Characterization of Co-blockers for Simple Perfect Matchings in a Convex Geometric Graph

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Abstract Consider the complete convex geometric graph on $2m$ vertices, $CGG(2m)$, i.e., the set of all boundary edges and diagonals of a planar convex $2m$ -gon P . In (Keller and Perles, Israel J Math 187:465–484, 2012), the smallest sets of edges that meet all the simple perfect matchings (SPMs) in $CGG(2m)$ (called “blockers”) are characterized, and it is shown that all these sets are caterpillar graphs with a special structure, and that their total number is $m \cdot 2^{m-1}$. In this paper we characterize the co-blockers for SPMs in $CGG(2m)$, that is, the smallest sets of edges that meet all the blockers. We show that the co-blockers are exactly those perfect matchings M in $CGG(2m)$ where all edges are of odd order, and two edges of M that emanate from two adjacent vertices of P never cross. In particular, while the number of SPMs and the number of blockers grow exponentially with m , the number of co-blockers grows super-exponentially.

Keywords Convex geometric graphs · Blockers · Semi-perfect matchings

1 Introduction

In this paper we consider convex geometric graphs (i.e., graphs whose vertices are points in convex position in the plane, and whose edges are segments connecting pairs of vertices), and in particular, the complete convex geometric graph on $2m$ vertices, denoted by $CGG(2m)$.

Definition 1.1 A simple perfect matching (SPM) in $CGG(2m)$ is a set of m pairwise disjoint edges (i.e., edges that do not intersect, not even in an interior point).

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In [3], Keller and Perles give a complete characterization of the smallest sets of edges in $CGG(2m)$ that meet all the SPMs, called *blockers*. It turns out that all the blockers are simple trees of size m admitting a special structure called *caterpillar graphs* [1,2], and that their number is $m \cdot 2^{m-1}$.

Following the result of [3], one may consider a sequence $\{A_n\}_{n=0}^\infty$, defined inductively as follows. A_0 is the family of all SPMs in $CGG(2m)$. Given A_k , define A_{k+1} as the family of all smallest sets of edges in $CGG(2m)$ that meet all of the elements of A_k . In particular, A_1 is the family of all blockers, characterized in [3].

A standard argument shows that $A_3 = A_1$, and thus $A_k = A_{k-2}$ for all $k \geq 3$. Thus, the only unknown element of the sequence is A_2 , i.e., the family of all smallest sets of edges of $CGG(2m)$ that meet all blockers, called in the sequel *co-blockers*. It is easy to show (see Sect. 3) that the size of any co-blocker is at least m , and on the other hand, any SPM meets every blocker by the definition of a blocker, and thus is a co-blocker. Therefore, the size of the co-blockers is m (like the size of the SPMs and of the blockers).

In this paper we give a complete characterization of the family of co-blockers:

Theorem 1.2 *For any $m \in \mathbb{N}$, the set of co-blockers in $CGG(2m)$ is the set of all perfect matchings in $CGG(2m)$, such that:*

- *All the edges of the matching have odd order (see Sect. 2 for a formal definition of the order of an edge in $CGG(2m)$).*
- *Two edges $[a, b]$ and $[a', c]$ of the matching whose end-points a, a' form a boundary edge of $CGG(2m)$ never cross.*

In the sequel, we call such matchings *semi-simple perfect matchings* (see Definition 2.5). Two examples of small co-blockers that are not SPMs are given in Fig. 1.

Using the characterization theorem and several further observations, we obtain lower and upper bounds on the number of co-blockers:

Theorem 1.3 *Denote the set of co-blockers in $CGG(2m)$ by $A_2(m)$. Then for all $m \in \mathbb{N}$,*

Fig. 1 Two small co-blockers that are not SPMs

$$\left(c_1 \frac{m}{(\log m)^2}\right)^m \leq |A_2(m)| \leq \left(c_2 \frac{m \log \log m}{\log m}\right)^m,$$

where c_1, c_2 are universal constants.

It is known that both the number of SPMs and the number of blockers grow only exponentially with m : It is easy to show that the number of SPMs is the Catalan number $C_m = \frac{1}{m+1} \binom{2m}{m}$, and it is shown in [3] that the number of blockers is $m \cdot 2^{m-1}$. Thus, Theorem 1.3 shows that the number of co-blockers is significantly larger than the numbers of SPMs and blockers.

The rest of this paper is organized as follows: In Sect. 2 we introduce some basic definitions and recall the properties of SPMs and blockers that are used in our proof. In Sect. 3 we present the proof of Theorem 1.2. Finally, in Sect. 4 we prove Theorem 1.3.

2 Preliminaries

In this section we introduce several basic definitions, and recall some properties of SPMs and of blockers presented in [3], which are used in the proof of Theorem 1.2.

2.1 Definitions and Notations

Throughout this paper, we use the following definitions and notations.

Notation 2.1 The set of vertices of $CGG(2m)$ is denoted by V , and is realized in the plane as the set of vertices of a convex $2m$ -gon P . The vertices are labelled cyclically from 0 to $2m - 1$.

Definition 2.2 The order of an edge $[i, j]$ is $\min(|j - i|, 2m - |j - i|)$. The boundary edges of P are, of course, of order 1. We call the non-boundary edges, i.e., the edges that are diagonals of P , *interior edges*.

Definition 2.3 The direction of an edge in $CGG(2m)$ is the sum (modulo $2m$) of the labels of its endpoints. That is, if $e = [i, j]$, then its direction is:

$$\text{Dir}(e) = i + j \pmod{2m} = \begin{cases} i + j, & i + j < 2m, \\ i + j - 2m, & i + j \geq 2m. \end{cases}$$

Two edges e, e' are *parallel* if $\text{Dir}(e) = \text{Dir}(e')$.¹

Definition 2.4 Two edges e, e' of $CGG(2m)$ are called *neighbors* if (at least) one endpoint of e is adjacent to (at least) one endpoint of e' on the boundary of P .

Definition 2.5 A perfect matching M of $CGG(2m)$ is called *semi-simple* if:

- All the edges of M are of odd order, and
- M does not contain a pair of crossing neighbors.

¹ Note that if P is regular, an equivalent definition is that e, e' are parallel as straight line segments in the plane.

2.2 Caterpillar Trees and the Structure of Blockers

Definition 2.6 A tree T is a *caterpillar* (or a *fishbone*) if the derived graph T' (i.e., the graph obtained from T by removing all leaves and their incident edges) is a path (or is empty). A geometric caterpillar is *simple* if it does not contain a pair of crossing edges. A longest path in a caterpillar T is called a *spine* of T . Given a spine of T , the edges of T that have one endpoint interior to the spine and the other endpoint exterior to the spine are called *legs* of T .

In [3], the blockers in $CGG(2m)$ are fully characterized by the following theorem:

Theorem 2.7 Any blocker of $CGG(2m)$ is a simple caterpillar graph whose spine lies on the boundary of P and is of length $t \geq 2$. If the spine “starts” with the vertex 0 and the edge $[0, 1]$, then the edges of the blocker are:

$$\{[i-1, i] : 1 \leq i \leq t\} \cup \{[t+j-1-\varepsilon_{t+j}, t+j+\varepsilon_{t+j}] : 1 \leq j \leq m-t\}, \quad (1)$$

where the ε_i are natural numbers satisfying $1 \leq \varepsilon_{t+1} < \varepsilon_{t+2} < \cdots < \varepsilon_m \leq m-2$.

Conversely, any set of m edges of the described form is a blocker in $CGG(2m)$.

The contents of Formula (1) can be described as follows:

1. For each pair e, e' of opposite boundary edges of P , the blocker contains exactly one edge parallel to e and e' .
2. Each leg of the caterpillar connects a vertex a interior to the spine to a vertex b exterior to the spine.
3. If $[a, b]$ and $[a', b']$ are two distinct legs of the caterpillar (where a, a' are on the spine and b, b' are not), then the distance between b and b' along the complement of the spine is larger than the distance between a and a' (within the spine). In other words, if the spine starts with the vertex 0 and $a < a'$, then

$$b - b' > a' - a. \quad (2)$$

An example of a blocker in $CGG(18)$ is depicted in Fig. 2.

In our proof we also use the following simple claim:

Claim 2.8 In $CGG(2m)$, the set of all edges of odd order emanating from a single vertex is a blocker. This blocker is called “a star blocker”.

The star blockers correspond to the case $t = 2$ in Theorem 2.7. The other extreme value $t = m$ yields blockers that are just one half of the boundary circuit of P .

3 Proof of Theorem 1.2

In this section we present the proof of our main theorem. We start by observing a simple necessary condition for co-blockers. As we shall see later, this condition is not so far from being sufficient.

Fig. 2 A blocker in $CGG(18)$

Lemma 3.1 *Let C be a co-blocker in $CGG(2m)$. Then C is a perfect matching, and all the edges of C are of odd order.*

Proof First, note that if there exists a vertex x that is not contained in any edge of C , then C does not meet the star blocker emanating from x , contradicting the assumption that C is a co-blocker. Thus, any vertex $x \in V$ is contained in an edge of C , and since C has only m edges (as noted in Sect. 1), this implies that C is a perfect matching.

Second, suppose on the contrary that C contains an edge $e = [x, y]$ of even order. Since e is the only edge of C that emanates from x , we again find that C does not meet the star blocker emanating from x , contradicting the assumption. Thus, all edges of C are of odd order. $\square$

We proceed by observing a property of semi-simple perfect matchings which will be crucial in our analysis:

Lemma 3.2 *Let M be a semi-simple perfect matching in $CGG(2m)$. Then the following holds:*

- *If e is an interior edge in M , then M contains a boundary edge in each of the two open half-planes determined by the straight line $\text{aff}(e)$.*
- *If e_1, e_2 are two crossing edges of M (i.e., edges which intersect in an interior point), then M contains a boundary edge in each of the four open quadrants determined by $\text{aff}(e_1)$ and $\text{aff}(e_2)$.*

Proof We begin with the first claim. Let $e = [a, b]$ and let H be one of the half-planes determined by $\text{aff}(e)$. H meets the boundary of P in a polygonal arc $\langle x_0, x_1, \dots, x_k \rangle$, where $x_0 = a$ and $x_k = b$. Consider the set of all edges of M both of whose endpoints are in $\{x_0, x_1, \dots, x_k\}$, like e . Among those edges, choose an edge $e' = [x_i, x_j]$ ($i < j$) that minimizes the difference $j - i$. We claim that e' is a boundary edge.

Indeed, if e' is not a boundary edge, then x_{i+1} is an internal vertex of the polygonal arc $\langle x_i, x_{i+1}, \dots, x_j \rangle$. Let e'' be the edge of M that contains x_{i+1} . By the minimality of e' , the other endpoint of e'' cannot be in $\{x_i, x_{i+1}, \dots, x_j\}$, and thus, e' and e'' are crossing neighbors in M , contradicting the assumption that M is semi-simple. Hence, e' is indeed a boundary edge, as asserted.

Fig. 3 Illustration of the proof of Lemma 3.2

Now we proceed to the second claim. Let $e_1 = [a, b]$, $e_2 = [c, d]$, and $e_1 \cap e_2 = \{z\}$. Let Q be the quadrant determined by the rays $\vec{za}$ and $\vec{zc}$. Q meets the boundary of P in a polygonal arc $\langle x_0, x_1, \dots, x_k \rangle$, where $x_0 = a$ and $x_k = c$. We proceed by induction on k . The case $k = 1$ is impossible, since otherwise e_1 and e_2 are crossing neighbors, which contradicts the assumption that M is semi-simple.

Thus, we may assume that $k \geq 2$, and, in particular, that x_1 is an internal vertex of the polygonal arc $\langle x_0, x_1, \dots, x_k \rangle$. Let $e' = [x_1, y]$ be the edge of M that contains x_1 . We consider four cases:

- If y is in $\{x_0, x_1, \dots, x_k\}$, then either e' is a boundary edge, or by the first claim, M contains a boundary edge e'' both of whose endpoints are in $\{x_1, x_2, \dots, x_{k-1}\}$, hence $e'' \subset \text{int}(Q)$.
- If y is on the boundary of P strictly between x_k and b , then the edge $[x_1, y]$ crosses the edge $[c, d]$ at some point $z' \in \text{int}(P)$ (see Fig. 3). The quadrant Q' determined by the rays $\vec{z'x}$ and $\vec{z'c}$ meets the boundary of P in a shorter polygonal arc $\langle x_1, x_2, \dots, x_k \rangle$. Thus, by the induction hypothesis, M contains a boundary edge in $\text{int}(Q')$, and that edge is (of course) contained in $\text{int}(Q)$.
- If $y = b$, then M contains two edges emanating from the same vertex, contradicting the assumption that M is a perfect matching.
- If y is not one of the above, then the edges $[a, b]$ and $[x_1, y]$ are crossing neighbors, contradicting the assumption that M is semi-simple.

This completes the proof. $\square$

Now we are ready to prove our main theorem.

Proof of Theorem 1.2 Necessity: Assume that M is a co-blocker. By Lemma 3.1, M is a perfect matching and all its edges are of odd order. Suppose on the contrary that M is not semi-simple, and thus w.l.o.g. contains the crossing neighbors $[0, 2l - 1]$ and $[2k, 2m - 1]$, where $0 < 2k < 2l - 1 < 2m - 1$. Then the blocker B whose spine is $\langle 2m - 2, 2m - 1, 0, 1 \rangle$ and whose legs are $[0, 2j - 1]$ for all $2 \leq j < l$ and

Fig. 4 A blocker that misses a perfect matching with crossing neighbors

$[2j, 2m - 1]$ for all $l \leq j < m - 1$ does not meet M , contradicting the assumption that M is a co-blocker. The blocker B is depicted in Fig. 4.

Sufficiency: Assume M is a semi-simple perfect matching, and suppose on the contrary that M misses some blocker B . Assume, without loss of generality, that the spine of B is $\{0, 1, 2, \dots, t\}$, where $2 \leq t \leq m$. (Note that by Theorem 2.7, the blocker is a caterpillar whose spine lies on the boundary of P .) For $i = 1, 2, \dots, t - 1$, denote by e_i the (unique) edge of M that emanates from i , and denote its other endpoint by y_i . We claim that the edges $e_1, e_2, \dots, e_{t-1}$ satisfy the following:

- For any $1 \leq i \leq t - 1$, we have $y_i \in \{t + 1, t + 2, \dots, 2m - 1\}$. In particular, the $t - 1$ edges $e_1, e_2, \dots, e_{t-1}$ are distinct.
- For any pair $i, j \in \{1, 2, \dots, t - 1\}$, the edges e_i and e_j do not cross.

The first claim follows from the first claim of Lemma 3.2. Indeed, if $y_i \in \{0, 1, \dots, t\}$, then by the lemma, M contains a boundary edge in the polygonal path between i and y_i , and by assumption, this edge is in the spine of B , which contradicts the assumption that M misses B . The second claim follows similarly from the second claim of Lemma 3.2.

The second claim implies that if $1 \leq i < i + 1 \leq t - 1$, then $y_{i+1} < y_i$, and therefore, $i + y_i \geq (i + 1) + y_{i+1}$. Thus, the function $g : i \mapsto i + y_i$ is monotone non-increasing in i for $1 \leq i \leq t - 1$. We can also bound the range of this function, namely,

$$2m - 1 = 1 + (2m - 2) \geq g(1) \geq g(i) \geq g(t - 1) \geq (t - 1) + (t + 2) = 2t + 1,$$

which implies that for all $1 \leq i \leq t - 1$,

$$\text{Dir}(e_i) = i + y_i \pmod{2m} = i + y_i = g(i).$$

Since, by Theorem 2.7, the blocker B contains a unique edge parallel to every edge of odd order in $\text{CGG}(2m)$, there is a unique edge f_i of B parallel to e_i . This edge

cannot lie on the spine of B , since for the edges on the spine of B , the direction takes the values $1, 3, \dots, 2t - 1$, and thus they are not parallel to the edges e_i . Hence, f_i is a leg of B , which can be represented as $f_i = [r_i, q_i]$, with $1 \leq r_i \leq t - 1$, and $t + 1 \leq q_i \leq 2m - 1$.

Note that we have $\text{Dir}(f_i) = r_i + q_i$. (The other option, $\text{Dir}(f_i) = r_i + q_i - 2m$, would yield $\text{Dir}(f_i) \leq t - 2$, whereas $\text{Dir}(f_i) = \text{Dir}(e_i) \geq 2t + 1$.) Thus, by the monotonicity of $\text{Dir}(e_i)$, we have:

$$r_{i+1} + q_{i+1} = \text{Dir}(f_{i+1}) = \text{Dir}(e_{i+1}) \leq \text{Dir}(e_i) = \text{Dir}(f_i) = r_i + q_i, \quad (3)$$

for $i = 1, 2, \dots, t - 2$.

Now we are ready to reach the contradiction. If $r_i = i$ for some i , then $f_i = e_i$, contrary to the assumption that $M \cap B = \emptyset$. Thus, $r_1 > 1$ and $r_{t-1} < t - 1$. Hence, there are two consecutive indices $1 \leq i < i + 1 \leq t - 1$ with $r_i > i$ and $r_{i+1} < i + 1$. It follows that $r_{i+1} < r_i$, and therefore, by Theorem 2.7, we must have $q_{i+1} - q_i > r_i - r_{i+1}$ (see Eq. (2)), which contradicts Eq. (3). This completes the proof. $\square$

4 The Number of Co-blockers

The characterization of the co-blockers given in Theorem 1.2 allows us to prove Theorem 1.3, giving upper and lower bounds on the number of co-blockers in $\text{CGG}(2m)$, as a function of m .

Theorem (Theorem 1.3) *Denote the set of co-blockers in $\text{CGG}(2m)$ by $A_2(m)$. Then for all $m \in \mathbb{N}$,*

$$\left(c_1 \frac{m}{(\log m)^2}\right)^m \leq |A_2(m)| \leq \left(c_2 \frac{m \log \log m}{\log m}\right)^m, \quad (4)$$

where c_1, c_2 are universal constants.

Proof Let us color the odd-labelled vertices of $\text{CGG}(2m)$ *white*, and the even-labelled vertices *black*. As all edges of a co-blocker are of odd order, each of them connects a white vertex with a black vertex.

In addition, throughout the proof we assume for sake of simplicity that all “large” numbers we deal with (such as $\frac{m}{\log m}$) are integers, as it is clear from the computations that the resulting inaccuracy can be absorbed into the constants c_1 and c_2 .

Proof of the upper bound. For $1 \leq t \leq m$, denote by $A_{2,t}(m)$ the set of co-blockers that contain exactly t boundary edges. It is clearly sufficient to prove that for any fixed t ,

$$|A_{2,t}(m)| \leq \left(c_2 \frac{m \log \log m}{\log m}\right)^m. \quad (5)$$

First, we note that for any t , we have

$$|A_{2,t}(m)| < \binom{2m}{m} (m - t)!. \quad (6)$$

Indeed, there are fewer than $\binom{2m}{m}$ possibilities to place t pairwise disjoint boundary edges, and after the boundary edges are chosen, the remaining edges can be viewed as a bijection between the sets of remaining white and black vertices. As these sets are of size $m - t$ each, the number of such bijections is $(m - t)!$. Inequality (6) implies that if $t \geq m \cdot \frac{\log \log m}{\log m}$, then

$$\begin{aligned} |A_{2,t}(m)| &< \binom{2m}{m} (m - t)! < 2^{2m} m^{m-t} \leq 2^{2m} m^{m-m \cdot \frac{\log \log m}{\log m}} \\ &= 2^{2m} e^{m \log m - m \log \log m} = 2^{2m} \left(\frac{m}{\log m} \right)^m = \left(\frac{4m}{\log m} \right)^m, \end{aligned}$$

which implies inequality (5). Hence, we can assume w.l.o.g. that

$$t < m \cdot \frac{\log \log m}{\log m}. \quad (7)$$

Let $C \in A_{2,t}(m)$ be a co-blocker with t boundary edges, denoted by $e_1, e_2, \dots, e_t$ (in cyclical order). The boundary edges divide the remaining boundary vertices into t blocks $V_1, V_2, \dots, V_t$ of consecutive vertices. Let $V'_i = \{v_1, v_2, \dots, v_{\ell_i}\}$ be the set of all white vertices of V_i (for some i), and consider the set $S_i = \{[v_1, w_1], [v_2, w_2], \dots, [v_{\ell_i}, w_{\ell_i}]\}$ of all edges of C that emanate from vertices of V'_i . We observe that S_i is non-intersecting. Indeed, if two edges $e = [v, w]$, $e' = [v', w']$ in S_i cross, then by Lemma 3.2, C must contain a boundary edge whose vertices are between v and v' , contradicting the assumption $v, v' \in V_i$. Therefore, for any choice of the neighbors $w_1, w_2, \dots, w_{\ell_i}$, only one of the $\ell_i!$ bijections between $\{v_1, \dots, v_{\ell_i}\}$ and $\{w_1, \dots, w_{\ell_i}\}$ can be used in a co-blocker C . This implies that Inequality (6) can be refined to

$$|A_{2,t}(m)| < \binom{2m}{m} \frac{(m - t)!}{\ell_1! \ell_2! \dots \ell_t!}, \quad (8)$$

where $\ell_i = |V'_i|$ for all i . Note that $\ell_1 + \dots + \ell_t = (|V_1| + \dots + |V_t|)/2 = m - t$. Subject to this condition, the denominator of (8) is minimized when $\ell_1 = \ell_2 = \dots = \ell_t$, and thus,

$$|A_{2,t}(m)| < \binom{2m}{m} \frac{(m - t)!}{\left(\left(\frac{m-t}{t}\right)!\right)^t}.$$

Using Assumption (7) and some simple calculations, this implies:

$$\begin{aligned} |A_{2,t}(m)| &< \binom{2m}{m} \frac{(m - t)!}{\left(\left(\frac{m-t}{t}\right)!\right)^t} < 2^{2m} \frac{(m - t)^{m-t}}{\left(\left(\frac{m-t}{et}\right)^{\frac{m-t}{t}}\right)^t} = 2^{2m} \frac{((m - t)et)^{m-t}}{(m - t)^{m-t}} \\ &< (4e)^m t^m < (4e \cdot m \cdot \frac{\log \log m}{\log m})^m, \end{aligned}$$

which implies inequality (5), and thus completes the proof of the upper bound.

Proof of the lower bound. In order to prove the left inequality in (4), we consider a specific family $\mathcal{F}$ of co-blockers. Let t be an odd integer close to $\log m - 1$. Assume m is a multiple of $\frac{t+1}{2}$, and let $\ell = \frac{2m}{t(t+1)}$. We divide the vertices of $CGG(2m)$ into t sets $V_0, \dots, V_{t-1}$ of $\frac{2m}{t}$ consecutive vertices each. We further subdivide each set V_i into ℓ blocks $V_{i,0}, \dots, V_{i,\ell-1}$ of $t+1$ consecutive vertices each. Denote the vertices of $V_{i,j}$ by $v_{i,j,0}, v_{i,j,1}, \dots, v_{i,j,t}$.

Connect the last two vertices $v_{i,j,t-1}, v_{i,j,t}$ of each block $V_{i,j}$ by a boundary edge. There will be no other boundary edges. Thus, the number of boundary edges in each $C \in \mathcal{F}$ is $(t+1)\ell \approx \frac{2m}{\log m}$. Define $V'_{i,j} = V_{i,j} \setminus \{v_{i,j,t-1}, v_{i,j,t}\}$.

The interior edges of all $C \in \mathcal{F}$ are of the form

$$[v_{i,j,k}, v_{i-k-1(\bmod t),j',t-2-k}], \quad (9)$$

where j' is the only index that varies freely between different elements of $\mathcal{F}$.

In words, the construction rule in the first “coordinate” means that for each set $V'_{i,j}$, the $t-1$ interior edges that emanate from it have their other endpoints in all $t-1$ other sets $V_{i'}$. Namely, the vertices $v_{i,j,0}, v_{i,j,1}, \dots, v_{i,j,t-2}$ are connected to vertices in the sets $V_{i-1}, V_{i-2}, \dots, V_{i-t+1} (= V_{i+1})$, respectively. This rule assures that no two of these edges cross each other. (Note that as we showed in the proof of the upper bound, this condition must be satisfied by any co-blocker.) In the third “coordinate”, the rule is simple: for each edge, the third coordinates of the two endpoints sum up to $t-2$. Since $t-2$ is odd and each set $V_{i,j}$ is of even size, this assures that all edges are of odd order. The second “coordinate” is the only “free” one. For each fixed value of i, k , the set of ℓ vertices $\{v_{i,0,k}, \dots, v_{i,\ell-1,k}\}$ is connected by a bijection to the set $\{v_{i-k-1(\bmod t),0,t-2-k}, \dots, v_{i-k-1(\bmod t),\ell-1,t-2-k}\}$, and each of the $\ell!$ possible bijections of this form can be used in the elements of $\mathcal{F}$. An example of an element of $\mathcal{F}$ is presented in Fig. 5.

We claim that the construction is well-defined, that all elements of $\mathcal{F}$ are co-blockers, and that

$$|\mathcal{F}| \geq \left(c_1 \frac{m}{(\log m)^2} \right)^m,$$

which proves the lower bound in (4).

In order to show well-definedness, we explain how exactly each element of $\mathcal{F}$ is constructed. We go over the sets $V_0, V_1, \dots, V_{t-1}$. For each set V_i , we consider sequentially sets of ℓ vertices of the form $\{v_{i,0,k}, v_{i,1,k}, \dots, v_{i,\ell-1,k}\}$ (where each set is defined for a fixed value of k , and k goes over the *even* numbers from 0 to $t-2$, in order to cover all *black* vertices of V_i). For the vertices of each such set, the first and third coordinates of the other endpoints are prescribed by the construction rule, and the second coordinates are selected by choosing one of the $\ell!$ possible bijections as prescribed above. After going over all values of i and k , the co-blocker is set.

Note that since in the construction process, all “other endpoints” are white vertices, it cannot happen that a vertex we encounter during the process already has a mate, and thus, the process indeed yields a perfect matching. Hence, the only property we have to check is that the construction rule is satisfied also for the white vertices (as

Fig. 5 A co-blocker $C \in \mathcal{F}$ on 36 vertices, with $t = \ell = 3$

for the black vertices, it is satisfied by the construction process). Let $v_{i,j,k}$ be a white vertex. Then by the construction rule, it is connected to exactly one of the ℓ black vertices of the form $v_{i+t-2-k+1(\text{mod } t), j', t-2-k}$, which indeed satisfies the rule (9), since $i+t-2-k+1 \equiv i-k-1(\text{mod } t)$. This shows that the construction is well-defined.

Let $C \in \mathcal{F}$. We claim that C is a co-blocker. As remarked above, by the construction, all edges of C are of odd order. Hence, it is sufficient to show that C is semi-simple. Let $[v, w]$, $[v', w']$ be two neighboring edges of C . If at least one of them is a boundary edge, then they clearly do not cross. If both are interior edges and v, v' are adjacent on the boundary of P , then there exist i, j such that $v, v' \in V_{i,j}$. (This holds since the sets $V'_{i,j}$ are separated by boundary edges). By the construction, the edges that emanate from the vertices $v_{i,j,0}, v_{i,j,1}, \dots, v_{i,j,t-2}$ do not intersect, and hence, $[v, w]$ and $[v', w']$ do not cross. This shows that C is semi-simple, hence a co-blocker.

Finally, each element of $\mathcal{F}$ is constructed by choosing, for each pair (i, k) such that k is even, one of the $\ell!$ possible bijections. Since these choices are independent, we have

$$\begin{aligned} |\mathcal{F}| &= (\ell!)^{\frac{t(t-1)}{2}} = \left(\left(\frac{2m}{(\log m - 1)(\log m)} \right)! \right)^{\frac{(\log m - 2)(\log m - 1)}{2}} \\ &\geq \left(\left(\frac{2m}{(\log m)^2} \right)! \right)^{\frac{(\log m - 2)^2}{2}} \geq \left(\left(\frac{2m}{e(\log m)^2} \right)^{\frac{2m}{(\log m)^2}} \right)^{\frac{(\log m - 2)^2}{2}} \\ &= \left(\frac{2m}{e(\log m)^2} \right)^{m \cdot \left(\frac{(\log m - 2)}{(\log m)} \right)^2} = \left(\frac{2m}{e(\log m)^2} \right)^{m \cdot \left(1 - \frac{2}{\log m} \right)^2} \end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{2m}{e(\log m)^2} \right)^{m \cdot \left(1 - \frac{4}{\log m}\right)} \geq \left(\frac{2m}{e(\log m)^2} \right)^m \cdot m^{-\frac{4m}{\log m}} \\
&= \left(\frac{2m}{e(\log m)^2} \right)^m \cdot e^{-4m} = \left(\frac{2}{e^5} \cdot \frac{m}{(\log m)^2} \right)^m.
\end{aligned}$$

This completes the proof of the theorem. $\square$

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