Size of Components of a Cube Coloring

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Abstract Suppose a d -dimensional lattice cube of size n^d is colored in several colors so that no face of its triangulation (subdivision of the standard partition into n^d small cubes) is colored in $m + 2$ colors. Then one color is used at least $f(d, m)n^{d-m}$ times.

Keywords Cube · Partition · Lattice · Skeleton · Triangulation

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1 Introduction

A theorem attributed to Lebesgue asserts that if a lattice cube of dimension d is colored in d colors then one of the colors has a connected component spanning two opposite facets of the cube. By the standard reasoning with nerves of coverings this means that the covering dimension of the d -dimensional cube is at least d . The reader may find more information on the Lebesgue theorem and the related notion of Gromov's waist in the remarkable survey [3].

There arises the following natural question: What happens if the number of colors is less than d ? In [1, 5] it was conjectured that the size of a monochromatic connected component has a lower bound of order n^{d-m} when $m + 1$ colors are used. For $m = d - 1$ this follows from the Lebesgue theorem and for $m = 1$ this conjecture is proved in [5].

Here we prove this conjecture in a slightly stronger form:

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Theorem 1.1 *Let a d -dimensional cube Q be partitioned into n^d small cubes in the standard way and then the $(m + 1)$ -dimensional skeleton Q_m of this partition is subdivided to the triangulation T . Suppose the vertices of T (equivalently, vertices of Q_m) are colored in several colors so that no $(m + 1)$ -face $\sigma \in T$ is colored in $m + 2$ different colors. Then one of the colors is used on at least $f(d, m)n^{d-m}$ vertices of T .*

Remark 1.2 It is also sufficient to assume that every cubic face of Q_m of dimension $m + 1$ has at most $m + 1$ colors on its vertices. Such a point of view allows not to use any triangulation T in the statement of the theorem.

Remark 1.3 If a color c has several connected components then every component can be assumed to be a separate color. So we obtain a monochromatic connected set of at least $f(d, m)n^{d-m}$ vertices of T . By *adjacent* vertices we mean two vertices in a single $(m + 1)$ -face of T . This remark also remains valid if we define adjacent vertices as contained in a single cubic $(m + 1)$ -face of Q_m .

A similar theorem was proved independently in [4]:

Theorem 1.4 ([4]) *Let the cube Q be partitioned into n^d small cubes in the standard way and let every vertex be colored in one of $m + 1$ colors. Call a pair of vertices adjacent if they are vertices of a single small cube. Then there exists a monochromatic connected set of vertices of cardinality at least $f(d, m)n^{d-m}$.*

This theorem follows from Theorem 1.1 if we color every monochromatic connected component in a separate color. Then, for every small cube q its vertices are colored in at most $m + 1$ colors, and therefore the vertices of any $(m + 1)$ -face of a triangulation of the $(m + 1)$ -skeleton are colored in at most $m + 1$ colors. Thus the new coloring satisfies the assumption of Theorem 1.1.

Remark 1.5 We do not establish any explicit values for $f(d, m)$. The reader may consult the paper [4] for this information.

In Sect. 5 we also give some corollaries of Theorem 1.1 about coverings of a cube or a torus.

2 Definitions

We are going to consider the cubic partition Q_m (the m -skeleton of the full cubic partition of Q) together with its further triangulation T , and define the following chain complexes: The group of k -chains $C_k(X)$ ($X = T, Q_m$) is the free abelian group generated by k -faces of T (or cubic k -faces of Q_m), every face is considered to be oriented in either way. The group of k -cochains $C^k(X)$ is the group of all group homomorphisms from $C_k(X)$ to $\mathbb{Z}$. The boundary operator $\partial_k : C_k \rightarrow C_{k-1}$ maps every k -face (cubic k -face) to the sum of its codimension 1 boundary faces with the signs arising from the induced orientation, and is extended linearly to all chains in C_k . The coboundary operator $\delta_k : C^k \rightarrow C^{k+1}$ is defined by the formula $\delta_k(a)(b) = a(\partial_{k+1}b)$, where $a \in C^k(X)$, $b \in C_{k+1}(X)$. A *cycle* is a chain with zero boundary; a *cocycle* is a cochain with zero coboundary.

Let $C = \{c_0, \dots, c_k\}$ be an ordered set of $k + 1$ colors. For every oriented k -face of T , we assign $+1$ if its vertices are colored with the colors of C in accordance with the orientation, -1 if the vertices are colored with the colors of C with opposite orientation, and 0 if the face is colored with other colors or some of the colors is used more than once. Thus we define a cochain $\chi(C) \in C^k(T)$ and observe the coboundary formula:

$$\delta\chi(C) = \sum_{c_{k+1}} \chi_{Cc_{k+1}}, \quad (2.1)$$

where we sum over all the colors and by Cc_{k+1} we mean the concatenation of C and the new color c_{k+1} . If c_{k+1} coincides with a color in C then $\chi_{Cc_{k+1}}$ is assumed to be zero. Indeed, the LHS and RHS of (2.1) are $k + 1$ -cochains, then it is sufficient to test the equality on the generators of C_{k+1} , that is on $(k + 1)$ -faces. This is done immediately by classifying the $(k + 1)$ -faces into (a) colored in $k + 2$ distinct colors containing all the colors of C , (b) colored in $k + 2$ distinct colors, containing not all the colors of C , and (c) having a pair of vertices colored in the same color.

3 Plan of the Proof

The proof will consist in mapping the set of vertices (of the cubic partition) of some $(d - m)$ -dimensional big face of Q (there are approximately n^{d-m} vertices) to a set of vertices of Q having the same color, so that the preimage of every vertex has cardinality bounded in terms of d and m . This map will be constructed after proving by induction a somewhat trickier statement. Evidently, such a map gives the required result. Possibly, this kind of argument is a new approach to Gromov's waist, compare [3].

In the rest of the proof we consider $(k + 1)$ -dimensional cubic subcomplexes $Q_k \subseteq Q_m$, which are $(k + 1)$ -dimensional cubic skeleta of some $(d - m + k)$ -dimensional faces (big faces, not faces of a partition) of the cube Q , for $k = m, m - 1, \dots, 0$. In other words, we consider a flag of cubes $Q_0 \subseteq Q_1 \subseteq \dots \subseteq Q_m$ such that Q_i is a big facet of Q_{i+1} , and every time we consider the groups of chains and cochains restricted to Q_i , etc.

In order to apply simplicial cochains to cubic chains we introduce the function $L: C_k(Q_m) \rightarrow C_k(T)$ that assigns to any k -face $\tau \in Q_m$ the sum of simplicial k -faces of T that partition τ with appropriate orientations. Obviously L commutes with the boundary map ∂ .

We are going to *balance* the complexes Q_k as follows:

Definition 3.1 For every k -face $\sigma \in Q_k$ we will assign a $(k + 1)$ -chain $B(\sigma) \in C_{k+1}(T)$ so that for any $(k + 1)$ -face $\tau \in Q_k$ and a set C of $k + 1$ colors the following holds:

$$\chi_C(L(\partial\tau) + \partial B(\partial\tau)) = 0, \quad (3.1)$$

where we assume that B is linearly extended to k -chains of Q_k .

We also put $A(\sigma) = L(\sigma) + \partial B(\sigma)$ and use (3.1) in the following form:

$$\chi_C(A(\partial\tau)) = 0. \quad (3.2)$$

Let us check that the $(m+1)$ -skeleton Q_m of Q is already balanced (so that we may put $B(\cdot) = 0$). Let $|C| = m+1$, then

$$\chi_C(L(\partial\tau)) = (\delta\chi_C, L(\tau)) = 0,$$

since $\delta\chi_C = 0$ by the formula (2.1) and the assumption of the theorem (no face of T is colored in $m+2$ different colors).

The plan of the remaining part of the proof is following:

- Denote the k -skeleton of any facet of the cube corresponding to Q_k by Q_{k-1} ;
- Balance Q_{k-1} by defining a suitable $B : C_{k-1}(Q_{k-1}) \rightarrow C_k(T)$;
- Make sure that in all the expressions $B(\sigma)$ for all $(k-1)$ -faces $\sigma \in Q_{k-1}$ every k -face $\beta \in T$ is used at most $C(d, k-1)$ times (counted with its multiplicity in the chains $B(\sigma)$).

If this plan passes then on the last stage we have a 1-dimensional skeleton of a $(d-m)$ -dimensional cube (containing n^{d-m} small cubes) Q_0 . To every vertex $v \in Q_0$ we assign a chain of 1-faces of T denoted by $B(v)$. Then 0-chains $A(v)$ are simply sets of vertices of T with integer multiplicities such that the sum of coefficients in every $A(v)$ is 1. For any color c by (3.2) we obtain that $\chi_c(A(v_1)) = \chi_c(A(v_2))$ for any pair of adjacent vertices v_1 and v_2 . Since Q_0 is a connected graph we obtain that the number $\chi_c(A(v)) = x_c$ does not depend on v . The sum over all colors is

$$\sum \chi_c(A(v)) = (1, A(v)) = 1,$$

so there exists a color c with nonzero x_c . Hence this color is used in every support of the 0-cycle $A(v)$ for every v . We have at least n^{d-m} different choices of v and any point colored in c is counted at most $C'C(d, 0)$ times (here C' is the maximal number of 1-faces incident to a vertex in T).

4 Proof of Theorem 1.1

So it remains to pass from the balancing of Q_k to the balancing of Q_{k-1} . Note that for every k -face $\tau \in Q_{k-1}$ we have to satisfy the equality (since $\partial^2 = 0$ and $\partial A(\tau) = \partial L(\tau) = L(\partial\tau)$):

$$\chi_C(\partial A(\tau) + \partial B(\partial\tau)) = \chi_C(L(\partial\tau) + \partial B(\partial\tau)) = 0. \quad (4.1)$$

In this formula $A(\tau)$ is already defined, and B is to be defined on $(k-1)$ -faces of Q_{k-1} . The equality (4.1) follows from the equality:

$$\chi_D(A(\tau) + B(\partial\tau)) = 0 \quad (4.2)$$

for every k -face $\tau \in Q_{k-1}$ and every set D of $k+1$ colors. Indeed, using (2.1) from (4.2) we obtain

$$\begin{aligned}\chi_C(\partial A(\tau) + \partial B(\partial \tau)) &= (\delta \chi_C, A(\tau) + B(\partial \tau)) \\ &= \left(\sum_{c_k} \chi_{C_{c_k}}, A(\tau) + B(\partial \tau) \right) = 0.\end{aligned}\quad (4.3)$$

Now we fix a set D of $k+1$ colors. Define by

$$\xi_D(\tau) = -\chi_D(A(\tau)) \quad (4.4)$$

a k -cocycle on Q_k since for every $(k+1)$ -face $\rho \in Q_k$ we have

$$\xi_D(\partial \rho) = -\chi_D(L(\partial \rho) + \partial B \partial \rho) = 0 \quad (4.5)$$

because Q_k is balanced. To make this cocycle zero (as required in (4.2)) we have to assign to some $(k-1)$ -faces $\sigma \in Q_{k-1}$ as $B(\sigma)$ some sets (with coefficients) of k -faces $\tau \in T$. Obviously, it suffices to use only those k -faces $\tau \in T$ that are colored exactly in the colors of D .

The map $\sigma \mapsto \xi_D(B(\sigma))$ is going to be a $(k-1)$ -dimensional cochain in $C^{k-1}(Q_{k-1})$, denote it by H . Let us show that if the coboundary of H is $\xi_D|_{Q_{k-1}}$, then the condition for D holds:

$$\chi_D(A(\tau) + B(\partial \tau)) = -\xi_D(\tau) + H(\partial \tau) = -H(\partial \tau) + H(\partial \tau) = 0. \quad (4.6)$$

In order to use any k -face (out of those colored in D) at most $C(d, k-1)$ times we have to check that the ratio between the norm (sum of absolute values) of some $(k-1)$ -cochain $\eta \in C^{k-1}(Q_{k-1})$ such that $\delta \eta = \xi_D$ on Q_{k-1} and the number of k -faces usable in $B(\sigma)$ (that is, colored in D) is bounded by a constant $C(d, k-1)$.

Let the norm of ξ_D as an element of $C^k(Q_k)$ equal M . By the assumption that in every $B(\tau)$ a $(k+1)$ -face of T is used at most $C(d, k)$ times we conclude that every k -face of T is used in all $A(\tau)$ at most $C(d, k)C'(k)$ times, where $C'(k)$ is the maximal number of $(k+1)$ -faces containing a given k -face of T (it can be bounded independently on the choice of a particular triangulation T). By the formula $\xi_D(\tau) = -\chi_D(A(\tau))$ we conclude that among the k -faces of T there do exist at least $\frac{M}{C(d, k)C'(k)}$ “candidates” for $B(\sigma)$. Now it suffices to solve the equation $\delta \eta = \xi_D$ on cochains on Q_{k-1} so that the norm $|\eta|$ is at most $C''(d, k-1)M$. After that we can assign to cubic faces of Q_{k-1} on which η is nonzero several faces of T on which ξ_D is nonzero.

Note that $|\xi_D| = M$ and for some codimension 1 cubic section Q' of Q_k (parallel to Q_{k-1}) we have $|\xi_D|_{Q'} \leq M/n$. Then we use the “filling inequality” (see for example [2], where filling inequalities are widely used):

Lemma 4.1 *For a k -dimensional cocycle α on the cubic partition of the d' -dimensional cube Q' (in terms of this proof) there exists a $(k-1)$ -dimensional cubic cochain β such that $\delta \beta = \alpha$ and $|\beta| \leq C_F(d', k)n|\alpha|$.*

By this lemma we select a $(k-1)$ -dimensional cochain β on Q' with norm at most $MC_F(d-m+k, k)$ with coboundary $\xi_D|_{Q'}$. Denote the part of ξ_D between Q' and Q_{k-1} by ξ'_D ; this is a cochain with norm at most M . As the required $(k-1)$ -dimensional cochain η on Q_{k-1} we may take

$$\eta = \beta + \pi_*(\xi'_D) \quad (4.7)$$

with norm at most $(C_F(d-m+k, k) + 1)M$. Here β is moved from Q' to Q_{k-1} by the translation and by $\pi_*(\xi'_D)$ we mean the *direct image* under the projection onto Q_{k-1} that drops the dimension by 1. The cochain $\pi_*(\xi'_D)$ can be defined explicitly (thanks to the cubic complexes that we use) as taking any $(k-1)$ -face $\sigma \in Q_{k-1}$ to the sum of values (with appropriate signs) of ξ'_D on k -faces of Q_k that project onto σ . In other words

$$\pi_*(\xi'_D)(\tau) = \xi'_D(\pi^{-1}(\tau)).$$

So we satisfy the equality (4.2) for a particular color set D . Now we can add the chains $B(\sigma)$ corresponding to different sets D . The expressions $B(\sigma)$ for every particular D contained exclusively k -faces colored in the colors in D . Hence for different D we use different faces, which guarantees the bounded multiplicity of the union of all $B(\sigma)$. The faces of $B(\sigma)$ colored in D do not affect the equality (4.2) for another set D' (not obtained from D by a permutation). Now to complete the proof it remains to prove the lemma.

Proof of Lemma 4.1 The proof is similar to the proof of [4, Lemma 2.6], which is stated in terms of cycles Poincaré dual to the cocycles in this proof.

Put $\alpha_0 = \alpha$. We are going to build a cocycle α_{i+1} out of α_i as follows. Take a hyperplane section Z of Q' parallel to a pair of its opposite facets so that $|\alpha_i|_Z| \leq |\alpha_i|/n$. This is possible by the Dirichlet principle.

Define a $(k-1)$ -cochain β_i as follows:

$$\beta_i(\tau) = \alpha_i([\tau, \pi_Z(\tau)]),$$

where $[\tau, \pi_Z(\tau)]$ is (at most k)-dimensional parallelepiped between τ and its projection onto Z with appropriate sign.

Now we put

$$\alpha_{i+1} = \alpha_i - \delta\beta_i$$

and note that $|\beta_i|$ is at most $n|\alpha_i|$. Note also that α_{i+1} takes the same values as the translation of $\alpha_i|_Z$ on sections parallel to Z and is zero on any face orthogonal to Z .

After several such operations for different directions of Z ($d'-k+1$ will be enough) the cocycle α_{i+1} becomes zero. We have the inequality:

$$|\alpha_{i+1}| \leq |\alpha_i| \leq \dots \leq |\alpha|.$$

If we take $\beta = \sum_i \beta_i$ then the required inequality holds with constant $C_F(d', k) \leq d' - k + 1$. $\square$

5 Some Corollaries

We give some topological corollaries of Theorem 1.1:

Corollary 5.1 ¹ *Let a d -dimensional cube Q be covered by a finite family of closed sets C_i so that no point is covered more than $m + 1$ times. Then one of the sets C_i intersects at least $d - m$ pairs of opposite facets of Q .*

Remark 5.2 This is a generalization of the Lebesgue theorem.

Proof Let us make a corresponding color for every covering set. Let r be a positive distance and x be a point in Q . Let $u(r, x)$ be the number of the covering sets intersecting the closed ball of radius r centered at x . Put $f(x) = \sup\{r : u(r, x) \leq m + 1\}$, which may attain $+\infty$ value.

Consider a point $x \in Q$. For every C_i not containing x there exists a neighborhood of x in Q disjoint from C_i (since C_i is closed). The intersection of the finite number of such neighborhoods makes a neighborhood of x intersecting at most $m + 1$ sets of C_i . Hence $f(x)$ is positive on Q . Moreover, if $f(x) = r > 0$, then $f(y) \geq r - |x - y|$, because the ball of radius $r - |x - y|$ centered at y is contained in the ball of radius r centered at x . Therefore f is lower semicontinuous on the compactum Q and therefore must attain its positive minimum somewhere on Q . Let $R > 0$ be this minimum.

The definition of R means that any open R -neighborhood of any $x \in Q$ intersects at most $m + 1$ sets of the covering. Now we partition the cube Q into n^d small cubes in the standard way and take n so large that every side of a small cube has length less than R/d . If we color every vertex of the cubic partition into the color, corresponding to any covering set that contains this vertex, then we obtain a coloring which uses at most $m + 1$ colors on the vertices of any small cube.

Let us include Q into the cube $2Q$ of size $(2n)^d$ and repeat the coloring of Q using reflections with respect to the halving hyperplanes of $2Q$. Then we extend the coloring onto the whole $\mathbb{Z}^d$ with translations by $\pm 2n$ along the coordinate axes. Let us see what happens with a color c_i . Following the discussion in the introduction (the comparison with Karasev's theorem) we assume that the color c_i makes a connected subset of Q . The vertices of $\mathbb{Z}^d$ colored in c_i can be decomposed into connected components; denote one of them by c'_i . If the component c_i spans a pair of opposite facets (orthogonal to a base vector e_j) in Q then c'_i is invariant under the translation by $\pm 2ne_j$. Otherwise c_i does not touch one of the facets orthogonal to e_j and c'_i is trapped between a pair of hyperplanes orthogonal to e_j at distance $2n$ from each other.

So the free Abelian group Λ_i of translational symmetries of c'_i has dimension exactly ℓ , where ℓ is the number of pairs of opposite facets of Q intersected by c_i and c'_i can be obtained from $2Q \cap c'_i$ by translations in Λ_i . If we intersect c'_i with a large cube Q' of size $(2nN)^d$ then the cardinality of $c'_i \cap Q'$ has the growth order N^ℓ for

¹ The statement of this corollary is suggested by R. Karasev as a simpler version of Corollary 5.3.

varying N . By Theorem 1.1 some $c'_i \cap Q'$ must have the number of vertices of order at least N^{d-m} ; so for some of c'_i we must have $\ell \geq d - m$. $\square$

Corollary 5.3 *Let a d -dimensional torus T^d be covered by open sets C_i so that no point is covered more than $m + 1$ times. Then for some C_i the image of $H_1(C_i)$ in $H_1(T^d) = \mathbb{Z}^d$ has dimension at least $d - m$.*

Remark 5.4 The sets have to be open so that the connectedness and the arcwise connectedness coincide.

Proof As in the previous proof we pass from the covering of T^d to a fine enough triangulation of a covering cube Q , which subdivides the cubic partition into n^d small cubes. Then open covering can be refined to a closed covering by the paracompactness of Q . Then we assume that the vertices of the triangulation are colored so that no face has more than $m + 1$ colors. Duplicating Q by translations we obtain a large cube Q_N with side length Nn and the corresponding coloring. By gluing the opposite facets of Q_N we obtain a torus naturally N^d -fold covering T^d .

By Theorem 1.1 in Q_N we have a monochromatic connected component S with size of order N^{d-m} . Let S_1 be the maximal intersection of S with a residue class modulo Q (points are equal modulo Q if the differences of their coordinates are divisible by n). Then $|S_1| \geq |S|/n^d \geq \frac{N^{d-m}}{n^d}$.

Note that a projection of a monochromatic path in S starting in a point of S_1 with coordinates $(x_1, \dots, x_d)$ and ending in a point of S_1 with coordinates $(y_1, \dots, y_d)$ is a monochromatic closed loop in T^d representing the homology class $(\frac{y_i - x_i}{n})_i$. So it suffices to show that the dimension of the linear space generated by pairwise differences of S_1 is at least $d - m$ (the covering set corresponding to S will be the one required). Equivalently, we have to show that the dimension of the affine hull of S_1 is at least $d - m$.

Assume the contrary: The dimension of the affine hull of S_1 is at most $d - m - 1$. Then S_1 is contained in at most $(d - m - 1)$ -dimensional affine subspace L and the number of vertices in $Q_N \cap L$ (and therefore the number of vertices in S_1) is at most $(Nn)^{d-m-1}$. For large enough N we obtain a contradiction with the inequality $|S_1| \geq \frac{N^{d-m}}{n^d}$. $\square$

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