Algebraic Boundaries of $\mathrm{SO}(2)$ -Orbitopes

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Abstract Let $X \subset \mathbb{A}^{2r}$ be a real curve embedded into an even-dimensional affine space. We characterise when the r th secant variety to X is an irreducible component of the algebraic boundary of the convex hull of the real points $X(\mathbb{R})$ of X . This fact is then applied to 4-dimensional $\mathrm{SO}(2)$ -orbitopes and to the so called Barvinok–Novik orbitopes to study when they are basic closed semi-algebraic sets. In the case of 4-dimensional $\mathrm{SO}(2)$ -orbitopes, we find all irreducible components of their algebraic boundary.

Keywords Convex hull · Secant variety · Basic closed semi-algebraic set

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1 Introduction

The *algebraic boundary* of a semi-algebraic set is the Zariski closure of its boundary in the euclidean topology. It encodes algebraic-geometric information. For convex semi-algebraic sets, the challenge is to relate convex-geometric information about a set with its algebraic boundary.

The algebraic boundary is a central object in convex algebraic geometry. It is for example closely related to the notion of a *spectrahedron*, an object of interest in semi-definite optimisation, by a result of Helton and Vinnikov (see [12, Sect. 3]). In this paper, we will use the algebraic boundary of a convex semi-algebraic set to study whether or not it is *basic closed*, i.e. can be described by finitely many simultaneous

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polynomial inequalities. A good characterisation of basic closed convex semi-algebraic sets is also important in applications to semi-definite programming. Firstly, because every spectrahedron is a basic closed semi-algebraic set. Secondly, for a given basic closed semi-algebraic set, Lasserre developed a relaxation method that approximates its convex hull by projections of spectrahedra (cf. [14]). This method and variations of it were investigated by Helton and Nie [10, 11]. Gouveia, Parrilo and Thomas studied a variation for the convex hull of real algebraic sets in [9] and [8]. Recently, Gouveia and Netzer [7] further investigated desirable exactness properties of these two methods.

A beautiful class of compact convex semi-algebraic sets are *orbitopes*. An orbitope is the convex hull of an orbit under a linear action of a compact real algebraic group on a real vector space. The question whether or not an orbitope is basic closed has been asked by Sanyal et al. in their paper [16], that initiated the study of orbitopes in their own right.

In this paper, we will restrict our attention to the special case of orbitopes of the group $SO(2)$ of real orthogonal 2×2 matrices of determinant 1, which is probably the simplest non-discrete case. Already in this case, the algebraic boundary is hard to describe. A first family of examples was done in [16]. The authors proved that an infinite family of $SO(2)$ -orbitopes called universal $SO(2)$ -orbitopes are spectrahedra ([16], Theorem 5.2). The spectrahedral representation gives a determinantal representation of the irreducible polynomial defining the algebraic boundary of these convex sets.

If the $SO(2)$ -orbitope is not universal, then the algebraic boundary tends to be reducible. We will focus on the question whether or not the secant variety to the $\mathbb{R}$ -Zariski closure of the orbit we started with is a component of the algebraic boundary. Our main result is the following statement, which deals with convex hulls of real curves in general.

Theorem *Let $X \subset \mathbb{A}^{2r}$ be an irreducible curve and assume that the real points $X(\mathbb{R})$ of X are Zariski-dense in X . Let C be the convex hull of $X(\mathbb{R}) \subset \mathbb{R}^{2r}$ and suppose that the interior of C is non-empty. Then the $(r - 1)$ th secant variety to X is an irreducible component of the algebraic boundary of C if and only if the set of all r -tuples of real points of X that span a face of C has dimension r .*

For two different infinite families of $SO(2)$ -orbitopes, namely 4-dimensional $SO(2)$ -orbitopes and the family of Barvinok–Novik orbitopes, we will apply this result to prove that the appropriate secant varieties are components of their algebraic boundaries. For 4-dimensional orbitopes, we use a complete description of their faces by Smilansky, see [17], to find all irreducible components of their algebraic boundary. In the case of Barvinok–Novik orbitopes, a result of [1] will be essential. In both cases, our results can be used to characterise when these semi-algebraic sets are basic closed.

Namely, we will prove for Barvinok–Novik orbitopes that they are not basic closed. For 4-dimensional $SO(2)$ -orbitopes, we prove the following statement:

Theorem *Let C be a 4-dimensional $SO(2)$ -orbitope. The following are equivalent:*

- (a) *C is linearly isomorphic to the universal $SO(2)$ -orbitope C_2 .*
- (b) *C is a spectrahedron.*
- (c) *C is a basic closed semi-algebraic set.*

Fig. 1 The intersection of B_4 with the two-dimensional coordinate subspace W is the set enclosed by the black lines

To show the basic ideas, we will consider the 4-dimensional Barvinok–Novik orbitope B_4 (all the statements will be proved): It is the convex hull of the symmetric trigonometric moment curve

$$\{(\cos(\vartheta), \sin(\vartheta), \cos(3\vartheta), \sin(3\vartheta)) : \vartheta \in [0, 2\pi)\}.$$

The Zariski closure X of this trigonometric curve is an algebraic curve of degree 6 in projective 4-space. The orbitope B_4 is a simplicial and centrally symmetric convex set. By a theorem of Barvinok and Novik ([1], Theorem 1.2; or alternatively, by Smilansky’s result on faces of 4-dimensional $\mathrm{SO}(2)$ -orbitopes in [17]), it is locally neighbourly, i.e. the convex hull of sufficiently close points on the trigonometric moment curve is a face of the orbitope. This allows us to explicitly compute the algebraic boundary of B_4 ; namely, it consists of two components, a quadratic hypersurface and the secant variety to the curve X , which is a hypersurface of degree 8 in this case. We will show that the secant component intersects the interior of the orbitope, which proves that it cannot be basic closed. This is easier to see if we slice the situation with the 2-dimensional coordinate subspace W spanned by the second and the last vector of the standard basis of $\mathbb{R}^4$. We get a 2-dimensional semi-algebraic set (see Fig. 1) whose algebraic boundary has now three components, two lines and a curve of degree 3 that goes through the origin. The explicit equations and an explanation of the line in gray can be found in Example 5.5.

This article is organised as follows: In Sect. 2, we will go through the basic definitions and some facts of the representation theory of $\mathrm{SO}(2)$. We will see that two orbitopes in the same representation are isomorphic if the orbits are generically chosen.

Section 3 is the crucial section where we apply methods from semi-algebraic geometry to prove our main result. It will be used in Sects. 4 and 5 to prove that the secant variety is a component of the algebraic boundary of the 4-dimensional $\mathrm{SO}(2)$ -orbitopes and the Barvinok–Novik orbitopes.

In Sect. 4, we deal with 4-dimensional $\mathrm{SO}(2)$ -orbitopes, using Smilansky’s characterisation of the faces and the material of Sect. 3 to compute its algebraic boundary. Universal orbitopes are used to compute explicit equations for algebraic boundaries in examples.

The final Sect. 5 contains the study of the Barvinok–Novik orbitopes. Again, a secant variety is a component of the algebraic boundary and the key object in the proof of the fact that a Barvinok–Novik orbitope is not basic closed. Our proof uses a theorem of Barvinok and Novik [1] on faces of Barvinok–Novik orbitopes from their paper.

2 Setup and Basic Facts

Definition 2.1 A *representation* of $\mathrm{SO}(2)$ is a pair (ρ, V) of a finite-dimensional real vector space V and a homomorphism $\rho: \mathrm{SO}(2) \rightarrow \mathrm{GL}(V)$ of real algebraic groups. This means, after choosing a basis of V and thereby identifying $\mathrm{GL}(V)$ with $\mathrm{GL}_n(\mathbb{R})$, that ρ is a group homomorphism defined by polynomials with real coefficients.

The dimension of the vector space V is called the *dimension* of the representation (ρ, V) . The representation (ρ, V) is called *irreducible* if V has no non-trivial invariant subspace.

2.2 We fix the following notation for representations of $\mathrm{SO}(2)$: For $j \in \mathbb{Z}$, $j \neq 0$, write

$$\rho_j : \left\{ \begin{array}{ccc} \mathrm{SO}(2) & \rightarrow & \mathrm{GL}_2(\mathbb{R}) \\ A = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix} & \mapsto & A^j = \begin{pmatrix} \cos(j\vartheta) & -\sin(j\vartheta) \\ \sin(j\vartheta) & \cos(j\vartheta) \end{pmatrix} \end{array} \right\}.$$

Denote by ρ_0 the trivial representation of $\mathrm{SO}(2)$, i.e. the representation $(\rho, \mathbb{R})$, where ρ is the constant group homomorphism. The set $\{\rho_j: j \in \mathbb{Z}\}$ is the family of all irreducible representations of $\mathrm{SO}(2)$ (up to linear isomorphism commuting with the group action on V via ρ). Since $\mathrm{SO}(2)$ is a compact group, every finite-dimensional representation of $\mathrm{SO}(2)$ is the sum of some of these irreducible representations. In particular, any representation of $\mathrm{SO}(2)$ that does not contain the trivial representation is even-dimensional.

Remark 2.3 (a) If we identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x, y) \mapsto x + iy$, then $\mathrm{SO}(2)$ gets identified with the unit circle $S^1 \subset \mathbb{C}$ in the complex plane by sending a rotation matrix as above to $\exp(i\vartheta)$. The representation ρ_j becomes multiplication by the exponential, i.e. for all $z \in \mathbb{C}$ and $\vartheta \in [0, 2\pi)$ we have

$$\rho_j(\exp(i\vartheta))z = \exp(i\vartheta)^j z = \exp(ij\vartheta)z.$$

(b) The complexification, i.e. the tensor product with $\mathbb{C}$ over $\mathbb{R}$ of $\mathrm{SO}(2)$ is

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\} =: \mathrm{SO}(2, \mathbb{C}).$$

The complexification $\rho_j \otimes \mathbb{C}$ of the representation ρ_j acts on $\mathbb{C}^2 = \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}$ via the same expression, i.e. $(\rho_j \otimes \mathbb{C})(A) = A^j \in \mathrm{GL}_2(\mathbb{C})$ for all $A \in \mathrm{SO}(2, \mathbb{C})$.

This representation is isomorphic to a representation of the algebraic torus $\mathbb{C}^\times$: Every matrix in $\mathrm{SO}(2, \mathbb{C})$ is diagonalisable with diagonal form

$$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix}.$$

Since this change of coordinates simultaneously diagonalizes $\mathrm{SO}(2, \mathbb{C})$, the group is conjugate in $\mathrm{GL}_2(\mathbb{C})$ to the subgroup

$$\left\{ \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\}$$

of $\mathrm{GL}_2(\mathbb{C})$ which is isomorphic to $\mathbb{C}^\times$. The base change in $\mathbb{C}^2$ that corresponds to the conjugation of $\mathrm{SO}(2, \mathbb{C})$ to this torus subgroup gives an isomorphism of the representation $\rho_j \otimes \mathbb{C}$ with the representation

$$\begin{aligned} \mathbb{C}^\times \times \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (z, (x, y)) &\mapsto (z^j x, z^{-j} y) \end{aligned}$$

of $\mathbb{C}^\times$. Here, the real form of the torus $\mathbb{C}^\times$ isomorphic to $\mathrm{SO}(2)$ is the unit circle.

Definition 2.4 Let (ρ, V) be a representation of $\mathrm{SO}(2)$. Take $w \in V$. The convex hull of the orbit of w by the action of $\mathrm{SO}(2)$ on V , i.e. the set

$$\mathrm{conv}(\rho(\mathrm{SO}(2))w) = \mathrm{conv}(\{\rho(A)w : A \in \mathrm{SO}(2)\})$$

is called the $\mathrm{SO}(2)$ -orbitope of w with respect to (ρ, V) .

Remark 2.5 Fix a representation (ρ, V) of $\mathrm{SO}(2)$.

- (a) If there is a vector $w \in V$ such that the $\mathrm{SO}(2)$ -orbitope of w with respect to (ρ, V) has non-empty interior then the representation (ρ, V) must be multiplicity-free and must not contain the trivial representation as an irreducible factor.
- (b) Any two $\mathrm{SO}(2)$ -orbitopes with respect to (ρ, V) and with non-empty interior are linearly isomorphic. Further, we can assume up to linear $\mathrm{SO}(2)$ -automorphism that the indices $j_1, \dots, j_r$ of the irreducible components of the representation $\rho = \rho_{j_1} \oplus \dots \oplus \rho_{j_r}$ are relatively prime. See [16], Lemma 5.1 for a proof of these statements.

The following proposition is a special case of [16], Proposition 2.2.

Proposition 2.6 Let (ρ, V) be a representation of $\mathrm{SO}(2)$ and let $C := \mathrm{conv}(\rho(\mathrm{SO}(2))w) \subset V$ be an $\mathrm{SO}(2)$ -orbitope. Then every point of the orbit of which C is the convex hull is an exposed point of C . In particular, the orbit is the set of extreme points of C .

Let us see some examples.

Example 2.7 The convex hull of the orbit of $(1, 0, 1, 0, \dots, 1, 0) \in \mathbb{R}^{2n}$ under the representation $\rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_n$ of $\mathrm{SO}(2)$, for some $n \in \mathbb{N}$, is called the *universal $\mathrm{SO}(2)$ -orbitope* of dimension $2n$. We will denote it by C_n . Explicitly, it is the convex hull of the trigonometric curve

$$\{(\cos(\vartheta), \sin(\vartheta), \cos(2\vartheta), \sin(2\vartheta), \dots, \cos(n\vartheta), \sin(n\vartheta)) \in \mathbb{R}^{2n} : \vartheta \in [0, 2\pi)\}.$$

Every $\mathrm{SO}(2)$ -orbitope is the projection of a universal $\mathrm{SO}(2)$ -orbitope. Sanyal et al. proved (cf. [16], Theorem 5.2) that the universal $\mathrm{SO}(2)$ -orbitope C_n is isomorphic to the spectrahedron of positive semi-definite hermitian Toeplitz matrices of size $(n+1) \times (n+1)$ via the linear map

$$\left\{ \begin{array}{ccc} \mathbb{R}^{2n} & \rightarrow & \mathbf{M}_{n+1}(\mathbb{C}) \\ (x_1, y_1, x_2, y_2, \dots, x_n, y_n) & \mapsto & \begin{pmatrix} 1 & x_1 + iy_1 & \dots & x_n + iy_n \\ x_1 - iy_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1 + iy_1 \\ x_n - iy_n & \dots & x_1 - iy_1 & 1 \end{pmatrix} \end{array} \right.$$

It follows from this theorem that C_n is an n -neighbourly, simplicial convex set. The maximal dimension of a face of C_n is $n-1$.

We now want to collect basic facts about an $\mathrm{SO}(2)$ -orbit from the point of view of real algebraic geometry. In this paper, a variety is a variety defined over $\mathbb{R}$. In the language of schemes, this means that a variety is a separated, reduced scheme of finite type over $\mathbb{R}$. In the classical language, it means that an affine variety is a subset of $\mathbb{C}^n$ defined by real polynomials. These sets define the $\mathbb{R}$ -Zariski topology on $\mathbb{C}^n$.

The real points of an affine variety $X \subset \mathbb{A}^n$, written as $X(\mathbb{R})$, are the points in X that are invariant under complex conjugation acting on $\mathbb{A}^n$. Analogously for a projective variety $X \subset \mathbb{P}^n$.

Proposition 2.8 *Let $\rho: \mathrm{SO}(2) \rightarrow \mathrm{GL}_n(\mathbb{R})$ be a representation of $\mathrm{SO}(2)$. Let O_w be the orbit of $w \in \mathbb{R}^n, w \neq 0$. Denote by X the Zariski closure of O_w in $\mathbb{A}^n$. The variety X is a rational curve and the real points of X are exactly the orbit O_w , i.e. $X(\mathbb{R}) = O_w$.*

Proof The variety $\mathrm{SO}(2)$ is isomorphic to the curve $\{(x, y) \in \mathbb{A}^2: x^2 + y^2 = 1\}$ which is rational over the reals via stereographic projection. Now the Zariski closure X of O_w is the closure of the image of the rational curve $\mathrm{SO}(2)$ under a morphism of real algebraic varieties and therefore also a rational curve. Using that $X = \rho(\mathrm{SO}(2, \mathbb{C}))w$ is smooth (cf. the proof of the following Proposition 2.10) and $\mathrm{SO}(2) = \mathrm{SO}(2, \mathbb{C})(\mathbb{R})$ is compact, the statement about the real points of X can be proven elementarily by birationality of $\mathrm{SO}(2, \mathbb{C})$ and X . $\square$

Definition 2.9 We call the curve $X = \mathrm{cl}_{\mathrm{Zar}}(O_w) \subset \mathbb{A}^n$ of the preceding proposition the curve associated with the $\mathrm{SO}(2)$ -orbitope $\mathrm{conv}(O_w)$. We denote the projective closure of X with respect to the embedding $\mathbb{A}^n \rightarrow \mathbb{P}^n, (x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$ by $\bar{X}$.

Proposition 2.10 *Let $\rho_{j_1} \oplus \rho_{j_2} \oplus \dots \oplus \rho_{j_r}$ be a representation of $\mathrm{SO}(2)$. Let C be the $\mathrm{SO}(2)$ -orbitope of $w \in \mathbb{R}^{2r}$ in this representation and assume that C has non-empty interior. Denote by d the greatest common divisor of $j_1, \dots, j_r$ and by $j = \max\{|j_1|, \dots, |j_r|\}$. Denote by $\bar{X}$ the projective curve associated with the orbitope C .*

(a) *The curve $\bar{X}$ is non-singular if and only if $j - d \in \{|j_1|, \dots, |j_r|\}$.*

(b) *The degree of the curve $\bar{X}$ is $2\frac{j}{d}$.*

Proof Setting $j'_i = |\frac{j_i}{d}|$, we assume that the j_i are relatively prime and $0 < j_1 < j_2 < \dots < j_r = j$. We complexify the situation as explained in Remark 2.3 and get the following parametrisation of the complex orbit of w after a complex change of coordinates:

$$\mathbb{C}^\times \rightarrow \mathbb{C}^{2r}, z \mapsto (z^{j_1}, z^{-j_1}, z^{j_2}, z^{-j_2}, \dots, z^{j_r}, z^{-j_r}).$$

(a) This map extends to a morphism $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^{2r}$ which is given by the equation $\varphi(1 : z) = (z^{j_r} : z^{j_r+j_1} : z^{j_r-j_1} : \dots : z^{2j_r} : 1)$ on the affine chart $\mathbb{P}^1 \setminus \{(0 : 1)\}$ and $\varphi(s : 1) = (s^{j_r} : s^{j_r-j_1} : s^{j_r+j_1} : \dots : 1 : s^{2j_r})$ on $\mathbb{P}^1 \setminus \{(1 : 0)\}$. This morphism is injective: If $y, z \in \mathbb{C}^\times$ with $(y^{j_r}, y^{j_r+j_1}, y^{j_r-j_1}, \dots, y^{2j_r}) = (z^{j_r}, z^{j_r+j_1}, z^{j_r-j_1}, \dots, z^{2j_r})$, then $(y/z)^{j_r} = 1$ and therefore, $(y/z)^{j_i} = 1$ for $i = 1, \dots, r$. Since the j_i are relatively prime, it follows that $(y/z) \in U(j_1) \cap \dots \cap U(j_r) = \{1\}$, where $U(n)$ denotes the group of the n th roots of unity.

The curve $\bar{X}$ is the image of this injective morphism φ . If $\bar{X}$ is smooth, then φ must be an isomorphism, because the inverse rational map extends to a morphism on the non-singular curve $\bar{X}$ ([5], Chap. 7, Corollary 1). In particular, if $\bar{X}$ is smooth, φ is an isomorphism of the structure sheaves and therefore, the differential is an isomorphism. This means that $\bar{X}$ is smooth if and only if the derivative of φ is non-zero at every point.

The derivative of φ is obviously non-zero at every point except for $(1 : 0)$ and $(0 : 1)$. It is non-zero at these points if and only if $j_r - 1 = j_{r-1}$, because only then $z^{j_r-j_{r-1}} = z$ and the gradient does not vanish.

(b) As for the degree, if we take a hyperplane $\mathcal{V}(a_0x_0 + a_1x_1 + b_1y_1 + \dots + a_rx_r + b_ry_r) \subset \mathbb{P}^{2r}$ and intersect it with the image of $\varphi|_{\mathbb{P}^1 \setminus \{(0:1)\}}$, we get the identity

$$a_0z^{j_r} + a_1z^{j_r+j_1} + b_1z^{j_r-j_1} + a_2z^{j_r+j_2} + b_2z^{j_r-j_2} + \dots + a_rz^{2j_r} + b_r = 0.$$

For a general choice of the hyperplane, this is a polynomial of degree $2j_r$ and therefore, it will have $2j_r = 2\frac{j}{d}$ roots in $\mathbb{C}$. $\square$

Remark 2.11 In fact, tracing all changes of coordinates, one can show that the points at infinity $\bar{X} \setminus X$ are always complex, i.e. $\bar{X}(\mathbb{R}) = X(\mathbb{R})$ is the $\mathrm{SO}(2)$ -orbit which is dense in X .

Example 2.12 The (projective closure of the) curve associated with the universal $\mathrm{SO}(2)$ -orbitope C_n is the rational normal curve of degree $2n$ in $\mathbb{P}^{2n}$. Over $\mathbb{C}$, we have seen this in the proof of Proposition 2.10. Over $\mathbb{R}$, it is parametrised by the Chebyshev polynomials of the first kind over the circle $\{(x, y) \in \mathbb{A}^2 : x^2 + y^2 = 1\}$.

3 Convex Hulls of Curves, Secant Varieties and Semi-algebraic Geometry

Definition 3.1 Let $S \subset \mathbb{R}^n$ be a semi-algebraic set.

- (a) The set S is called *basic closed* if there are polynomials $g_1, \dots, g_r \in \mathbb{R}[X_1, \dots, X_n]$ such that

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}.$$

- (b) The *algebraic boundary* $\partial_a S$ of $S \subset \mathbb{R}^n$ is the Zariski closure in $\mathbb{A}^n$ of its boundary ∂S in the euclidean topology.
 (c) The set S is called *regular* if it is contained in the closure of its interior.

Note that every convex semi-algebraic set with non-empty interior is regular and its complement is also regular.

Lemma 3.2 *Let $\emptyset \neq S \subset \mathbb{R}^n$ be a regular semi-algebraic set and suppose that its complement $\mathbb{R}^n \setminus S$ is also regular and non-empty.*

- (a) *The algebraic boundary of S is a variety of pure codimension 1.*
 (b) *If the interior of S intersects the algebraic boundary of S in a regular point then S is not basic closed.*

Proof (a) By [3], Proposition 2.8.13, $\dim(\partial S) \leq n - 1$. Conversely, we prove that every point in the boundary ∂S of S has local dimension $n - 1$ in ∂S : Let $x \in \partial S$ be a point and take $\varepsilon > 0$. Then $\text{int}(S) \cap B(x, \varepsilon)$ and $\text{int}(\mathbb{R}^n \setminus S) \cap B(x, \varepsilon)$ are non-empty, because both S and $\mathbb{R}^n \setminus S$ are regular. Applying [3], Lemma 4.5.2, yields that

$$\dim(\partial S \cap B(x, \varepsilon)) = \dim(B(x, \varepsilon) \setminus (\text{int}(S) \cup (\mathbb{R}^n \setminus \bar{S}))) \geq n - 1.$$

Therefore, all components of $\partial_a S = \text{cl}_{\text{Zar}}(\partial S)$ have dimension $n - 1$.

(b) Assume that S is basic closed, i.e. there are polynomials $g_1, \dots, g_r \in \mathbb{R}[x_1, \dots, x_n]$ such that $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$. Since S is regular, in every point of the boundary of S , at least one polynomial $g_k \in \{g_1, \dots, g_r\}$ must change sign. Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial defining an irreducible component of $\partial_a S$ intersecting the interior of S in a regular point. There is a Zariski dense subset $M \subset \mathcal{V}(h)_{\text{reg}}(\mathbb{R})$ that is contained in ∂S . Since $M = \bigcup_{k=1}^r \mathcal{V}(g_k) \cap M$, we get

$$\mathcal{V}(h) = \overline{M}^{\text{Zar}} \subset \bigcup_{k=1}^r \mathcal{V}(g_k) \cap \overline{M}^{\text{Zar}}.$$

It follows from the irreducibility of $\mathcal{V}(h)$ that $\mathcal{V}(h) \subset \mathcal{V}(g_j)$ for some g_j that changes sign along M . Consequently, g_j changes sign in every point of the set $\mathcal{V}(h)_{\text{reg}} \cap \text{int}(S) \neq \emptyset$, which is a contradiction to $S \subset \{x \in \mathbb{R}^n : g_j(x) \geq 0\}$. $\square$

Example 3.3 Let $g := x^2 + y^2 - 1 \in \mathbb{R}[x, y]$. The union of the closed disc $\{(x, y) \in \mathbb{R}^2 : g(x, y) \leq 0\}$ with the line defined by $y = 0$ is basic closed, defined by the inequality $y^2 g \leq 0$ and the algebraic boundary of this union has two components, namely the circle $\mathcal{V}(g)$ and the line $\mathcal{V}(y)$. The origin is a regular point of this hypersurface. This shows that the assumption on S being regular in the above lemma cannot be dropped in (b).

For statement (a), we just have to do the same example in $\mathbb{R}^3$: Write $h := x^2 + y^2 + z^2 - 1 \in \mathbb{R}[x, y, z]$. The union of the ball $\{(x, y, z) \in \mathbb{R}^3 : h(x, y, z) \leq 0\}$ with the line defined by $y = 0$ and $z = 0$ is basic closed, defined by the two inequalities $y^2 h \leq 0$ and $z^2 h \leq 0$. The algebraic boundary of this union consists of the sphere $\mathcal{V}(h)$ and the line $\mathcal{V}(y, z)$. It is a hypersurface with a lower dimensional component.

We want to characterise, when the secant variety is a component of the algebraic boundary of the convex hull of a curve.

Definition 3.4 Let $X \subset \mathbb{P}^n$ be an embedded quasi-projective variety. A *secant k -plane* to X is a k -dimensional linear space in $\mathbb{P}^n$ that is spanned by $k + 1$ points on X . The *k th secant variety* $S_k(X)$ of X is the Zariski closure of the union of all secant k -planes to X .

Before we can state the theorem, we want to observe that the set of all k -tuples of points spanning a face is semi-algebraic:

Remark 3.5 Let $N \subset \mathbb{R}^n$ be a semi-algebraic set and $r \in \mathbb{N}$. The subset $M \subset N \times \cdots \times N$ of the r -fold product of N which contains all r -tuples of points whose convex hull is a face of the convex hull of N is a semi-algebraic set: The set M is the set of all points where a first order formula in the language of ordered fields is satisfied, namely the definition of a face, i.e. for all $x, y \in \text{conv}(N)$ if $\frac{1}{2}(x + y)$ is in the convex hull of the free variables $x_1, \dots, x_r \in \mathbb{R}^n$, then so are x and y .

We now come to the most important result of this section. It will be used in the following sections to show that the secant variety is an irreducible component of the algebraic boundary of certain $\text{SO}(2)$ -orbitopes.

Theorem 3.6 Let $X \subset \mathbb{A}^{2r}$ be an irreducible curve and assume that the real points $X(\mathbb{R})$ of X are Zariski-dense in X . Let C be the convex hull of $X(\mathbb{R}) \subset \mathbb{R}^{2r}$ and suppose that the interior of C is non-empty. Let $M \subset X \times X \times \cdots \times X$ be the semi-algebraic subset of the r -fold product of X defined as the set of all r -tuples of real points whose convex hull is a face of C . Then the $(r - 1)$ -th secant variety to X is an irreducible component of the algebraic boundary of C if and only if the dimension of M is r .

Proof The $(r - 1)$ th secant variety $S_{r-1}(X)$ to X is a hypersurface (cf. [13]), because it follows from the assumption that C has non-empty interior that the curve is not contained in any hyperplane. Note that $S_{r-1}(X)$ is irreducible as the secant variety to an irreducible curve. It is contained in the algebraic boundary of C if and only if the dimension of its intersection $S_{r-1}(X) \cap \partial C$ with the boundary of C has codimension 1 as a semi-algebraic set.

Set $M_0 := M \setminus V(\mathbb{R})$ where $V \subset X \times \cdots \times X$ is the subvariety of all r -tuples of points on X which are affinely dependent. If it is non-empty, it is a semi-algebraic set of dimension $\dim(M)$. Consider the map

$$\Phi: \begin{cases} M_0 \times \Delta_{r-1} & \rightarrow \mathbb{R}^{2r} \\ ((x_1, \dots, x_r), (\lambda_1, \dots, \lambda_r)) & \mapsto \sum_{i=1}^r \lambda_i x_i \end{cases}$$

This is a semi-algebraic map and the image under Φ of $M_0 \times \Delta_{r-1}$ is contained in the intersection $S_{r-1}(X) \cap \partial C$ by definition of M_0 . We claim that $\dim(\Phi(M_0 \times \Delta_{r-1})) = 2r - 1$ if and only if $\dim(S_{r-1}(X) \cap \partial C) = 2r - 1$: If the dimension of $S_{r-1}(X) \cap \partial C$ is $2r - 1$, then there exist $x \in S_{r-1}(X) \cap \partial C$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \cap S_{r-1}(X)$ is contained in ∂C . Since it is also dense in $S_{r-1}(X)$, every Zariski-open subset of $S_{r-1}(X)$ intersects this set in a non-empty set, which is then open in the euclidean topology. So $S_{r-1}(X) \cap \partial C$ contains general points of the $(r - 1)$ -th secant variety. Since the union of all secant $(r - 1)$ -planes to X is a constructible set in the Zariski topology, it contains a Zariski open subset of the $(r - 1)$ th secant variety. Therefore, there is a point $x \in S_{r-1}(X) \cap \partial C$ which lies on a secant $(r - 1)$ -plane to X and an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap S_{r-1}(X)$ is contained in the euclidean boundary of C and the image of Φ , i.e. these points all lie on secant $(r - 1)$ -planes to X spanned by real points. Therefore, the image of Φ has dimension $2r - 1$. The converse of the claimed equivalence is trivial, because $\Phi(M_0 \times \Delta_{r-1}) \subset S_{r-1}(X) \cap \partial C$. From the claim, it follows that, if $S_{r-1}(X) \subset \partial_a C$, the dimension of M_0 is r by a count of dimensions in the source of Φ and [3], Theorem 2.8.8.

Conversely, assume that the dimension of M_0 is r . Denote by $\text{Gr}(\Phi)$ the graph of the map Φ in $(M_0 \times \Delta_{r-1}) \times \mathbb{R}^{2r}$ and by π_2 the projection of this product to the second factor $\mathbb{R}^{2r}$. The fibre of a generic real point in $S_{r-1}(X)$ under this projection is finite, because a general point on this secant variety lies on only finitely many secant $(r - 1)$ -planes to X . This implies that the image of Φ , which is the same as $\pi_2(\text{Gr}(\Phi))$, is locally homeomorphic to the graph of Φ . This can be seen by a cylindrical decomposition of the semi-algebraic set $\text{Gr}(\Phi)$ adapted to the projection π_2 (cf. [2], Chap. 5.1): Over every open cell of the decomposition of $S_{r-1}(X)(\mathbb{R})$ into semi-algebraic sets, there are only graphs and no bands, so the projection π_2 is a local homeomorphism of $\text{Gr}(\Phi_0)$ with the image of Φ . Since the graph of Φ is in turn homeomorphic to the source of Φ , it follows that the dimension of $\Phi(M_0 \times \Delta_{r-1}) \subset S_{r-1}(X) \cap \partial C$ is $r + r - 1 = 2r - 1$. $\square$

We will mostly use this more explicit corollary to the above theorem.

Corollary 3.7 *Let $X \subset \mathbb{A}^{2r}$ be an irreducible curve and assume that the real points of X are Zariski-dense in X . Set $C := \text{conv}(X(\mathbb{R})) \subset \mathbb{R}^{2r}$ and suppose that C has non-empty interior. Then the $(r - 1)$ -th secant variety to X is an irreducible component of the algebraic boundary of C if and only if there are r real points $x_1, \dots, x_r \in X(\mathbb{R})$ of X and semi-algebraic neighbourhoods $U_j \subset X(\mathbb{R})$ of x_j for $j = 1, \dots, r$ such that for all $(y_1, \dots, y_r) \in U_1 \times \dots \times U_r$, the convex hull $\text{conv}(y_1, \dots, y_r)$ is a face of C .*

Proof That the $(r - 1)$ th secant variety to X is an irreducible component of $\partial_a C$ means that M as in the above notation has dimension r . The euclidean topology of $X(\mathbb{R}) \times \dots \times X(\mathbb{R})$ is the product topology. So M contains a set of the form $U_1 \times \dots \times U_r$ for open semi-algebraic sets $U_j \subset X$ if and only if it has dimension r . $\square$

For the universal $\text{SO}(2)$ -orbitopes, the result [16], Theorem 5.2, of Sanyal et al. gives a complete description of the algebraic boundary (see also 2.7).

Example 3.8 The algebraic boundary of the universal $\mathrm{SO}(2)$ -orbitope of dimension $2n$ is defined by the vanishing of the determinant

$$\det \begin{pmatrix} 1 & x_1 + iy_1 & \dots & x_n + iy_n \\ x_1 - iy_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1 + iy_1 \\ x_n - iy_n & \dots & x_1 - iy_1 & 1 \end{pmatrix}.$$

It has real coefficients and is the (dehomogenisation of the) equation of the $(n - 1)$ th secant variety to the curve $\tilde{X}_n$ associated with C_n .

More generally, for $k < n$, the k th secant variety to the curve $\tilde{X}_n$ is defined by the $(k + 2) \times (k + 2)$ minors of that matrix. The union of all k -dimensional faces of C_n is Zariski dense in the k th secant variety to $\tilde{X}_n$.

We take a closer look at the real points of the secant variety. We eventually show that every real point on a secant spanned by regular real points is a central point.

Definition 3.9 Let X be a variety. A real point $x \in X(\mathbb{R})$ of X is called a *central point* of X if it has full local dimension in the set of real points, i.e.

$$\dim_x(X(\mathbb{R})) = \dim(X).$$

Corollary 3.10 Let $X \subset \mathbb{P}^n$ be an irreducible variety that is not contained in any hyperplane. Assume that the real points of X are Zariski-dense in X . Take $x_0, \dots, x_k \in X_{\mathrm{reg}}(\mathbb{R})$ to be regular real points of X that span a secant k -plane Λ to X . Then every real point $y \in \Lambda$ is a central point of the k -th secant variety:

$$\dim_y(S_k(X)(\mathbb{R})) = \dim(S_k(X)).$$

In particular, the union of all k -dimensional real projective spaces spanned by $k + 1$ real points of X is a Zariski-dense subset of $S_k(X)$.

Proof The statement follows from upper semi-continuity of the local dimension if the points $x_0, \dots, x_k$ are general, because in that case, Terracini's Lemma (cf. [4], Proposition 4.3.2) says that the general point on the secant k -plane spanned by these points is a regular point of $S_k(X)$. And upper semi-continuity of the local dimension follows for example from the fact that every closed semi-algebraic set can be locally triangulated (cf. [3], Sect. 9.2, Theorem 9.2.1).

If we take regular points $x_0, \dots, x_k \in X_{\mathrm{reg}}(\mathbb{R})$, then, since the real points of the curve X are Zariski-dense in X , we can find for every $\varepsilon > 0$ a tuple $x'_0, \dots, x'_k \in X_{\mathrm{reg}}(\mathbb{R})$ of general real points such that $\|x_j - x'_j\| < \varepsilon$ (the reason is that $B(x_j, \varepsilon) \cap X(\mathbb{R})$ is Zariski-dense in X for all $\varepsilon > 0$ by [3], Proposition 7.6.2).

Now, if $y = \sum_{j=0}^k \lambda_j x_j$ with $\lambda_j \in \mathbb{R}$, $\sum_{j=0}^k \lambda_j = 1$, then

$$\left\| y - \sum_{j=0}^k \lambda_j x'_j \right\| = \left\| \sum_{j=0}^k \lambda_j (x_j - x'_j) \right\| \leq \sum_{j=0}^k \lambda_j \|x_j - x'_j\| < \sum_{j=0}^k \lambda_j \varepsilon = \varepsilon.$$

Therefore, we can find a regular real point of $S_k(X)$ in every euclidean neighbourhood of y . By the preceding remark, this is equivalent to the claim. $\square$

4 Four-Dimensional $\mathrm{SO}(2)$ -Orbitopes

Smilansky completely characterised the face lattice of an arbitrary 4-dimensional $\mathrm{SO}(2)$ -orbitope in his paper [17]. Let $p, q \in \mathbb{N}$ be relatively prime integers with $p < q$, let $\rho = \rho_p \oplus \rho_q$ be the corresponding 4-dimensional representation of $\mathrm{SO}(2)$. Denote by C_{pq} the convex hull of the orbit of $(1, 0, 1, 0)$. He proved that C_{pq} always has a 2-dimensional family of edges. Furthermore, if $q \geq 3$, there is a 1-dimensional family of 2-dimensional q -gons among the faces of C_{pq} . The vertices of these q -gons correspond to the q th roots of unity. Analogously, if $p \geq 3$, there is also a 2-dimensional family of p -gons among the faces of C_{pq} . This theorem has two immediate consequences that we want to emphasise:

- (1) A 4-dimensional $\mathrm{SO}(2)$ -orbitope C_{pq} is simplicial (i.e. all faces are simplices) if and only if $(p, q) = (1, 2)$ or $(p, q) = (1, 3)$.
- (2) A 4-dimensional $\mathrm{SO}(2)$ -orbitope has non-exposed faces (also called facelets) if and only if $(p, q) \neq (1, 2)$. Combining this with the theorem of Sanyal et al. about universal orbitopes (cf. [16], Theorem 5.2) and the fact that every face of a spectrahedron is exposed (cf. [15], Corollary 1), it is immediate that a 4-dimensional $\mathrm{SO}(2)$ -orbitope is a spectrahedron if and only if it is universal. An even stronger statement is true (cf. Corollary 4.2).

We investigate the algebraic boundary of 4-dimensional $\mathrm{SO}(2)$ -orbitopes.

Theorem 4.1 *Let p and q be relatively prime integers, $q > p$. Choose the coordinates $\mathbb{R}^4 \subset \mathbb{A}^4 = \{(w, x, y, z)\}$ and denote by X_{pq} the curve associated with C_{pq} . The algebraic boundary of C_{pq} is*

$$\partial_a C_{pq} = \begin{cases} S_1(X_{pq}) & \text{if } p = 1, q = 2, \\ S_1(X_{pq}) \cup \mathcal{V}(y^2 + z^2 - 1) & \text{if } p \in \{1, 2\}, q \geq 3, \\ S_1(X_{pq}) \cup \mathcal{V}(w^2 + x^2 - 1) \cup \mathcal{V}(y^2 + z^2 - 1) & \text{if } p \geq 3. \end{cases}$$

Proof The fact that the secant variety to the curve X_{pq} associated with C_{pq} is a component of the algebraic boundary of C_{pq} follows from Theorem 3.6 and the list of 1-dimensional faces of C_{pq} because there is always a 2-dimensional family of edges.

The case of the universal 4-dimensional orbitope, i.e. $p = 1, q = 2$, follows from [16], Theorem 5.2 (cf. Example 3.8).

Next, consider the case $p = 1$ or $p = 2$ and $q \geq 3$. Then the boundary of C_{pq} consists of a 2-dimensional family of edges and a 1-dimensional family of regular q -gons. The union of the q -gons is a semi-algebraic set of dimension 3: Consider the semi-algebraic map

$$\begin{cases} (0, 1) \times \text{relint}(\Delta_2) \rightarrow \mathbb{R}^4 \\ (t, (\lambda_0, \lambda_1, \lambda_2)) \mapsto \lambda_0 z(t) + \lambda_1 z\left(t + \frac{1}{q}\right) + \lambda_2 z\left(t + \frac{2}{q}\right) \end{cases}$$

which is injective, because three vertices of a regular q -gon are affinely independent and the relative interiors of the q -gons in the boundary of C_{pq} are disjoint. By [3], Theorem 2.8.8, it follows that the image has dimension 3. To calculate the Zariski closure of this set, note that the last two components of the vectors $z(t)$, $z(t + \frac{1}{q})$ and $z(t + \frac{2}{q})$ are equal, and therefore the same is true for every element in the convex hull of these three points. This implies that the image is contained in the hypersurface $\mathcal{V}(y^2 + z^2 - 1)$, which is irreducible. Therefore, the Zariski closure of the image is this hypersurface. This shows $S_1(X_{pq}) \cup \mathcal{V}(y^2 + z^2 - 1) \subset \partial_a C_{pq}$ and since every face of C_{pq} is contained in this variety, there are no further components in this case.

The case $p \geq 3$ is completely analogous to the last case. The new component $\mathcal{V}(w^2 + x^2 - 1)$ is the Zariski closure of the regular p -gons that lie in the boundary of C_{pq} . $\square$

Corollary 4.2 *Let C be a 4-dimensional $\mathrm{SO}(2)$ -orbitope. The following are equivalent:*

- (a) C is linearly isomorphic to the universal $\mathrm{SO}(2)$ -orbitope C_2 .
- (b) C is a spectrahedron.
- (c) C is a basic closed semi-algebraic set.

Proof The implication from (a) to (b) is [16], Theorem 5.2, (b) to (c) is linear algebra: A spectrahedron $\{(x_1, \dots, x_n) \in \mathbb{R}^n : A_0 + x_1 A_1 + \dots + x_n A_n \geq 0\}$ can be defined in terms of polynomial inequalities by simultaneous sign conditions on the minors of the matrix inequality. We prove the implication from (c) to (a) by contraposition:

Let C be a 4-dimensional $\mathrm{SO}(2)$ orbitope which is not linearly isomorphic to the universal orbitope. Then the algebraic boundary consists of at least two components, one of which is the secant variety to the curve X associated with the orbitope C . Smilansky's list of all faces shows that there is a line segment joining two points $X_{\mathrm{reg}}(\mathbb{R})$ of the orbit associated with C that intersects the interior of C . This point has full local dimension in the real points of the secant variety $S_1(X)(\mathbb{R})$ to X by Corollary 3.10. By Lemma 3.2 we conclude that C is not basic closed. $\square$

Remark 4.3 The degree of the algebraic boundary of a 4-dimensional $\mathrm{SO}(2)$ -orbitope can be computed if the curve associated with it is smooth, or more precisely its projective closure: If $X \subset \mathbb{P}^n$ is a smooth curve of degree d and genus g , $n \geq 4$, then the secant variety to X has degree

$$\deg(S_1(X)) = \frac{(d-1)(d-2)}{2} - g$$

(see [4], Sect. 8.2, p. 259). So, if C_{pq} is a 4-dimensional $\mathrm{SO}(2)$ -orbitope and $p = q - 1$, then the (projective closure of the) curve X_{pq} associated with it is smooth, has degree $2q$ and genus 0 (cf. Proposition 2.10). In this case, the algebraic boundary of C_{pq} has degree

$$\deg(\partial_a C_{pq}) = \frac{(2q-1)(2q-2)}{2} + 4$$

for $p \geq 3$ and degree 12 for $p = 2$, $q = 3$.

Example 4.4 We explicitly compute the algebraic boundary of the 4-dimensional Barvinok–Novik orbitope $B_4 = C_{13}$ —we will introduce the family of Barvinok–Novik orbitopes in Sect. 5. This means that we have to compute the equation of the secant variety to the curve X_{13} associated with C_{13} . We will use the ideal defining the secant variety to the curve associated with the universal $\mathrm{SO}(2)$ -orbitope C_3 , which is given by the 3×3 minors of the linear matrix inequality defining C_3 , cf. Example 3.8.

The union of all lines joining two general real points of X_{13} is a Zariski-dense subset of the secant variety $S_1(\bar{X}_{13})$ because the real points of X_{13} are by definition Zariski-dense in X_{13} (cf. Corollary 3.10). The projection from $\mathbb{R}^6$ to $\mathbb{R}^4$ that projects C_3 onto C_{13} gives a bijection if restricted to the union of all lines joining two real points of the curve X_3 associated with C_3 because it is a bijection, if restricted to the orbit $X_{3,\mathrm{reg}}(\mathbb{R})$. Therefore, the secant variety $S_1(X_{13})$ is the image of the secant variety $S_1(X_3)$ under this projection. This leads to an elimination problem. The author solved it using the computer algebra system Macaulay2 [6]. In the coordinates $\mathbb{A}^4 = \{w, x, y, z\}$, the equation of the secant variety is the following polynomial f of degree 8 and 47 terms:

$$\begin{aligned} f = & -36w^4x^2y^2 + 24w^2x^4y^2 - 4x^6y^2 + 24w^5xyz - 80w^3x^3yz \\ & + 24wx^5yz - 4w^6z^2 + 24w^4x^2z^2 - 36w^2x^4z^2 + 4w^6 + 12w^4x^2 \\ & + 12w^2x^4 + 4x^6 - 12w^5y + 24w^3x^2y + 36wx^4y + 12w^4y^2 + 24w^2x^2y^2 \\ & + 12x^4y^2 - 4w^3y^3 + 12wx^2y^3 - 36w^4xz - 24w^2x^3z + 12x^5z \\ & - 12w^2xy^2z + 4x^3y^2z + 12w^4z^2 + 24w^2xz^2 + 12x^4z^2 - 4w^3yz^2 \\ & + 12wx^2yz^2 - 12w^2xz^3 + 4x^3z^3 - 3w^4 - 6w^2x^2 - 3x^4 + 8w^3y - 24wx^2y \\ & - 6w^2y^2 - 6x^2y^2 + y^4 + 24w^2xz - 8x^3z - 6w^2z^2 - 6x^2z^2 + 2y^2z^2 + z^4. \end{aligned}$$

5 Barvinok–Novik Orbitopes

Definition 5.1 For any odd integer $n \in \mathbb{N}$, we consider the direct sum of representations $\rho = \rho_1 \oplus \rho_3 \oplus \cdots \oplus \rho_n$ of $\mathrm{SO}(2)$ indexed by all odd integers from 1 to n . The $(n+1)$ -dimensional Barvinok–Novik orbitope, denoted by B_{n+1} , is the convex hull of the orbit of $(1, 0, 1, 0, \dots, 1, 0) \in (\mathbb{R}^2)^{\frac{n+1}{2}}$ under the representation ρ . Explicitly, it is the convex hull of the symmetric trigonometric moment curve

$$\{(\cos(\vartheta), \sin(\vartheta), \cos(3\vartheta), \sin(3\vartheta), \dots, \cos(n\vartheta), \sin(n\vartheta)) : \vartheta \in [0, 2\pi]\}.$$

Proposition 5.2 Every Barvinok–Novik orbitope is a simplicial compact convex set.

Proof Let $n \geq 3$ be an odd integer. We will prove that any $n+1$ points on the orbit

$$\{(z, z^3, \dots, z^n) : z \in \mathbb{C}^*, z\bar{z} = 1\}$$

are $\mathbb{R}$ -affinely linearly independent. Take pairwise distinct points $z_0, \dots, z_n \in \mathbb{C}^*$ with $z_j\bar{z}_j = 1$. The corresponding points $(z_j, z_j^3, \dots, z_j^n)$ on the orbit are $\mathbb{R}$ -affinely

linearly independent if and only if we can conclude from the equations

$$\sum_{j=0}^n a_j = 0,$$

$$\sum_{j=0}^n a_j \begin{pmatrix} z_j \\ z_j^3 \\ \vdots \\ z_j^n \end{pmatrix} = 0$$

and $a_j \in \mathbb{R}$ ($j = 0, \dots, n$) that all the coefficients a_j are zero. This is true if the $(n+2) \times (n+1)$ -matrix

$$\begin{pmatrix} \overline{z_0}^n & \overline{z_1}^n & \dots & \overline{z_n}^n \\ \vdots & \vdots & & \vdots \\ \overline{z_0}^3 & \overline{z_1}^3 & \dots & \overline{z_n}^3 \\ \overline{z_0} & \overline{z_1} & \dots & \overline{z_n} \\ 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_n \\ z_0^3 & z_1^3 & \dots & z_n^3 \\ \vdots & \vdots & & \vdots \\ z_0^n & z_1^n & \dots & z_n^n \end{pmatrix}$$

has full rank $n+1$ over $\mathbb{C}$. By using Vandermonde's rule, we prove that the determinant of the matrix obtained from the one above by deleting the row of ones does not vanish. To see this, we first rescale the j th column of the new $(n+1) \times (n+1)$ matrix, i.e. the column with z_{j-1} , by z_{j-1}^{-1} . Using the identity $z_j^{-1} \overline{z_j} = z_j^{-2}$, we get the matrix

$$\begin{pmatrix} z_0^{-2} \overline{z_0}^{n-1} & z_1^{-2} \overline{z_1}^{n-1} & \dots & z_n^{-2} \overline{z_n}^{n-1} \\ \vdots & \vdots & & \vdots \\ z_0^{-2} \overline{z_0}^2 & z_1^{-2} \overline{z_1}^2 & \dots & z_n^{-2} \overline{z_n}^2 \\ z_0^{-2} & z_1^{-2} & \dots & z_n^{-2} \\ 1 & 1 & \dots & 1 \\ z_0^2 & z_1^2 & \dots & z_n^2 \\ \vdots & \vdots & & \vdots \\ z_0^{n-1} & z_1^{n-1} & \dots & z_n^{n-1} \end{pmatrix}$$

This gives a Vandermonde matrix if we rescale the j th column by z_{j-1}^{n+1} and substitute $y_j := z_j^2$. We conclude that the determinant of this matrix does not vanish. This proves the claim. $\square$

We have already seen that the 4-dimensional Barvinok–Novik orbitope B_4 is not basic closed (cf. Corollary 4.2). We proved this by calculating the algebraic boundary

of B_4 . By the description of its faces, we saw that the algebraic boundary of this convex set intersects the interior in regular points. The component on which these points lie is the secant variety to the curve associated with B_4 . We will now generalise this to higher dimensions by examining higher secant varieties.

The essential ingredient from convex geometry is the result [1], Theorem 1.2, stating that Barvinok–Novik orbitopes are locally $(n-1)/2$ -neighbourly, i.e. the convex hull of any $(n-1)/2$ points on the trigonometric moment curve form a face if these points lie sufficiently close together.

Theorem 5.3 *Let $n \in \mathbb{N}$ be an odd integer greater than 2. Denote by X_{n+1} the curve associated with the $(n+1)$ -dimensional Barvinok–Novik orbitope B_{n+1} . The $\frac{n-1}{2}$ -th secant variety to $\bar{X}_{n+1}$ is an irreducible component of the algebraic boundary of B_{n+1} .*

Proof Set $k := \frac{n-1}{2}$. Firstly, the origin is an interior point of the Barvinok–Novik orbitope B_{n+1} because it is an interior point of all universal $\text{SO}(2)$ -orbitopes (cf. [16], Theorem 5.2) and B_{n+1} is a linear projection of C_n . Therefore, X_{n+1} is not contained in any hyperplane. So by [13], the dimension of $S_k(X_{n+1})$ equals

$$2k+1 = 2\frac{n-1}{2} + 1 = n.$$

Because it is the secant variety to an irreducible curve, it is irreducible.

To see that it is a component of the algebraic boundary of B_{n+1} , observe that the result [1], Theorem 1.2, states that the semi-algebraic set M in Theorem 3.6 has non-empty interior. $\square$

Corollary 5.4 *No Barvinok–Novik orbitope is a basic closed semi-algebraic set.*

Proof The $\frac{n-1}{2}$ -th secant variety to the curve X_{n+1} associated with B_{n+1} is a component of the algebraic boundary. The origin lies on this component because it lies on the line joining $(1, 1, 1, \dots, 1) \in X_{n+1}(\mathbb{R})$ and $(-1, -1, -1, \dots, -1) \in X_{n+1}(\mathbb{R})$. It is a central point of $S_k(X_{n+1})$ by Corollary 3.10, i.e. in every euclidean neighbourhood of the origin there is a regular point of $S_k(X_{n+1})$. By Lemma 3.2, this implies that the Barvinok–Novik orbitope is not basic closed. $\square$

In the special case of B_4 , we look into this argument more concretely by considering a fortunately chosen slice of the convex set.

Example 5.5 We intersect B_4 with the subspace $W := \{(0, x, 0, z) \in \mathbb{R}^4 : x, z \in \mathbb{R}\}$. The polynomials defining the irreducible components of $\partial_a B_4$ restricted to this subspace factor $0^2 + z^2 - 1 = (z+1)(z-1)$ and $f(0, x, 0, z) = (x+z)^3(4x^3 - 3x + z)$ (cf. Fig. 1). The polynomial $4x^3 - 3x + z$ is part of the algebraic boundary of the convex and semi-algebraic set $W \cap B_4$ but the origin is an interior point of $W \cap B_4$ and a regular point of the hypersurface $\mathcal{V}(4x^3 - 3x + z)$. Using Lemma 3.2, we can conclude from this that B_4 is not basic closed.

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