applying Brouwer's Fixed Point Theorem

Proposition 10 *If $d_{\text{BM}}(X, \ell_\infty^d) < 3/2$, then there exists a maximal equilateral set with $d + 1$ elements. As a consequence, $m(X) \leq d + 1$.*

Proof As preparation for the proof, we first exhibit a 2-equilateral set A of $d + 1$ points in ℓ_∞ such that $S(\mathbf{o}, 1)$ is the unique sphere (of any radius) that passes through A . For $i \in [d + 1]$ and $n \in [d]$, let

$$p_i^{(n)} := \begin{cases} -1 & \text{if } n = i, \\ 0 & \text{if } n > i, \\ 1 & \text{if } n < i, \end{cases}$$

and set $A = \{p_1, \dots, p_{d+1}\}$. Suppose that $A \subset S(\mathbf{x}, r)$ for some $\mathbf{x} \in X$ and $r > 0$. Then for each $n \in [d]$, $|x^{(n)} \pm 1| \leq r$, hence $|x^{(n)}| \leq r - 1$ and $r \geq 1$. If we can show that $r = 1$, we would also get $\mathbf{x} = \mathbf{o}$. Suppose for the sake of contradiction that $r > 1$.

We first show that $\mathbf{x} = (r-1, r-1, \dots, r-1)$. If not, let m be the smallest index such that $\mathbf{x}^{(m)} \neq r-1$. Then for all $n < m$, $|\mathbf{x}^{(n)} - \mathbf{p}_m^{(n)}| = |r-1-1| < r$, and for $n > m$, $|\mathbf{x}^{(n)} - \mathbf{p}_m^{(n)}| = |\mathbf{x}^{(n)}| \leq r-1$. It follows that $r = \|\mathbf{x} - \mathbf{p}_m\|_\infty = |\mathbf{x}^{(m)} + 1|$. Thus $\mathbf{x}^{(m)} = -1 \pm r$, which contradicts $|\mathbf{x}^{(n)}| \leq r-1$ and the choice of m . Therefore, $\mathbf{x} = (r-1, r-1, \dots, r-1)$.

Since $r = \|\mathbf{x} - \mathbf{p}_{d+1}\|_\infty = |r-1-1| < r$, we have obtained a contradiction. Therefore, A lies on a unique sphere. As this sphere has radius 1, A is maximal equilateral. This shows that $m(\ell_\infty^d) \leq d+1$.

We now prove the general result. Let $D := d_{\text{BM}}(X, \ell_\infty^d) < 3/2$, and assume without loss of generality that $X = (\mathbb{R}^d, \|\cdot\|)$ such that

$$\|\mathbf{x}\| \leq \|\mathbf{x}\|_\infty \leq D\|\mathbf{x}\| \quad \text{for all } \mathbf{x} \in \mathbb{R}^d. \quad (2)$$

We will prove that $m(X) \leq d+1$ by finding a perturbation of the above set A that will be maximal equilateral in X . We use Brouwer's fixed-point theorem as in [2] and [17]. Consider the space $\mathbb{R}^{\binom{d+1}{2}}$ of vectors indexed by unordered pairs of elements from $[d+1]$. Write $\mathbf{z}^{\{i,j\}}$ for the coordinate of $\mathbf{z} \in \mathbb{R}^{\binom{d+1}{2}}$ indexed by $\{i, j\}$. Given $\mathbf{z} \in I := [0, 1]^{\binom{d+1}{2}} \subset \mathbb{R}^{\binom{d+1}{2}}$, define $d+1$ points $\mathbf{p}_1(\mathbf{z}), \dots, \mathbf{p}_{d+1}(\mathbf{z}) \in \mathbb{R}^d$ as follows. For $i \in [d+1]$ and $n \in [d]$, let

$$\mathbf{p}_i^{(n)}(\mathbf{z}) := \begin{cases} -1 & \text{if } n = i, \\ 0 & \text{if } n > i, \\ 1 + \mathbf{z}^{\{n,i\}} & \text{if } n < i. \end{cases} \quad (3)$$

Define the mapping $\varphi : I \rightarrow I$ by setting

$$\varphi^{\{i,j\}}(\mathbf{z}) := \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|_\infty - \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\| = 2 + \mathbf{z}^{\{i,j\}} - \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|$$

for each $\{i, j\} \in \binom{[d+1]}{2}$. Then by (2), $\varphi^{\{i,j\}}(\mathbf{z}) \geq 0$ and

$$\begin{aligned} \varphi^{\{i,j\}}(\mathbf{z}) &\leq \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|_\infty - \frac{1}{D} \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|_\infty \\ &= \left(1 - \frac{1}{D}\right)(2 + \mathbf{z}^{\{i,j\}}) < \left(1 - \frac{2}{3}\right)(2+1) = 1. \end{aligned}$$

Thus φ is well-defined. It is clearly continuous, and so has a fixed point $\mathbf{z}_0 \in I$ by Brouwer's theorem:

$$2 + \mathbf{z}_0^{\{i,j\}} - \|\mathbf{p}_i(\mathbf{z}_0) - \mathbf{p}_j(\mathbf{z}_0)\| = \mathbf{z}_0^{\{i,j\}} \quad \text{for all } \{i, j\} \in \binom{[d+1]}{2}$$

Therefore, $\{\mathbf{p}_1(\mathbf{z}_0), \dots, \mathbf{p}_{d+1}(\mathbf{z}_0)\}$ is 2-equilateral in X .

From now on, write $\mathbf{p}_i$ for $\mathbf{p}_i(\mathbf{z}_0)$. Suppose that $\{\mathbf{p}_1, \dots, \mathbf{p}_{d+1}\}$ is not maximal equilateral. Then there exists $\mathbf{x} \in X$ such that $\|\mathbf{x} - \mathbf{p}_i\| = 2$ for each $i \in [d+1]$. We first show that all $|\mathbf{x}^{(n)}| < 2$, then that all $\mathbf{x}^{(n)} \geq 1$, and then obtain a contradiction.

By (2),

$$2 \leq \|x - p_i\|_\infty \leq 2D \quad \text{for each } i \in [d+1].$$

In particular, $|x^{(n)} - p_n^{(n)}| = |x^{(n)} + 1| \leq 2D$, which gives $x^{(n)} \leq 2D - 1 < 2$ for all $n \in [d]$. Also, $|x^{(n)} - p_{n+1}^{(n)}| \leq 2D$, that is, $|x^{(n)} - 1 - z_0^{\{n, n+1\}}| \leq 2D$, which gives

$$x^{(n)} \geq 1 + z_0^{\{n, n+1\}} - 2D > -2.$$

It follows that $|x^{(n)}| < 2$ for all $n \in [d]$.

Next let m be the smallest index such that $x^{(m)} < 1$. For all $n < m$,

$$x^{(n)} - p_m^{(n)} = x^{(n)} - (1 + z_0^{\{n, m\}}) \geq -z_0^{\{n, m\}} \geq -1$$

and

$$x^{(n)} - p_m^{(n)} < 2 - (1 + z_0^{\{n, m\}}) = 1 - z_0^{\{n, m\}} \leq 1,$$

hence $|x^{(n)} - p_m^{(n)}| \leq 1$. For all $n > m$, $|x^{(n)} - p_m^{(n)}| = |x^{(n)}| < 2$. It follows that $2 \leq \|x - p_m\|_\infty = |x^{(m)} + 1|$. However, $x^{(m)} + 1 < 2$ by assumption and $x^{(m)} + 1 > -2 + 1$, so we obtain a contradiction.

It follows that $x^{(n)} \geq 1$ for all $n \in [d]$. Then $|x^{(n)} - p_{d+1}^{(n)}| = |x^{(n)} - 1 - z_0^{\{n, d+1\}}| \leq 1$ since $1 \leq x^{(n)} < 2$ and $0 \leq z_0^{\{n, d+1\}} \leq 1$. It follows that $\|x - p_{d+1}\|_\infty \leq 1$, a contradiction. $\square$

4 Using Graphs

In their studies of neighbourly axis-parallel boxes, Zaks [19] and Alon [1] modelled a certain geometric problem as a problem about covering a complete graph by complete bipartite subgraphs. We use the same technique when showing that an arbitrary equilateral set of at most d points in ℓ_∞^d can be extended to a larger equilateral set. Our proof in fact shows that any collection of at most d pairwise touching, axis-parallel boxes in $\mathbb{R}^d$ can be extended to a pairwise touching collection of $d+1$ axis-parallel boxes.

The graph-theoretical result needed is the following simple lemma which states in particular that if the edges of a complete graph on k vertices are covered by at least k complete bipartite subgraphs G_i , then for each G_i we may choose one of its two parts such that the chosen parts cover all k vertices. For technical reasons we have to allow one, but not more than one, of the classes of the complete bipartite subgraphs to be empty. Thus we define the *join* of $A, B \subseteq [k]$ to be $A \vee B := \{\{a, b\} : a \in A, b \in B\}$ whenever $A \cap B = \emptyset$ and $A \cup B \neq \emptyset$.

Lemma 11 *Let $d \geq k \geq 1$ be integers. For each $n \in [d]$ let $A_n^0, A_n^1 \subseteq [k]$ be given such that $A_n^0 \cap A_n^1 = \emptyset$ and $A_n^0 \cup A_n^1 \neq \emptyset$. Suppose that $\bigcup_{n \in [d]} (A_n^0 \vee A_n^1) = \binom{[k]}{2}$. Then there exist $\sigma_1, \dots, \sigma_d \in \{0, 1\}$ such that $A_1^{\sigma_1} \cup \dots \cup A_d^{\sigma_d} = [k]$.*

Proof We use induction on $k \in \mathbb{N}$. The case $k = 1$ is trivial, so we assume that $k \geq 2$ and that the theorem holds for $k - 1$. If for each $j \in [k]$, some $A_n^0 \vee A_n^1 = \emptyset \vee \{j\}$, take σ_n such that $A_n^{\sigma_n} = \{j\}$ for each of these n . Then choose all remaining σ_n arbitrarily to obtain the required covering of $[k]$.

Thus assume without loss of generality that $\emptyset \vee \{k\}$ does not occur as a $A_n^0 \vee A_n^1$. Without loss of generality, $\{1, k\} \in A_d^0 \vee A_d^1$ (note $k \geq 2$). Thus $k \in A_d^{\sigma_d}$ for some $\sigma_d \in \{0, 1\}$. Set $B_n^0 := A_n^0 \setminus \{k\}$ and $B_n^1 := A_n^1 \setminus \{k\}$ for each $n \in [d - 1]$. Then $\bigcup_{n \in [d-1]} (B_n^0 \vee B_n^1) = [k - 1]$. Since all $A_n^0 \vee A_n^1 \neq \emptyset \vee \{k\}$, we still have $B_n^0 \cup B_n^1 \neq \emptyset$, so we may apply the induction hypothesis to obtain $B_n^{\sigma_n}$, $n \in [d - 1]$, with union $[k - 1]$. Together with $A_d^{\sigma_d}$ we have obtained the required covering of $[k]$. $\square$

Proposition 12 $m(\ell_\infty^d) \geq d + 1$.

Proof We show that any 1-equilateral set $\{\mathbf{p}_1, \dots, \mathbf{p}_k\} \subset \ell_\infty^d$ of size at most $k \leq d$ can be extended. Without loss of generality, $k \geq 1$.

Since $|\mathbf{p}_i^{(n)} - \mathbf{p}_j^{(n)}| \leq 1$ for all $\{i, j\} \in \binom{[k]}{2}$ and $n \in [d]$, we may assume after a suitable translation that all $\mathbf{p}_i \in [0, 1]^d$. For each $n \in [d]$, define $A_n^0 := \{i : \mathbf{p}_i^{(n)} = 0\}$ and $A_n^1 := \{i : \mathbf{p}_i^{(n)} = 1\}$. Again by making a suitable translation we may assume that each $A_n^0 \cup A_n^1 \neq \emptyset$.

Since $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ is 1-equilateral, each $\{i, j\} \in \binom{[k]}{2}$ is in some $A_n^0 \vee A_n^1$, $n \in [d]$. Indeed, since $\|\mathbf{p}_i - \mathbf{p}_j\|_\infty = 1$, there exists an $n \in [d]$ with $|\mathbf{p}_i^{(n)} - \mathbf{p}_j^{(n)}| = 1$. Since $0 \leq \mathbf{p}_i^{(n)}, \mathbf{p}_j^{(n)} \leq 1$, it follows that $\mathbf{p}_i^{(n)}, \mathbf{p}_j^{(n)} = \{0, 1\}$, which gives $\{i, j\} \in A_n^0 \vee A_n^1$.

By Lemma 11 $A_1^{\sigma_1} \cup \dots \cup A_d^{\sigma_d} = [k]$ for some $\sigma_1, \dots, \sigma_d \in \{0, 1\}$. Define $\mathbf{q} = (1, 1, \dots, 1) - (\sigma_1, \dots, \sigma_d)$. We show that for each $i \in [k]$, $\|\mathbf{p}_i - \mathbf{q}\|_\infty = 1$. Since $\mathbf{q} \in [0, 1]^d$, $\|\mathbf{p}_i - \mathbf{q}\|_\infty \leq 1$. There exists $n \in [d]$ such that $i \in A_n^{\sigma_n}$, i.e., $\mathbf{p}_i^{(n)} = \sigma_n$. It follows that $|\mathbf{p}_i^{(n)} - \mathbf{q}^{(n)}| = 1$, which gives $\|\mathbf{p}_i - \mathbf{q}\|_\infty = 1$. $\square$

5 A Calculation

We omit the simple proof of the following lemma.

Lemma 13 For any $p \geq 1$ and $\lambda > 0$ the function $f(x) = |x + \lambda|^p - |x|^p$, $x \in \mathbb{R}$, is increasing, and strictly increasing if $p > 1$.

Proposition 14 For any $p > 1$ and $d \geq 3$, the $2^{1/p}$ -equilateral set of standard unit vectors $S = \{\mathbf{e}_i : i \in [d]\}$ in ℓ_p^d can be extended in exactly two ways to equilateral sets $S \cup \{\mathbf{p}\}$ and $S \cup \{\mathbf{q}\}$. Furthermore, $\|\mathbf{p} - \mathbf{q}\|_p > 2^{1/p}$.

Proof Let $\mathbf{p}$ be equidistant to all points of S , say $\|\mathbf{p} - \mathbf{e}_i\|_p = c$ for all $i \in [d]$ where $c > 0$ is fixed. Then $|\mathbf{p}^{(i)}|^p - |\mathbf{p}^{(i)} - 1|^p = \|\mathbf{p}\|_p^p - c^p$ for all i . By Lemma 13, $\mathbf{p}^{(1)} = \dots = \mathbf{p}^{(d)}$, i.e., $\mathbf{p}$ is a multiple of $\mathbf{j} := (1, 1, \dots, 1) \in \mathbb{R}^d$.

Suppose now $\mathbf{p} = x\mathbf{j}$ satisfies $\|\mathbf{p} - \mathbf{e}_i\|_p = 2^{1/p}$ for all $i \in [d]$. It follows that

$$|x - 1|^p + (d - 1)|x|^p = 2. \quad (4)$$

Consider the function $f(x) = |x - 1|^p + (d - 1)|x|^p$. It is easily checked that f has a unique minimum at a point $x_0 \in (0, 1)$ and is strictly decreasing on $(-\infty, x_0)$ and strictly increasing on (x_0, ∞) . Since $f(-1) = 2^p + d - 1 > 2$, $f(0) = 1$ and $f(1) = d - 1 \geq 2$, Eq. (4) has a unique negative solution $x = -\mu \in (-1, 0)$ and a unique positive solution $x = \lambda \in (0, 1]$. We have to show that $\|-\mu \mathbf{j} - \lambda \mathbf{j}\|_p > 2^{1/p}$, that is, $\lambda + \mu > (2/d)^{1/p}$. Since λ is a solution to (4), it follows that $2 = (1 - \lambda)^p + (d - 1)\lambda^p < 1 + d\lambda^p$, hence $\lambda > (1/d)^{1/p}$. It remains to show that $\mu \geq (2^{1/p} - 1)/d^{1/p}$. Suppose to the contrary that

$$\mu < \frac{2^{1/p} - 1}{d^{1/p}}. \quad (5)$$

Since $x = -\mu$ is a solution of (4),

$$\begin{aligned} 2 &= (1 + \mu)^p + (d - 1)\mu^p \\ &< (1 + \mu)^p - \mu^p + (2^{1/p} - 1)^p \quad \text{by (5),} \end{aligned}$$

hence

$$(2^{1/p} - 1 + 1)^p - (2^{1/p} - 1)^p < (\mu + 1)^p - \mu^p.$$

By Lemma 13, $2^{1/p} - 1 < \mu$, which contradicts (5). $\square$

Proof of Theorem 4 We have already observed above that $m(X) = 3$ for any two-dimensional X , so we may assume that $d \geq 3$. We have also observed that $m(\ell_1^d) \leq 4$ for all $d \geq 3$, so we may assume that $p > 1$. Then the theorem follows from Proposition 14. $\square$

Proposition 15 *Let $1 < p < \infty$, $d \geq 3$, $0 < \varepsilon \leq (2d - 4)^{-1/(p-1)}$, and $R = (1 + \frac{p-1}{2}\varepsilon)^{1/p}$. Suppose that $X = (\mathbb{R}^d, \|\cdot\|)$ is given such that*

$$\|x\| \leq \|x\|_p \leq R\|x\| \quad \text{for all } x \in \mathbb{R}^d.$$

Then X has a λ -equilateral set $\{\mathbf{p}_1, \dots, \mathbf{p}_d\}$, where $\lambda = (2 + (d - 2)\varepsilon^p)^{1/p}$, such that $\mathbf{p}_i^{(i)} = 1$ for all $i \in [d]$, $-\varepsilon < \mathbf{p}_i^{(j)} < 0$ for all $i, j \in [d]$ with $j < i$, and $\mathbf{p}_i^{(j)} = 0$ for all $i, j \in [d]$ with $j > i$.

Proof Let $\beta, \gamma > 0$ be arbitrary (to be fixed later). For $i \in [d]$ define $\mathbf{p}_i: \mathbb{R}^{\binom{d}{2}} \rightarrow \mathbb{R}^d$ by setting for each $n \in [d]$,

$$\mathbf{p}_i^{(n)}(\mathbf{z}) = \begin{cases} z^{\{n, i\}} & \text{if } n < i, \\ -\gamma & \text{if } n = i, \\ 0 & \text{if } n > i. \end{cases}$$

That is,

$$\mathbf{p}_i(\mathbf{z}) = (z^{\{1, i\}}, \dots, z^{\{i-1, i\}}, -\gamma, 0, \dots, 0).$$

Let $I = [0, \beta]^{\binom{[d]}{2}}$ and define $\varphi: I \rightarrow I$ by

$$\varphi^{\{i,j\}}(\mathbf{z}) = 1 + z^{\{i,j\}} - \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\| \quad \text{for each } \{i, j\} \in \binom{[d]}{2}.$$

It is clear that φ is continuous. We next show that φ is well defined if R , β , and γ are chosen appropriately. Let $\mathbf{z} \in I$. Then $0 \leq z^{\{i,j\}} \leq \beta$ for all $\{i, j\} \in \binom{[d]}{2}$. We first bound $\|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|_p$. Without loss of generality, $i < j$. Then

$$\begin{aligned} \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|_p^p &= \sum_{k=1}^{i-1} |z^{\{k,i\}} - z^{\{k,j\}}|^p + |\gamma + z^{\{i,j\}}|^p \\ &\quad + \sum_{k=i+1}^{j-1} |z^{\{k,j\}}|^p + \gamma^p \\ &\leq (i-1)\beta^p + (\gamma + z^{\{i,j\}})^p + (j-1-i)\beta^p + \gamma^p \\ &= (j-2)\beta^p + \gamma^p + (\gamma + z^{\{i,j\}})^p \end{aligned} \quad (6)$$

and

$$\|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|_p^p \geq \gamma^p + (\gamma + z^{\{i,j\}})^p. \quad (7)$$

Thus

$$\varphi^{\{i,j\}}(\mathbf{z}) \geq 1 + z^{\{i,j\}} - ((j-2)\beta^p + \gamma^p + (\gamma + z^{\{i,j\}})^p)^{1/p}.$$

Let $f(x) = 1 + x - ((j-2)\beta^p + \gamma^p + (\gamma + x)^p)^{1/p}$, $0 \leq x \leq \beta$. Then

$$f'(x) = 1 - \left(\frac{(\gamma + x)^p}{(j-2)\beta^p + \gamma^p + (\gamma + x)^p} \right)^{1-1/p} > 0.$$

It follows that f is strictly increasing, which gives that

$$\begin{aligned} \varphi^{\{i,j\}}(\mathbf{z}) &\geq f(z^{\{i,j\}}) \geq f(0) = 1 - ((j-2)\beta^p + 2\gamma^p)^{1/p} \\ &\geq 1 - ((d-2)\beta^p + 2\gamma^p)^{1/p}. \end{aligned}$$

If we require that

$$(d-2)\beta^p + 2\gamma^p = 1 \quad (8)$$

then $\varphi^{\{i,j\}} \geq 0$ for all $\mathbf{z} \in I$. Also,

$$\begin{aligned} \varphi^{\{i,j\}}(\mathbf{z}) &\leq 1 + z^{\{i,j\}} - \frac{1}{R} \|\mathbf{p}_i(\mathbf{z}) - \mathbf{p}_j(\mathbf{z})\|_p \\ &\leq 1 + z^{\{i,j\}} - \frac{1}{R} (\gamma^p + (\gamma + z^{\{i,j\}})^p)^{1/p}. \end{aligned}$$

Let $g(x) = 1 + x - \frac{1}{R}(\gamma^p + (\gamma + x)^p)^{1/p}$, $0 \leq x \leq \beta$. Then

$$g'(x) = 1 - \frac{1}{R} \left(\frac{(\gamma + x)^p}{\gamma^p + (\gamma + x)^p} \right)^{1-1/p} > 1 - \frac{1}{R} > 0.$$

Therefore, g is strictly increasing, which gives that

$$\varphi^{[i,j]}(z) \leq g(z^{[i,j]}) \leq g(\beta) = 1 + \beta - \frac{1}{R}(\gamma^p + (\gamma + \beta)^p)^{1/p}.$$

In order to conclude that $\varphi^{[i,j]}(z) \leq \beta$, it is sufficient to require that

$$\gamma^p + (\gamma + \beta)^p \geq R^p. \quad (9)$$

Suppose for the moment that we can find β , $\gamma > 0$ and $R > 1$ such that (8) and (9) are satisfied. Then φ is well defined, and by Brouwer's fixed point theorem φ has a fixed point, that is, for some $z_0 \in I$, $\varphi(z_0) = z_0$, which implies that $\{p_i(z_0) : i \in [d]\}$ is 1-equilateral. Since $p_i^{(i)}(z_0) = -\gamma$, we have to scale each $p_i(z_0)$ by $-\gamma$. Set $\gamma = 1/\lambda$ and $\beta = \gamma\varepsilon$. Then $\{-(1/\gamma)p_i(z_0) : i \in [d]\}$ is λ -equilateral, the requirement (8) becomes the definition of λ ,

$$(d-2)\varepsilon^p + 2 = \lambda^p$$

and the requirement (9) becomes

$$\frac{1 + (1 + \varepsilon)^p}{2 + (d-2)\varepsilon^p} \geq R^p. \quad (10)$$

It remains to verify (10) given that $\varepsilon \leq (2(d-2))^{-1/(p-1)}$ and $R^p = 1 + \frac{p-1}{2}\varepsilon$. Since

$$(d-2)\varepsilon^p = (d-2)\varepsilon^{p-1}\varepsilon \leq \varepsilon/2$$

and $\varepsilon \leq 2^{-1/(p-1)} < 1$, it is sufficient to show that

$$f(\varepsilon) := 1 + (1 + \varepsilon)^p - \left(1 + \frac{p-1}{2}\varepsilon\right)\left(2 + \frac{\varepsilon}{2}\right) \geq 0$$

for all $\varepsilon \in [0, 1]$. A calculation gives that $f(0) = 0$, $f'(0) = 1/2$, $f''(\varepsilon) = p(p-1)(1+\varepsilon)^{p-2} - (p-1)/2$, $f''(0) = (p-\frac{1}{2})(p-1) > 0$ and $f''(1) = \frac{p-1}{2}(p2^{p-1}-1) > 0$. Thus f'' is monotone and positive at the endpoints of $[0, 1]$, hence positive on the whole $[0, 1]$, and it follows that f' is positive on $[0, 1]$ and $f(\varepsilon) \geq 0$ for all $\varepsilon \in [0, 1]$. $\square$

Proof of Theorem 5 Suppose that the theorem is false. Then for some fixed $p \in (1, \infty)$ and $d \geq 3$ and for all $c > 1$, there exists a d -dimensional X such that $d_{\text{BM}}(X, \ell_p^d) < c$

and $m(X) \geq d + 2$. Choose a sequence $X_n = (\mathbb{R}^d, \|\cdot\|_{(n)})$ such that $m(X_n) \geq d + 2$ and

$$\|x\|_{(n)} \leq \|x\|_p \leq \left(1 + \frac{1}{n}\right)^{1/p} \|x\|_{(n)} \quad \text{for all } x \in \mathbb{R}^d.$$

If n is sufficiently large, in particular if

$$n > \frac{2(2d - 4)^{1/(p-1)}}{p - 1},$$

and if we choose $\varepsilon = \frac{2}{n(p-1)}$, then $\frac{1}{n} = \frac{p-1}{2}\varepsilon$ and $\varepsilon < (d-2)^{-1/(p-1)}$, and we may apply Proposition 15 to obtain an equilateral set $\{p_i(n) : i \in [d]\}$ in X_n such that $p_i^{(i)}(n) = 1$ for all $i \in [d]$ and $-\varepsilon < p_i^{(j)}(n) \leq 0$ for all $i, j \in [d], i \neq j$. Since $m(X_n) \geq d + 2$, there exist points $p_{d+1}(n), p_{d+2}(n) \in X_n$ such that $\{p_i(n) : i \in [d+2]\}$ is equilateral. By passing to a subsequence we may assume without loss of generality that $p_{d+1}(n) \rightarrow p$ and $p_{d+2}(n) \rightarrow q$ as $n \rightarrow \infty$. Since $p_i(n) \rightarrow e_i$ and $d_{\text{BM}}(X_n, \ell_p^d) \rightarrow 1$ as $n \rightarrow \infty$, it follows that $\{e_1, \dots, e_d, p, q\}$ is equilateral in ℓ_p^d . This contradicts Proposition 14. $\square$

6 A Construction with Hadamard Matrices

In [15] Hadamard matrices were used to construct equilateral sets in ℓ_p^d of cardinality greater than $d + 1$, for all $p \in (1, 2)$ and sufficiently large d depending on p . The construction used here is a more involved version of this idea. Before introducing the properties of Hadamard matrices that will be needed, we first consider a special case to illustrate the construction.

Lemma 16 *Let $1 \leq p \leq 2$. For each $\lambda \in [2^{1-1/p}, 2^{1/p}]$ there exist unit vectors $u, v \in \ell_p^2$ such that $\|u + v\|_p = \|u - v\|_p = \lambda$.*

Proof Let $u = (\alpha, \beta)$ and $v = (-\beta, \alpha)$ where $\alpha, \beta \geq 0$ and $\alpha^p + \beta^p = 1$. Then $\|u \pm v\|_p^p = |\alpha + \beta|^p - |\alpha - \beta|^p$, which ranges from 2 when $\alpha = 0$ and $\beta = 1$, to 2^{p-1} when $\alpha = \beta = 2^{-1/p}$. $\square$

Lemma 17 (Monotonicity lemma) *Let u and v be linearly independent unit vectors in a strictly convex 2-dimensional normed space. Let $p \neq o$ be any point such that u is between $\frac{1}{\|p\|}p$ and v on the boundary of the unit ball. Then $\|p - u\| < \|p - v\|$.*

For a proof, see [8, Proposition 31]. For non-strictly convex norms the above lemma still holds with a non-strict inequality. On the other hand, the following corollary of the monotonicity lemma is false when the norm is not strictly convex, as easy examples show.

Lemma 18 *Let u and v be linearly independent unit vectors in a strictly convex 2-dimensional normed space. Suppose that x is such that $\|x - u\| = \|x + u\|$ and $\|x - v\| = \|x + v\|$. Then $x = o$.*

Proof Without loss of generality, $\mathbf{x} = \alpha \mathbf{u} + \beta \mathbf{v}$ with $\alpha, \beta \geq 0$. If $\mathbf{x} \neq \mathbf{o}$, then by Lemma 17,

$$\|\mathbf{x} - \mathbf{v}\| < \|\mathbf{x} + \mathbf{u}\| = \|\mathbf{x} - \mathbf{u}\| < \|\mathbf{x} + \mathbf{v}\|,$$

a contradiction. $\square$

Proposition 19 *Let X be any normed space, $d \geq 4$, $q \in [1, \infty)$, and $1 \leq p < \frac{\log 5/2}{\log 2}$. Then $m(\ell_p^d \oplus_q X) \leq 5$. If $p = \frac{\log 5/2}{\log 2}$, then $m(\ell_p^d \oplus_q X) \leq 6$.*

Proof Consider the following subset of $\ell_p^d \oplus_q X = (\ell_p^4 \oplus_p \ell_p^{d-4}) \oplus_q X$:

$$S = \left\{ \begin{pmatrix} 1 & 1 & 1 & 0 & \mathbf{o} & \mathbf{o} \\ 1 & -1 & -1 & 0 & \mathbf{o} & \mathbf{o} \\ -1 & 1 & -1 & 0 & \mathbf{o} & \mathbf{o} \\ -1 & -1 & 1 & 0 & \mathbf{o} & \mathbf{o} \\ 0 & 0 & 0 & \lambda & \mathbf{o} & \mathbf{o} \end{pmatrix} \right\}.$$

By setting $\lambda = (2^{p+1} - 3)^{1/p}$, S becomes a $2^{1+1/p}$ -equilateral set. We show that S is maximal equilateral if $p < \log(5/2)/\log 2$ and can be uniquely extended if $p = \log(5/2)/\log 2$.

Suppose that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \mathbf{x}, \mathbf{y})$ has distance $2^{1+1/p}$ to each point in S . Then $(\alpha_1, \alpha_2, \alpha_3)$ has the same distance in ℓ_p^3 to the points

$$(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1),$$

which gives

$$\|(\alpha_1, \alpha_2) - (1, 1)\|_p = \|(\alpha_1, \alpha_2) - (-1, -1)\|_p$$

and

$$\|(\alpha_1, \alpha_2) - (1, -1)\|_p = \|(\alpha_1, \alpha_2) - (-1, 1)\|_p.$$

It follows (from Lemma 18 if $p > 1$) that $(\alpha_1, \alpha_2) = (0, 0)$. Thus $|\alpha_3 - 1| = |\alpha_3 + 1|$, which gives $\alpha_3 = 0$ and $3 + |\alpha_4|^p = |\alpha_4 - \lambda|^p$. By Lemma 13, the function $f(x) = 3 + |x|^p - |x - \lambda|^p$ is increasing (strictly increasing if $p > 1$). Since $f(\alpha_4) = 0$ and $f(-\lambda) = 2^{p+1}(\frac{5}{2} - 2^p) \geq 0$ (> 0 if $p = 1$), it follows that $\alpha_4 \leq -\lambda$. Then by assumption,

$$\begin{aligned} 2^{1+1/p} &= \|(0, 0, 0, \alpha_4, \mathbf{x}, \mathbf{y}) - (1, 1, 1, 0, \mathbf{o}, \mathbf{o})\| \\ &= \left((3 + |\alpha_4|^p + \|\mathbf{x}\|_p^p)^{q/p} + \|\mathbf{y}\|_q^q \right)^{1/q} \\ &\geq (3 + \lambda^p)^{1/p} = 2^{1+1/p}, \end{aligned}$$

and equality holds throughout, which implies that $\alpha_4 = -\lambda$, $\mathbf{x} = \mathbf{o}$ and $\mathbf{y} = \mathbf{o}$. Also, by assumption, $2^{1+1/p} = \|(0, 0, 0, -\lambda, \mathbf{o}, \mathbf{o}) - (0, 0, 0, \lambda, \mathbf{o}, \mathbf{o})\| = 2\lambda$, which implies $p = \frac{\log 5/2}{\log 2}$. Therefore, S is a maximal equilateral set unless $p = \frac{\log 5/2}{\log 2}$, and then $S \cup \{(0, 0, 0, -\lambda, \mathbf{o}, \mathbf{o})\}$ is maximal. $\square$

An $n \times n$ matrix H is called an *Hadamard matrix* of order n if each entry equals ± 1 and $HH^T = nI$. It is well known [18, Chap. 18] that if an Hadamard matrix of order n exists, then $n = 1$, $n = 2$ or n is divisible by 4. It is conjectured that there exist Hadamard matrices of all orders divisible by 4. This is known for all multiples of 4 up to 664 [7]. The next lemma summarises the only results (all well known) on the existence of Hadamard matrices that we will use.

Lemma 20 (1) *There exist Hadamard matrices of orders 1, 2, 4, 8, 12.*

(2) *Let $x \geq 1$. The interval $(x, 2x)$ contains the order of some Hadamard matrix iff $x \notin \{1, 2, 4\}$.*

(3) *Let $H(x)$ be the largest order n of an Hadamard matrix with $n < x$. Then*

$$\lim_{x \rightarrow \infty} H(x)/x = 1.$$

Proof Given Hadamard matrices H_1 of order n_1 and H_2 of order n_2 , the Kronecker product $H_1 \otimes H_2$ is an Hadamard matrix of order $n_1 n_2$ [18, Chap. 18]. Starting with the unique Hadamard matrices of orders 2 and 12, we obtain Hadamard matrices of orders 2^k and $6 \cdot 2^k$ for all $k \geq 1$. This is sufficient to cover every interval $(x, 2x)$ except for $(1, 2)$, $(2, 4)$ and $(4, 8)$.

The Paley construction [18, Chap. 18] gives an Hadamard matrix of order $q + 1$ for any prime power $q \equiv 3 \pmod{4}$. By the prime number theorem for arithmetic progressions [5] the number of primes less than x that are congruent to 3 modulo 4 is $(1 + o(1))x/(2 \ln x)$. This implies that the largest such prime less than x is $\geq (1 + o(1))x$, which gives $H(x)/x \rightarrow 1$ as $x \rightarrow \infty$. $\square$

An Hadamard matrix is *normalised* if its first column are all +1s. Note that any Hadamard matrix can be normalised by multiplying each row by its first entry. If

$$H = \begin{bmatrix} 1 & \mathbf{h}_1 \\ 1 & \mathbf{h}_2 \\ \vdots & \vdots \\ 1 & \mathbf{h}_n \end{bmatrix}$$

is a normalised Hadamard matrix, we say that $\{\mathbf{h}_1, \dots, \mathbf{h}_n\} \subset \mathbb{R}^{n-1}$ is an *Hadamard simplex*. In the sequel we repeatedly use the fact that each $\mathbf{h}_i$ has $n - 1$ coordinates, each ± 1 , and that any two $\mathbf{h}_i$ differ in exactly $n/2$ coordinates. In particular, the distance between any two vertices of an Hadamard simplex in ℓ_p^{n-1} equals $n^{1/p} 2^{1-1/p}$ and all vertices lie on a sphere with centre $\mathbf{o}$ and radius $(n - 1)^{1/p}$. The next lemma shows in particular (by taking $X = \mathbb{R}$) that an Hadamard simplex cannot lie on any other sphere of ℓ_p^{n-1} if $p \in [1, \infty)$.

Lemma 21 Let $\mathbf{h}_1, \dots, \mathbf{h}_n \subset \mathbb{R}^{n-1}$ be an Hadamard simplex, $p \in [1, \infty)$, X a normed space and $\mathbf{u} \in X$. Suppose that

$$(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \in \overbrace{X \oplus_p \cdots \oplus_p X}^{n-1 \text{ summands}}$$

has the same distance to each point of $\{\mathbf{h}_i \otimes \mathbf{u} : i \in [n]\} \subset X \oplus_p \cdots \oplus_p X$. Then $\|\mathbf{x}_i - \mathbf{u}\| = \|\mathbf{x}_i + \mathbf{u}\|$ for all $i \in [n]$.

Proof Let $\mathbf{h}_i = [h_{i,1}, h_{i,2}, \dots, h_{i,n-1}]$ for $i \in [n]$. We may assume without loss of generality that $\mathbf{h}_1 = [-1, -1, \dots, -1]$. Since $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-1})$ is equidistant to all $\mathbf{h}_i \otimes \mathbf{u}$, there exists $D \geq 0$ such that $\sum_{j=1}^{n-1} \|\mathbf{x}_j - h_{i,j} \mathbf{u}\|^p = D^p$ for each $i \in [n]$. Subtract the first of these equations from the others to obtain the system

$$\begin{bmatrix} \mathbf{h}_2 - \mathbf{h}_1 \\ \mathbf{h}_3 - \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_n - \mathbf{h}_1 \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_1 - \mathbf{u}\|^p - \|\mathbf{x}_1 + \mathbf{u}\|^p \\ \|\mathbf{x}_2 - \mathbf{u}\|^p - \|\mathbf{x}_2 + \mathbf{u}\|^p \\ \vdots \\ \|\mathbf{x}_{n-1} - \mathbf{u}\|^p - \|\mathbf{x}_{n-1} + \mathbf{u}\|^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (11)$$

The Hadamard matrix H is invertible. If we subtract the first row from all the other rows, the resulting matrix

$$\begin{bmatrix} 1 & \mathbf{o} \\ 0 & \mathbf{h}_2 - \mathbf{h}_1 \\ \vdots & \vdots \\ 0 & \mathbf{h}_n - \mathbf{h}_1 \end{bmatrix}$$

is still invertible. It follows that (11) has the unique solution

$$\|\mathbf{x}_j - \mathbf{u}\|^p - \|\mathbf{x}_j + \mathbf{u}\|^p = 0 \quad \text{for all } j \in [n-1].$$

□

Lemma 22 Let $\mathbf{h}_1, \dots, \mathbf{h}_n \subset \mathbb{R}^{n-1}$ be an Hadamard simplex and $p \in [1, \infty)$. Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent unit vectors in a strictly convex 2-dimensional normed space X . Suppose that

$$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \in \overbrace{X \oplus_p \cdots \oplus_p X}^{n-1 \text{ summands}}$$

has the same distance in the p -norm to each point of $\{\mathbf{h}_i \otimes \mathbf{u} : i \in [n]\}$, and the same distance to each point of $\{\mathbf{h}_i \otimes \mathbf{v} : i \in [n]\}$. Then $\mathbf{x} = \mathbf{o}$.

Proof Combine Lemmas 18 and 21. □

Proposition 23 Let $p \in [1, 2)$, $q \in [1, \infty)$, and X any normed space. Let $k_1, k_2 \in \mathbb{N}$ be such that there exist Hadamard matrices of orders k_1 and k_2 and such that

$$2 - 2^{p-1} < \frac{1}{k_1} + \frac{1}{k_2} < 4 - 2^p, \quad (12)$$

$$(1 - 2^{-p})(2 - 2^{p-1}) < (1 - 2^{1-p})\frac{1}{k_1} + \frac{1}{k_2}, \quad (13)$$

$$\text{and } (1 - 2^{-p})(2 - 2^{p-1}) < \frac{1}{k_1} + (1 - 2^{1-p})\frac{1}{k_2}. \quad (14)$$

Then $m(\ell_p^d \oplus_q X) \leq 2(k_1 + k_2)$ for all $d \geq 2(k_1 + k_2 - 1)$.

Proof It is sufficient to construct an equilateral set S of cardinality $2(k_1 + k_2)$ in $\ell_p^{2(k_1+k_2-1)}$ that does not lie on any sphere. Then $(S \oplus \{\mathbf{o}\}) \oplus \{\mathbf{o}\}$ will be maximal equilateral in $\ell_p^d \oplus_q X = (\ell_p^{2(k_1+k_2-1)} \oplus_p \ell_p^{d-2(k_1+k_2-1)}) \oplus_q X$ for any $q \in [1, \infty)$.

Let $\alpha_1, \alpha_2, \lambda_1, \lambda_2 \in \mathbb{R}$ (to be fixed later) such that

$$\alpha_1, \alpha_2 > 0 \quad \text{and} \quad 2^{1-1/p} \leq \lambda_1, \lambda_2 \leq 2^{1/p}. \quad (15)$$

By Lemma 16 there exist unit vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \ell_p^2$ such that $\|\mathbf{u}_i \pm \mathbf{v}_i\|_p = \lambda_i$, $i = 1, 2$. Let $\{\mathbf{g}_i : i \in [k_1]\} \subset \ell_p^{k_1-1}$ and $\{\mathbf{h}_i : i \in [k_2]\} \subset \ell_p^{k_2-1}$ be Hadamard simplices. Consider the following subsets of $\ell_p^{2(k_1+k_2-1)} = \mathbb{R} \oplus_p \ell_p^{2(k_1-1)} \oplus_p \mathbb{R} \oplus_p \ell_p^{2(k_2-1)}$:

$$\begin{aligned} S_1^- &= \{(-\alpha_1, k_1^{-1/p} \mathbf{g}_i \otimes \mathbf{u}_1, 0, \mathbf{o}) : i \in [k_1]\}, \\ S_1^+ &= \{(\alpha_1, k_1^{-1/p} \mathbf{g}_i \otimes \mathbf{v}_1, 0, \mathbf{o}) : i \in [k_1]\}, \\ S_2^- &= \{(0, \mathbf{o}, -\alpha_2, k_2^{-1/p} \mathbf{h}_i \otimes \mathbf{u}_2) : i \in [k_2]\}, \\ S_2^+ &= \{(0, \mathbf{o}, \alpha_2, k_2^{-1/p} \mathbf{h}_i \otimes \mathbf{v}_2) : i \in [k_2]\}. \end{aligned}$$

We would like to choose $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ so as to make $S = S_1^- \cup S_1^+ \cup S_2^- \cup S_2^+$ equilateral and non-spherical. Note that $|S| = 2(k_1 + k_2)$.

The p th power of the distance between points

- in the same set $S_1^\pm$ equals

$$\frac{k_1}{2} \frac{1}{k_1} \|2\mathbf{u}_1\|_p^p = \frac{k_1}{2} \frac{1}{k_1} 2^p = 2^{p-1},$$

- in the same set $S_2^\pm$ similarly equals $\frac{k_2}{2} \frac{1}{k_2} 2^p = 2^{p-1}$,
- in S_1^- and S_1^+ equals

$$(2\alpha_1)^p + (k_1 - 1) \frac{1}{k_1} \|\mathbf{u}_1 \pm \mathbf{v}_1\|_p^p = (2\alpha_1)^p + \left(1 - \frac{1}{k_1}\right) \lambda_1^p,$$

- in S_2^- and S_2^+ similarly equals $(2\alpha_2)^p + (1 - \frac{1}{k_2})\lambda_2^p$,
- in $S_1^- \cup S_1^+$ and $S_2^- \cup S_2^+$ equals

$$\alpha_1^p + \frac{k_1 - 1}{k_1} + \alpha_2^p + \frac{k_2 - 1}{k_2} = \alpha_1^p + \alpha_2^p + 2 - \left(\frac{1}{k_1} + \frac{1}{k_2}\right).$$

For S to be equilateral, we therefore need

$$(2\alpha_1)^p + \left(1 - \frac{1}{k_1}\right)\lambda_1^p = 2^{p-1}, \quad (2\alpha_2)^p + \left(1 - \frac{1}{k_2}\right)\lambda_2^p = 2^{p-1} \quad (16)$$

and

$$\alpha_1^p + \alpha_2^p + 2 - \left(\frac{1}{k_1} + \frac{1}{k_2}\right) = 2^{p-1}. \quad (17)$$

The set S will lie on some sphere iff some $(\beta, \mathbf{x}, \gamma, \mathbf{y})$ is equidistant to S . This implies that $\mathbf{x}$ is equidistant to all $k_1^{-1/p} \mathbf{g}_i \otimes \mathbf{u}_1$ and also equidistant to all $k_1^{-1/p} \mathbf{g}_i \otimes \mathbf{v}_1$. By Lemma 22, $\mathbf{x} = \mathbf{o}$. Similarly, $\mathbf{y} = \mathbf{o}$. Then $|\alpha_1 - \beta| = |\alpha_1 - \beta|$, which gives $\beta = 0$. Similarly, $\gamma = 0$. Thus S can only lie on a sphere with centre $\mathbf{o}$ and radius $(\alpha_1^p + 1)^{1/p} = (\alpha_2^p + 1)^{1/p}$. Therefore, for S not to lie on a sphere, we need

$$\alpha_1 \neq \alpha_2. \quad (18)$$

It turns out that the hypotheses (12), (13), (14) are sufficient for the three simultaneous equations (16) and (17) to have a solution in $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ given the constraints (15) and (18). This can be seen as follows. First use (16) to eliminate α_1 and α_2 from (15), (17) and (18), and set $x_1 = (1 - \frac{1}{k_1})\lambda_1^p$ and $x_2 = (1 - \frac{1}{k_2})\lambda_2^p$ to obtain that the simultaneous solvability of (15), (16), (17) and (18) is equivalent to the existence of $x_1, x_2 \in \mathbb{R}$ such that

$$x_1 + x_2 = 2^p \left(3 - 2^{p-1} - \frac{1}{k_1} - \frac{1}{k_2}\right), \quad (19)$$

$$2^{p-1} \left(1 - \frac{1}{k_1}\right) \leq x_1 \leq 2 \left(1 - \frac{1}{k_1}\right), \quad (20)$$

$$2^{p-1} \left(1 - \frac{1}{k_2}\right) \leq x_2 \leq 2 \left(1 - \frac{1}{k_2}\right), \quad (21)$$

$$x_1, x_2 < 2^{p-1}, \quad (22)$$

$$\text{and } x_1 \neq x_2. \quad (23)$$

Geometrically, it is sufficient to show that the line ℓ in the $x_1 x_2$ -plane with equation (19) intersects the interior of the rectangle R with bottom-left corner $\mathbf{a} = 2^{p-1} \left(1 - \frac{1}{k_1}, 1 - \frac{1}{k_2}\right)$ and top-right corner

$$\mathbf{b} = \left(\min \left\{ 2^{p-1}, 2 \left(1 - \frac{1}{k_1} \right) \right\}, \min \left\{ 2^{p-1}, 2 \left(1 - \frac{1}{k_2} \right) \right\} \right).$$

Indeed, if ℓ intersects the interior of R then it intersects R in infinitely many points, among which one can be chosen that satisfies (23). That ℓ intersects the interior of R is equivalent to $\mathbf{a}$ lying below ℓ :

$$2^{p-1} \left(1 - \frac{1}{k_1} \right) + 2^{p-1} \left(1 - \frac{1}{k_2} \right) < 2^p \left(3 - 2^{p-1} - \frac{1}{k_1} - \frac{1}{k_2} \right) \quad (24)$$

and $\mathbf{b}$ lying above ℓ :

$$\begin{aligned} 2^p \left(3 - 2^{p-1} - \frac{1}{k_1} - \frac{1}{k_2} \right) &< \min \left\{ 2^{p-1}, 2 \left(1 - \frac{1}{k_1} \right) \right\} \\ &+ \min \left\{ 2^{p-1}, 2 \left(1 - \frac{1}{k_2} \right) \right\}. \end{aligned} \quad (25)$$

The inequality (24) is equivalent to the second inequality in (12).

The inequality (25) can be split into four simultaneous inequalities:

$$2^p \left(3 - 2^{p-1} - \frac{1}{k_1} - \frac{1}{k_2} \right) < 2 \left(1 - \frac{1}{k_1} \right) + 2 \left(1 - \frac{1}{k_2} \right), \quad (25a)$$

$$2^p \left(3 - 2^{p-1} - \frac{1}{k_1} - \frac{1}{k_2} \right) < 2^{p-1} + 2^{p-1}, \quad (25b)$$

$$2^p \left(3 - 2^{p-1} - \frac{1}{k_1} - \frac{1}{k_2} \right) < 2 \left(1 - \frac{1}{k_1} \right) + 2^{p-1}, \quad (25c)$$

$$2^p \left(3 - 2^{p-1} - \frac{1}{k_1} - \frac{1}{k_2} \right) < 2^{p-1} + 2 \left(1 - \frac{1}{k_2} \right). \quad (25d)$$

Each of (25a) and (25b) is equivalent to the first inequality in (12). Inequality (25c) is equivalent to (13) and (25d) is equivalent to (14).

It follows that $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ can be chosen so that S is equilateral and non-spherical. $\square$

Proof of Theorem 6 By Lemma 20 the interval $(2/(4 - 2^p), 4/(4 - 2^p))$ contains the order of some Hadamard matrix except if $2/(4 - 2^p) \in \{1, 2, 4\}$. Therefore, unless

$$p \in \left\{ 1, \frac{\log 3}{\log 2}, \frac{\log(7/2)}{\log 2} \right\},$$

if we let $k_1 = k_2 = k$ be the order of such an Hadamard matrix, conditions (12), (13) and (14) are satisfied, and Proposition 23 gives a maximal equilateral set of size $C(p) = 2(k_1 + k_2) = 4k$ and of dimension at least $d_0(p) = 2(k_1 + k_2 - 1) = 4k - 2$.

Asymptotically when $p \rightarrow 2$, we may choose k to be the largest order of an Hadamard matrix such that $k < 4/(4 - 2^p)$. Then by Lemma 20, $4k < \frac{16}{4-2^p} \sim \frac{4}{(2-p)\ln 2}$, which gives the required asymptotic upper bounds for $C(p)$ and $d_0(p)$.

The exceptional case $p = 1$ has already been dealt with in Proposition 19. For $p = \log 3 / \log 2$ we may use $(k_1, k_2) = (2, 4)$ in Proposition 23 and for $p = \log(7/2) / \log 2$ we may use $(k_1, k_2) = (4, 8)$. This covers the existence of $C(p)$ and $d_0(p)$ for all $p \in [1, 2)$.

The last column of Table 1 indicates how each line in that table is obtained: Proposition 19 covers the case $1 \leq p \leq \frac{\log 5/2}{\log 2}$, and in the remaining cases Proposition 23 is applied with Hadamard matrices of various orders k_1 and k_2 . $\square$

Proof of Corollary 7 Let $p \in [1, 2)$ and let Γ be any set. If Γ is infinite, then by Theorem 6 where X is chosen to be $\ell_p(\Gamma)$ and $q = p$, $m(\ell_p^d \oplus_p \ell_p(\Gamma)) \leq C(p)$ for large enough d . However, since Γ is infinite, $\ell_p^d \oplus_p \ell_p(\Gamma)$ is isometrically isomorphic to $\ell_p(\Gamma)$.

For the remaining case where Γ is finite, it is sufficient to bound $m(\ell_p^d)$ for all d . Again by Theorem 6 where X is chosen to be the zero space, $m(\ell_p^d) \leq C(p)$ for all $d \geq d_0(p)$. For $d < d_0(p)$, we apply the Danzer–Grünbaum–Petty–Soltan result to obtain $m(\ell_p^d) \leq 2^{d_0(p)}$.

Thus for fixed $p \in [1, 2)$, we have found an upper bound to $m(\ell_p(\Gamma))$ uniform over all Γ . $\square$

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