Highly Incident Configurations with Chiral Symmetry

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Abstract A geometric k -configuration is a collection of points and straight lines in the plane so that k points lie on each line and k lines pass through this point. We introduce a new construction method for constructing k -configurations with non-trivial dihedral or chiral (i.e., purely rotational) symmetry, for any $k \geq 3$; the configurations produced have 2^{k-2} symmetry classes of points and lines. The construction method produces the only known infinite class of symmetric geometric 7-configurations, the second known infinite class of symmetric geometric 6-configurations, and the only known 6-configurations with chiral symmetry.

Keywords Configurations · Incidence geometry · Symmetry

1 Introduction

A k -configuration, also known as a (n_k) geometric configuration, is a collection of n points and n straight lines, typically in the Euclidean or projective plane, so that each point lies on k lines and each line passes through k points. Constructing new configurations by using geometric symmetries has proven to be a highly productive technique; see, e.g., [3, 7, 9, 10, 12]. However, most of the previous work on configurations has focused on constructions that produce 3- and 4-configurations; relatively little work has been done on producing or classifying k -configurations for $k > 4$.

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Fig. 1 Two versions of the (20_4) configuration. **a** Branko Grünbaum's original diagram for a (20_4) configuration, from [11]. **b** The same configuration, labelled to show it corresponds to $\mathcal{A}(5; 2, 2; 1', 3)$; this symbol will be described in the text (Color figure online)

The configurations in this paper were initially found when one of the authors was trying to reconstruct the single known geometric (20_4) configuration, which was found by Branko Grünbaum a few years ago. The first published diagram of the configuration appeared in 2008, in [11]; that diagram is reproduced as our Fig. 1. In that article, he very briefly described his construction technique, saying “What I first managed to do was to construct a (20_4) configuration of pseudolines, by merging two (10_3) configurations. Looking at the result, I wondered whether one could straighten the pseudolines. As it turned out, this is possible...”

The construction outlined in this paper is a significantly different construction from the one used by Grünbaum. In fact, his configuration is the first in an infinite family of configurations. Not only is the configuration generalizable to other dihedrally symmetric 4-configurations (which Grünbaum had noticed very soon after the previous discovery (private communication, and also discussed in [12, Chap. 3])), but the construction algorithm described in this paper also extends to produce chirally symmetric 4-configurations, and to k -configurations for arbitrary k . The construction technique, with several worked-out examples, will be described in the next section, and subsequent sections will prove the validity of the technique.

In particular, this new construction technique produces the first known 6-configurations with chiral symmetry (the infinite family of 6-configurations constructed in [4, 5] have dihedral symmetry) and the first known infinite class of symmetric 7-configurations at all.

2 The Configuration Construction Algorithm

Throughout this paper, the notation $\mathcal{C}_O(Q)$ refers to a circle centered at point O and passing through point Q . The notation $\mathcal{C}(P, Q, R)$ refers to the circumcircle of points

P , Q , and R . The line between points P and Q is denoted by either $P \vee Q$ or $\overline{PQ}$, and the intersection of lines ℓ and m is denoted $\ell \wedge m$.

Definition 1 Given a regular convex m -gon of points cyclically labelled as w_i , a *line of span s* is any line $w_i \vee w_{i+s}$.

Definition 2 Given a regular convex m -gon of points centered at O , cyclically labelled as w_i , the *circle of span s* through w_j is the circumcircle $\mathcal{C}(w_j, O, w_{j-s})$; note that the line $w_{j-s} \vee w_j$ is a span s line.

Definition 3 Given a regular convex m -gon of points centered at O , cyclically labelled as w_i , and integers a, b with $1 \leq a, b \leq \lfloor \frac{m}{2} \rfloor$, an integer d is *well-defined* if the circle $\mathcal{C}(w_d, O, w_{d-b})$ intersects the line $w_0 \vee w_a$ in at least one point.

The algorithm for the construction of a configuration with symbol $\mathcal{A}(m; a, b; d_1, d_2, \dots, d_{k-2})$ is iterative. The parameters m, a, b and k are all positive integers, with $1 \leq a, b < \frac{m}{2}, k \geq 3$; $d_1, \dots, d_{k-2}$ are integers between 0 and $m-1$, modulo m . A number of nontrivial constraints are required to ensure the incidence structure produced by the algorithm is a configuration. A detailed discussion of parameters occurs in Sect. 6. The configurations $\mathcal{A}(m; a, b; d)$ were discussed by Grünbaum in [12, Sect. 2.7] as a class of chiral *astral* 3-configurations; that is, as 3-configurations with chiral geometric symmetries and precisely two symmetry classes of points and lines. In that reference, they were denoted $m\#(b, c; d)$.

2.1 The Construction Algorithm

To construct $\mathcal{A}(m; a, b; d_1, \dots, d_{k-2})$ where each d_j is a well-defined parameter, let $N = k - 2$, and define $S = \{1, 2, \dots, N\}$ to be the index set associated with $\{d_1, \dots, d_N\}$. For each subset $\sigma \subseteq S$, we will construct a symmetry class of points and a symmetry class of lines indexed by σ . All indices are taken modulo m .

Step 0: Construct a regular convex m -gon with center O , and label the points cyclically as

$$(v)_0, (v)_1, \dots, (v)_{m-1}.$$

Here, we will interpret $(v)_i$ as $(v_\emptyset)_i$, but for notational convenience, we suppress the $\emptyset$ subscript.

Then construct span a lines $(L)_i = (v)_i \vee (v)_{i+a}$. (Again, by $(L)_i$, we mean $(L_\emptyset)_i$.)

Step 1: For each d_j , construct the span b circle $c_j := \mathcal{C}((v)_{d_j}, O, (v)_{d_j-b})$. For each c_j , let $(v_j)_0$ be a point of intersection of c_j and $(L_0)_0$; since the parameter d_j is well-defined, such an intersection point must exist. Again, we interpret v_j as $v_{\{j\}}$ but suppress the brackets for convenience. Define $(v_j)_i$ to be the rotation of $(v_j)_0$ by $\frac{2\pi i}{m}$ about O . Then construct span b lines $(L_j)_i = (v_j)_i \vee (v_j)_{i+b}$.

Step n , for $1 < n \leq k-2$: Let σ be any subset of S of size n . For any q in σ , define σ_q to be the set $\sigma \setminus \{q\}$.

Construct vertex $(v_\sigma)_0$ to be the common intersection of all lines $(L_{\sigma_q})_0$ for each $q \in \sigma$. (It remains to show that this vertex $(v_\sigma)_0$ is well-defined.) Define $(v_\sigma)_i$ to be the rotation of $(v_\sigma)_0$ by $\frac{2\pi i}{m}$ about $\mathcal{O}$.

Next, define $(L_\sigma)_i := (v_\sigma)_i \vee (v_\sigma)_{i+\delta_n}$, where

$$\delta_n = \begin{cases} a & \text{if } n \text{ is even;} \\ b & \text{if } n \text{ is odd.} \end{cases}$$

Definition 4 A well-defined parameter d_j is *valid* if the circle c_j intersects line $(L)_0$ at a point which does not lie on any $(L)_{-t}$ other than $(L)_0$.

In Sect. 4, we show that the points $(v_\sigma)_i$ are well-defined and discuss properties of the lines $(L_\sigma)_i$. In Sect. 5, we show that, given valid parameters, the algorithm produces a k -configuration with 2^{k-2} symmetry classes of points and lines. A discussion of parameters follows in Sect. 6.

Note that for each set of discrete parameters, there may be multiple configurations corresponding to the same symbol, depending on which intersection of c_j with $(L)_0$ is chosen; we indicated which intersection should be used by the use of primes (or not) on the d_j . Intuitively, d'_j indicates that we choose the leftmost intersection, while d_j means that we choose the rightmost intersection. More formally, if we parameterize $(L)_0$ as $(L)_0 = (1-t)(v)_0 + t(v)_a$ and c_j intersects $(L)_0$ at two distinct points associated with parameter values $t = \lambda_j, \lambda'_j$ with $\lambda_j < \lambda'_j$, then the primed notation d'_j indicates that we choose the intersection point $(v_1)_0 = (1 - \lambda'_j)(v)_0 + \lambda'_j(v)_a$, while the unprimed notation d_j indicates that we choose $(v_1)_0 = (1 - \lambda_j)(v)_0 + \lambda_j(v)_a$. For example, Fig. 5 shows the configuration $\mathcal{A}(9; 4, 2; 1, 3')$, which is not geometrically congruent to the configuration $\mathcal{A}(9; 4, 2; 1', 3')$, although the two configurations are combinatorially isomorphic; the construction of configuration $\mathcal{A}(9; 4, 2; 1', 3')$ is described carefully in Example 2 and shown in Fig. 4.

2.2 Three Examples

Here, we outline the steps in constructing the 3-configuration $\mathcal{A}(5; 2, 2; 1')$, the 4-configuration $\mathcal{A}(9; 4, 2; 1', 3')$, and the 5-configuration $\mathcal{A}(10; 3, 3; 1, 2, 4)$.

Example 1 The 3-configuration $\mathcal{A}(5; 2, 2; 1')$.

In this configuration, $m = 5$, $a = b = 2$, and $d_1 = 1'$; the configuration is shown in Fig. 2. To construct the configuration, begin with the vertices of a regular 5-gon centered at $\mathcal{O}$, labelled consecutively as $(v)_i$, and construct lines of span 2 with respect to these points: that is, construct the lines $(L)_i := (v)_i \wedge (v)_{i+2}$.

Secondly, since $d_1 = 1$, construct the circle $\mathcal{C}_1$ passing through $(v)_1$, $\mathcal{O}$, and $(v)_{1-2} = (v)_{-1} = (v)_4$. This circle intersects line $(L)_0$ in two places. Figure 2 shows both possibilities. Since the symbol d_1 is primed, choose the left-most intersection point, which is labelled as $(v_1)_0$. Next, the points $(v_1)_i$ are constructed as the images under rotation by $\frac{2\pi i}{m}$ of $(v_1)_0$. Adding lines $(L_1)_i = (v_1)_i \vee (v_1)_{i+2}$ of span 2 completes the construction.

Fig. 2 The 3-configurations $\mathcal{A}(5; 2, 2; 1')$ (top right) and $\mathcal{A}(5; 2, 2; 1)$ (bottom right), along with the circumcircle intersection that generates each configuration. Note they are geometrically similar (Color figure online)

Notice that in this case, the two configurations $\mathcal{A}(5; 2, 2; 1)$ and $\mathcal{A}(5; 2, 2; 1')$ are geometrically similar (as well as combinatorially isomorphic), although the labels and colors are different.

Example 2 The 4-configuration $\mathcal{A}(9; 4, 2; 1', 3')$.

Here, since $N = 2$, we will be constructing a 4-configuration; the various steps in the construction are illustrated in Fig. 3, and the final configuration is shown in Fig. 4. Note that $m = 9$, $a = 4$, $b = 2$, $d_1 = 1'$, and $d_2 = 3'$.

Since $m = 9$, begin by constructing the vertices $(v)_i$ of a regular convex 9-gon, colored black. The vertex $(v)_0$ is shown larger and black in Fig. 3a. Next, since $a = 4$, construct lines $(L)_i$ of span 4 with respect to these vertices. The line $(L)_0$ is shown thick and black in Fig. 3b.

Now construct two circumcircles of span $b = 2$ passing through vertices $(v)_1$ (since $d_1 = 1$) and $(v)_3$ (since $d_2 = 3$), respectively. That is, construct the circles $c_1 = \mathcal{C}((v)_1, \mathcal{O}, (v)_{-1})$ (Fig. 3c) and $c_2 = \mathcal{C}((v)_3, \mathcal{O}, (v)_1)$ (Fig. 3d). To construct

Fig. 3 Intermediate steps in the construction of $\mathcal{A}(9; 4, 2; 1', 3')$ (Color figure online)

the next two classes of points, we look to see if the d_i are primed. Both are, so we will choose the left-most intersection of circle c_j with line $(L)_0$ to define as $(v_j)_0$. The points $(v_1)_0$ and $(v_2)_0$ are colored blue and red, respectively, and both are shown larger. The points $(v_j)_i$ are constructed as the images under rotation by $\frac{2\pi i}{m}$ of $(v_j)_0$ (Fig. 3c, d).

Now, we construct the lines $(L_1)_i$ of span $b = 2$ with respect to the points $(v_1)_i$ and the lines $(L_2)_i$ of span 2 with respect to the points $(v_2)_i$ (Fig. 3e).

Next, we construct the point $(v_{12})_0$ (shown green in Fig. 3f) as the intersection of lines $(L_1)_0$ and $(L_2)_0$ and then construct the rotated images $(v_{12})_i$. When the line $(L_{12})_0$ of span $a = 4$ is constructed, it passes through points $(v_1)_{d_2}$ and $(v_2)_{d_1}$. (This is not obvious and will be proved in Sect. 4.) The resulting incidence structure has four lines passing through each point and four points lying on each line, so it is a 4-configuration, shown in Fig. 4. It has ninefold rotational symmetry. Note this configuration is not geometrically congruent to $\mathcal{A}(9; 4, 2; 1, 3')$, shown in Fig. 5, although it is combinatorially isomorphic.

Example 3 The 5-configuration $\mathcal{A}(10; 3, 3; 1, 2, 4)$

In this example, we construct the 5-configuration $\mathcal{A}(10; 3, 3; 1, 2, 4)$; intermediate steps in the construction are shown in Fig. 6 and the completed configuration is shown in Fig. 7. As before, this configuration begins with the construction of the vertices of a regular m -gon centered at $\mathcal{O}$, where in this case, $m = 10$. Since $a = 3$, we

Fig. 4 The completed configuration $\mathcal{A}(9; 4, 2; 1', 3')$. Notice that the line $(L_{12})_0$, which is of span $a = 4$ with respect to the $(v_{12})_j$, passes through the point $(v_1)_{d_2} = (v_1)_3$ and the point $(v_2)_{d_1} = (v_2)_1$ (Color figure online)

construct all lines of span 3. Here, $b = 3$ as well, and $d_1 = 1, d_2 = 2, d_3 = 4$, so we construct three circumcircles, each of span 3, beginning at the points $(v_1)_1, (v_1)_2, (v_1)_4$ respectively (shown in dark blue, medium blue, and light blue in Fig. 6a). Since none of the d_j are primed, we choose the right-most intersection of each circle with line $(L)_0$, again using dark, medium and light blue to construct the points $(v_1)_0, (v_2)_0, (v_3)_0$, respectively (also shown in Fig. 6a).

Next, the points $(v_i)_j$ are constructed along with lines of span b with respect to each of the $(v_j)_i$. In Fig. 6b, the lines $(L_1)_0, (L_2)_0, (L_3)_0$ are shown in dark, medium, and light blue, respectively. Also in Fig. 6b, we construct the point $(v_{12})_0$ (red) as the intersection of lines $(L_1)_0$ and $(L_2)_0$, the point $(v_{13})_0$ (orange) as the intersection of lines $(L_1)_0$ and $(L_3)_0$, and the point $(v_{23})_0$ (yellow) as the intersection of lines $(L_2)_0$ and $(L_3)_0$.

In Fig. 6c, we construct the rotated images $(L_j)_i$ and $(v_{jk})_i$ by rotating appropriate elements by $\frac{2\pi i}{m}$ about $\mathcal{O}$. Notice that the points $(v_{23})_i$ do not lie on lines $(L_1)_i$, although they are very close.

Fig. 5 The configuration $\mathcal{A}(9; 4, 2; 1, 3')$. Here, $(v_1)_0$ is chosen to be the right-most point of intersection between circle c_1 and line $(L_0)_0$. The configuration is not geometrically congruent to $\mathcal{A}(9; 4, 2; 1', 3')$, although it is combinatorially isomorphic (Color figure online)

In the next step of the construction, shown in Fig. 6d, we construct the lines $(L_{12})_0$, $(L_{13})_0$ and $(L_{23})_0$, where, for example, $(L_{12})_0$ is constructed to pass through $(v_{12})_0$ and $(v_{12})_3$, all shown in red. Notice that $(L_{12})_0$ passes through a dark blue point, which is in fact $(v_1)_2$, and a medium blue point, $(v_2)_1$, and recall that $d_1 = 1$ and $d_2 = 2$. That is, $(L_{12})_0$ passes through $(v_1)_{d_2}$ and $(v_2)_{d_1}$; similar relationships can be verified for $(L_{13})_0$ (orange) and $(L_{23})_0$ (yellow). Furthermore, notice that the three lines $(L_{12})_0$, $(L_{13})_0$, and $(L_{23})_0$ all intersect in a single point, which we will call $(v_{123})_0$, indicated in green.

In Fig. 7, we show the completed configuration, formed by first constructing the rotated images of the lines $(L_{12})_0$, $(L_{23})_0$ and $(L_{13})_0$ and the point $(v_{123})_0$. Finally, we construct one more symmetry class of lines, called $(L_{123})_i$, which are lines that pass through $(v_{123})_i$ and $(v_{123})_{i+3}$; notice that line $(L_{123})_0$ (green) passes through $(v_{12})_4 = (v_{12})_{d_3}$ (red), $(v_{13})_2 = (v_{13})_{d_2}$ (orange), and $(v_{23})_1 = (v_{23})_{d_1}$ (yellow).

Fig. 6 Intermediate steps in the construction of $\mathcal{A}(10; 3, 3; 1, 2, 4)$ (Color figure online)

There are a number of near misses, indicated by pink circles, where symmetry classes of points are almost, but not quite, incident with various symmetry classes of lines.

3 Preliminary Geometric Lemmas

To prove the validity of the construction, we will need several elementary geometric lemmas. The first two are well-known results from elementary Euclidean Geometry

Fig. 7 The completed configuration $\mathcal{A}(10; 3, 3; 1, 2, 4)$. Points which have been circled indicate “near misses”; they are almost—but not quite—on unexpected lines. (Yellow does not lie on dark blue; medium blue does not lie on green; black does not lie on orange) (Color figure online)

(see, e.g., [1]). Recall that a cyclic quadrilateral is a quadrilateral whose vertices are concyclic—that is, they all lie on a circle.

Lemma 1 *A quadrilateral $ABCD$ is cyclic if and only if one pair of opposite angles is supplementary.*

Lemma 2 (Cyclic Quad Lemma) *Quadrilateral $ABCD$ is cyclic if and only if $\angle ABD = \angle ACD$.*

We will also need the following:

Lemma 3 (Equal Spans Lemma) *Let $\mathcal{C}, \mathcal{D}$ be two concentric circles with center O , and let u, u' be on $\mathcal{C}$. Let $\angle uOu' = \alpha$. Let R be any point on $\mathcal{D}$ that does not lie on the line $O \vee u$, and construct R' so that $\angle ROR' = \alpha$. Finally let Q be the intersection of $\overline{uu'}$ and $\overline{RR'}$. Then the four points u, R, Q, O are concyclic.*

Fig. 8 Equal Spans Lemma (Color figure online)

Proof There are several cases to consider. First $\mathcal{C}$ or $\mathcal{D}$ may have larger radius. Second, either point R or R' may be within the sector determined by uOu' or both may be outside it. Finally, the point Q may occur on the line one side or the other of segment $R \vee R'$ or it may occur between points R and R' . (Similarly, for line $u \vee u'$.) All of these cases have effectively the same proof: an application of Lemma 1. We will give a detailed proof in one case, shown in Fig. 8, where $\mathcal{D}$ is the smaller circle, R and R' are both outside sector uOu' , and the point Q occurs past R and u' .

By construction, $\triangle uOu'$ and $\triangle ROR'$ are both isosceles with apex angle α , so the two triangles are similar; hence, their base angles are equal (shown blue double-marked). Note that angle $\angle ORQ$ (shown green triple-marked) is supplementary to $\angle R'RO$, since R', R, Q are collinear. Therefore, since all the base angles are equal, $\angle ORQ$ is supplementary to $\angle u'uO$, which equals $\angle QuO$ because Q, u', u are collinear. It follows that O, u, Q, R form a cyclic quadrilateral and hence are concyclic. $\square$

Grünbaum used a specialized case of the forward direction of the following lemma in [12, Sect. 2.8], in his discussion of a geometric construction for chirally symmetric astral 3-configurations. In [6], one author [LWB] discusses ways to construct a number of infinite families of symmetric 3-configurations using a modified version of the forward direction of the lemma.

Lemma 4 (Configuration Construction Lemma) *Let R and R' be two points on circle $C_O(R)$ centered at O , and let $\angle ROR' = \alpha$. If Q is any point on the circumcircle $\mathcal{C}(R, O, R')$ not on line $O \vee R$, and Q' is on $C_O(Q)$ so that $\angle QOQ' = \alpha$, then $\overline{QQ'}$ passes through R' . Conversely, if Q and Q' are any two points on a circle centered at O so that $\angle QOQ' = \alpha$ and Q, R', Q' are all collinear, then Q lies on the circumcircle $\mathcal{C}(R, O, R')$.*

Proof [$\implies$] Observe that $C_O(R)$ and $C_O(Q)$ are two concentric circles such that $\angle ROR' = \angle QOQ'$ and Q does not lie on line $O \vee R$. Thus, the Equal Spans Lemma implies that the point $P = \overline{QQ'} \cap \overline{RR'}$ is concyclic with O, R , and Q . Since we assumed O, R and Q lie on $C_O(R)$, the point P must also lie on $C_O(R)$. Now, P lies on $C_O(R)$ and on the line $\overline{RR'}$ which forces $P = R'$. Thus, line $\overline{QQ'}$ passes through R' .

Fig. 9 The Circumcircle-Span Lemma (Color figure online)

[$\Leftarrow$] Again, circles $C_O(R)$ and $C_O(Q)$ are concentric and $\angle ROR' = \angle QOQ'$. Further, we assume Q , Q' and R' are collinear. Hence, $\overline{QQ'} \cap \overline{RR'} = R'$. Thus, the Equal Spans Lemma implies that O , R , Q , and R' are concyclic. Thus, Q must lie on $C_O(R)$. $\square$

Lemma 5 (Circumcircle-Span Lemma) *If points $v_1, v_2, \dots, v_n, O$ lie on a single circle C , and for each i , v_i' is constructed so that $Ov_i = Ov_i'$ and $\angle v_i Ov_i' = \alpha$ for some fixed angle α , then the lines $v_i \vee v_i'$ all intersect in a single point. Moreover, this point lies on C .*

Proof By induction on n . For each j , define $L_j = v_j \vee v_j'$. Define Z as the intersection of L_1 with C other than v_1 (see Fig. 9).

Base case: Clearly, L_1 passes through Z .

Induction hypothesis: Suppose that for all j with $1 \leq j \leq n-1$ that L_j passes through Z . We need to show that L_n also passes through Z .

By construction, v_1 and v_1' lie on a circle centered at O , v_n and v_n' lie on a different circle centered at O , and $\angle v_1 Ov_1' = \angle v_n Ov_n'$. Thus, if Q is the intersection of L_n and L_1 , then the Equal Spans Lemma implies that v_1 , v_n , O and Q are concyclic. But v_1 , v_n and O already lie on C , and the other intersection of L_1 and C is Z . Hence $Z = Q$. $\square$

4 The Validity of the Construction

Theorem 6 *For any $k \geq 3$, given a positive integer $m \geq 5$ and a choice of integers $1 \leq a, b < \frac{m}{2}$ and $k-2$ valid discrete parameters $d_1, d_2, \dots, d_{k-2}$, it is possible to construct a symmetric k -configuration $\mathcal{A}(m; a, b; d_1, \dots, d_{k-2})$, with $m(2^{k-2})$ points and lines, either C_m or D_m symmetry, and 2^{k-2} symmetry classes of points and lines.*

To prove this, we will show the following results: (1) For each $\sigma \subseteq S$, the vertex $(v_\sigma)_0$ is well-defined; (2) for each $\sigma \subseteq S$, the line $(L_\sigma)_0$ passes through the points $(v_{\sigma_q})_{d_q}$ for each $q \in \sigma$ where $\sigma_q = \sigma \setminus \{q\}$; (3) every point $(v_\sigma)_0$ has k lines passing through it; (4) every line $(L_\sigma)_0$ has k points lying on it; (5) the construction method produces a total of 2^{k-2} symmetry classes of points and lines; (6) the resulting configuration has the desired symmetry group (and thus the desired total number of points and lines).

Theorem 7 *For each $\sigma \subseteq S$, the vertex $(v_\sigma)_0$ is well-defined: that is, it lies on the common intersection point of all lines $(L_{\sigma_q})_0$, for each $q \in \sigma$. In addition, the set of points $\{(v_{\sigma_q})_0 : q \in \sigma\} \cup \{\mathcal{O}, (v_\sigma)_0\}$ are all concyclic, on a circle named $\mathcal{C}_\sigma$.*

Proof By induction on n , the size of the set σ .

Base cases:

$n = 0$: In this case, the vertex $(v_\emptyset)_0 = (v)_0$ is well-defined as the vertex of a regular convex m -gon. In addition, there are infinitely many circles passing through $(v)_0$ and the origin.

$n = 1$: Let $\sigma = \{j\} = j$, for ease of notation. Each vertex $(v_1)_0, (v_2)_0, \dots, (v_N)_0$ is constructed to lie on the line $(L)_0$, so these vertices are well defined. In this case, the set $\{(v_{\sigma_q})_0 : q \in \sigma\} \cup \{\mathcal{O}, (v_\sigma)_0\} = \{(v)_0, \mathcal{O}, (v_j)_0\}$ which is clearly concyclic. Note that the circle $\mathcal{C}_{\{j\}}$ defined here is not the same as the circle c_j used to determine the points $(v_j)_0$ in the construction algorithm.

$n = 2$: Choose any subset $\sigma = \{x, y\}$ of S . In the construction algorithm, we defined $(v_{xy})_0$ to be the intersection of lines $(L_x)_0$ and $(L_y)_0$.

Since by construction lines $(L_x)_0$ and $(L_y)_0$ are both lines of span b passing through points $(v_x)_0$ and $(v_y)_0$, respectively, by the Equal Spans Lemma, the points $(v_x)_0, (v_y)_0, \mathcal{O}$ and the intersection of $(L_x)_0$ and $(L_y)_0$, which we called $(v_{xy})_0$, are all concyclic, on a circle we define to be $\mathcal{C}_{xy}$.

Induction Hypothesis: For any j with $0 \leq j \leq n - 1$, let ψ be any subset of S with $|\psi| = j$, and assume that $(v_\psi)_0$ is defined to be the common intersection of lines $(L_{\psi_q})_0$ for all $q \in \psi$. In addition, assume that the set of points $\{(v_{\psi_q})_0 : q \in \psi\} \cup \{\mathcal{O}, (v_\psi)_0\}$ are all concyclic, on the circle named $\mathcal{C}_\psi$.

We need to show that for any $2 < n \leq N$ and for any $\sigma \subset S$ with $|\sigma| = n$, that $(v_\sigma)_0$ lies on the common intersection of the lines $(L_{\sigma_q})_0$ for all $q \in \sigma$. It suffices to show that for any distinct p, q, r in σ , that the lines $(L_{\sigma_p})_0, (L_{\sigma_q})_0$ and $(L_{\sigma_r})_0$ intersect in a single point. To do this, we will first show that the points $(v_{\sigma_r})_0, (v_{\sigma_p})_0, (v_{\sigma_q})_0, \mathcal{O}$ are concyclic.

Let $\varphi = \sigma \setminus \{p, q, r\}$, and let $\varphi x := \varphi \cup \{x\}$, for $x \subseteq \{p, q, r\}$. Note that $\varphi pqr = \sigma$ and $\varphi pq = \sigma_r, \varphi pr = \sigma_q, \varphi qr = \sigma_p$.

In the remainder of the proof, we will suppress the “0” subscript to simplify notation; that is, $v_{\varphi q}$ indicates the point $(v_{\varphi q})_0$, and $L_{\varphi q}$ indicates the line $(L_{\varphi q})_0$, etc.

By the Construction Algorithm and the induction hypothesis, since $|\varphi| = n - 3$, we have constructed (see Figure 10) line L_φ (black), which contains vertices $v_{\varphi p}, v_{\varphi q}$, and $v_{\varphi r}$ (blue, green, cyan, respectively). Also by the algorithm, we have constructed lines $L_{\varphi p}$ (blue), $L_{\varphi q}$ (green), and $L_{\varphi r}$ (cyan), and by the induction hypothesis, these intersect pairwise in points $v_{\varphi pq}$ (red), $v_{\varphi pr}$ (magenta), $v_{\varphi qr}$ (orange). Moreover, the

Fig. 10 Illustrating the inductive step for vertex construction. Note that for clarity, the “0” subscript on each vertex and line of the drawing has been suppressed. That is, in the diagram, the point $v_{\phi q}$ indicates the point $(v_{\phi q})_0$ (Color figure online)

points $v_{\phi p}, v_{\phi q}, v_{\phi pq}, O$ are concyclic on circle $C_{\phi pq}$ (red), and similarly for $C_{\phi pr}$ (magenta), $C_{\phi qr}$ (orange).

Considering the segment $Ov_{\phi p}$ and circle $C_{\phi pr}$ (magenta), using the Cyclic Quad Lemma we conclude that $\angle Ov_{\phi r} v_{\phi p} = \angle Ov_{\phi pr} v_{\phi p}$ (gray triple-marked angles). Using the same segment and circle $C_{\phi pq}$ (red), we conclude that $\angle Ov_{\phi q} v_{\phi p} = \angle Ov_{\phi pq} v_{\phi p}$ (blue double-marked angles). Since $v_{\phi pq}, v_{\phi pr}, v_{\phi p}$ all lie on line $L_{\phi p}$, and $v_{\phi q}, v_{\phi r}, v_{\phi p}$ all lie on line L_{ϕ} , $\angle v_{\phi pq} v_{\phi pr} O = \angle v_{\phi q} v_{\phi r} O$ (red single-marked angles), since they are both supplementary to equal angles.

Therefore, $\triangle Ov_{\phi pq} v_{\phi pr} \sim \triangle Ov_{\phi q} v_{\phi r}$ (gray triangles), since they have two pairs of equal angles. Hence the third angles in each triangle are equal (bottom green quadruple-marked angles); that is, $\angle v_{\phi r} Ov_{\phi q} = \angle v_{\phi pr} Ov_{\phi pq}$.

However, using segment $v_{\phi r} v_{\phi q}$ and circle $C_{\phi qr}$ (orange), $\angle v_{\phi r} v_{\phi qr} v_{\phi q} = \angle v_{\phi r} Ov_{\phi q}$ (green quadruple-marked angles). But $\angle v_{\phi r} v_{\phi qr} v_{\phi q}$ is determined by lines $L_{\phi q}$ and $L_{\phi r}$, so $\angle v_{\phi r} v_{\phi qr} v_{\phi q} = \angle v_{\phi pq} v_{\phi qr} v_{\phi pr}$. Thus, we can conclude that

$$\angle v_{\phi pq} v_{\phi qr} v_{\phi pr} = \angle v_{\phi pr} Ov_{\phi pq}.$$

Hence by the Cyclic Quad Lemma, we conclude that $v_{\phi pq} = v_{\sigma_r}, v_{\phi qr} = v_{\sigma_p}, v_{\phi pr} = v_{\sigma_q}, O$ are concyclic, on a circle (green dashed) which we shall name $C_{\phi prq} = C_{\sigma}$.

Therefore, if we construct lines $L_{\varphi pq} = L_{\sigma_r}$, $L_{\varphi pr} = L_{\sigma_q}$, $L_{\varphi qr} = L_{\sigma_p}$ of span δ_{n-1} (with respect to $\mathcal{O}$) through vertices $v_{\varphi pq} = v_{\sigma_r}$, $v_{\varphi pr} = v_{\sigma_q}$, $v_{\varphi qr} = v_{\sigma_p}$, respectively, by the Circumcircle-Span Lemma, these three lines will intersect at a single point $v_{\varphi pqr} = v_{\sigma}$ which also lies on $\mathcal{C}_{\sigma}$. $\square$

Lemma 8 *Let σ be a nonempty subset of S with $|\sigma| = n$. Then the circle $\mathcal{C}_{\sigma}$ contains $(v_{\sigma})_{-(\delta_{n-1})}$. That is, $\mathcal{C}_{\sigma}$ is a circle of span δ_{n-1} with respect to the class of points v_{σ} .*

Proof Let σ be a nonempty subset of S of cardinality n . For ease of reference, let $\delta = \delta_{n-1}$. Recall from Theorem 7 that $\mathcal{C}_{\sigma}$ contains the points $(v_{\sigma})_0$, $\mathcal{O}$, and $(v_{\sigma_q})_0$ for every $q \in \sigma$.

Let q be an arbitrary element in σ and consider the concentric circles $\mathcal{C}_{\mathcal{O}}((v_{\sigma})_0)$ and $\mathcal{C}_{\mathcal{O}}((v_{\sigma_q})_0)$, and points $(v_{\sigma_q})_0$, $(v_{\sigma_q})_{\delta}$, $(v_{\sigma})_{-\delta}$, and $(v_{\sigma})_0$. Clearly, $\angle(v_{\sigma_q})_0\mathcal{O}(v_{\sigma_q})_{\delta} = \angle(v_{\sigma})_{-\delta}\mathcal{O}(v_{\sigma})_0$. Thus, by the Equal Spans Lemma, the point of intersection $Q = \overline{(v_{\sigma_q})_0(v_{\sigma_q})_{\delta}} \wedge \overline{(v_{\sigma})_{-\delta}(v_{\sigma})_0}$ is concyclic with the points $(v_{\sigma_q})_0$, $\mathcal{O}$, and $(v_{\sigma})_{-\delta}$. Call this circle $\mathcal{C}'$.

Note that by definition of $(L_{\sigma_q})_0$, the line $(L_{\sigma_q})_0 := \overline{(v_{\sigma_q})_0(v_{\sigma_q})_{\delta}}$ and contains the point $(v_{\sigma})_0$, by Theorem 7. Hence, $Q = (v_{\sigma})_0$. But now circle $\mathcal{C}'$ contains the three points: $(v_{\sigma})_0$, $\mathcal{O}$, and $(v_{\sigma_q})_0$ and so $\mathcal{C}' = \mathcal{C}_{\sigma}$. Thus, $\mathcal{C}_{\sigma}$ includes the point $(v_{\sigma})_{-\delta}$, which was what we wanted to show. $\square$

Theorem 9 *For each $\sigma \subseteq S$ with $|\sigma| = n$, the line $(L_{\sigma})_0$ passes through the points $(v_{\sigma_q})_{d_q}$ for each $q \in \sigma$ (as well as through the two points $(v_{\sigma})_0$ and $(v_{\sigma})_{\delta_n}$).*

Proof Again, we proceed by induction on the size of σ .

Base cases:

$n = 0$: If $n = 0$, then $\sigma = \emptyset$, and by construction, line $(L_{\sigma})_0$ passes through $(v_{\sigma})_0$ and $(v_{\sigma})_{\delta_n}$, and there are no other vertices to consider.

$n = 1$: The line $(L_j)_0$ is defined to pass through $(v_j)_0$ and $(v_j)_b$. We need to show that it also passes through $(v)_{d_j}$. But this follows immediately from the Circumcircle Construction Lemma: line $(L_j)_0$ is a span b line, and by construction, point $(v_j)_0$ lies on the circumcircle $c_j := \mathcal{C}((v)_{d_j}, \mathcal{O}, (v)_{d_j-b})$, so $(L_j)_0$ also passes through $(v)_{d_j}$.

Induction Hypothesis: Suppose that for any $\psi \subseteq S$ with $|\psi| = r$ for any $0 \leq r \leq n - 1$ that line $(L_{\psi})_0 := (v_{\psi})_0 \vee (v_{\psi})_{\delta_r}$ passes through each point $(v_{\psi_q})_{d_q}$ for all $q \in \psi$.

We need to show for any $\sigma \subseteq S$ with $|\sigma| = n$ that $(L_{\sigma})_0$ passes through the points $(v_{\sigma_p})_{d_p}$ for each $p \in \sigma$.

Choose any $p, q \in \sigma$. Let $\sigma_{pq} = \sigma \setminus \{p, q\} = (\sigma_q)_p = (\sigma_p)_q$ and let $\delta = \delta_n = \delta_{n-2}$. Note that by construction, $(v_{\sigma_p})_0$ and $(v_{\sigma_q})_0$ both lie on $(L_{\sigma_{pq}})_0$, which is of span δ . Construct $(L_{\sigma_q})_0$ passing through $(v_{\sigma_q})_0$. By the induction hypothesis applied to σ_q which has size $n - 1$, we conclude that since $\sigma_{pq} \cup \{p\} = \sigma_q$, $(v_{(\sigma_q)_p})_{d_p} = (v_{\sigma_{pq}})_{d_p}$ lies on $(L_{\sigma_q})_0$. See Fig. 11 to follow along in the diagram.

Now let $C = \mathcal{C}((v_{\sigma_p})_{d_p}, \mathcal{O}, (v_{\sigma_p})_{d_p-\delta})$ (shown red and thick). If we rotate C by angle $-\frac{2\pi d_p}{m}$ about $\mathcal{O}$, then we get the circle $C' = \mathcal{C}((v_{\sigma_p})_0, \mathcal{O}, (v_{\sigma_p})_{-\delta})$ (red and dotted). Recall that circle $\mathcal{C}_{\sigma_p}$ contains $(v_{(\sigma_p)_q})_0$, along with $(v_{\sigma_p})_0$ and $\mathcal{O}$, and by

Now apply the Configuration Construction Lemma using the points $(v_{\sigma_p})_{d_p}$ and $(v_{\sigma_p})_{d_p-\delta}$, which are both on C , along with $Z = (v_{\sigma})_0$, which is now also on C , to conclude that the line $ZZ' = (L_{\sigma})_0$ passes through $(v_{\sigma_p})_{d_p}$, which is what we needed to show. $\square$

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count the number of symmetry classes of points and lines, completing the proof of Theorem 6.

Choose any $\sigma \subseteq S$ (where $|S| = k - 2$), and suppose $|\sigma| = n$. Consider $(v_\sigma)_0$. By symmetry, it suffices to show that $(v_\sigma)_0$ has k lines passing through it.

Note that if $|\sigma| = 0$, then $\sigma = \emptyset$, and $(v_\emptyset)_0$ lies only on the two lines $(L_\emptyset)_0$ and $(L_\emptyset)_{-a}$ and the $(k - 2)$ subsequently defined lines $(L_{\{p\}})_{-d_p}$ for $p \in S$. If $|\sigma| = k - 2$, then $\sigma = S$ and $(v_S)_0$ lies only on the $k - 2$ previously defined lines $(L_{S_q})_0$ for $q \in S$ and the two lines $(L_S)_0$ and $(L_S)_{-\delta_{k-2}}$.

In the other cases, by construction, $(v_\sigma)_0$ lies on the n previously defined lines $(L_{\sigma_q})_0$ for each $q \in \sigma$, the pair of lines $(L_\sigma)_0$ and $(L_\sigma)_{-\delta_n}$ (for $\delta_n = a$ or b depending on the parity of n), and the $(k - 2) - n$ subsequently defined lines $(L_{\sigma_p})_{-d_p}$ for each $p \notin \sigma$, for a total of $n + 2 + (k - 2) - n = k$ lines passing through each point.

Similarly, consider $(L_\sigma)_0$. By symmetry, it suffices to show that $(L_\sigma)_0$ has k points lying on it. If $|\sigma| = 0$, then $(L_\emptyset)_0$ contains the two points $(v_\emptyset)_0$ and $(v_\emptyset)_a$ and the $(k - 2)$ subsequently defined points $(v_{\{p\}})_0$ for $p \in S$. If $|\sigma| = k - 2$, then $(L_S)_0$ only passes through the $k - 2$ previously defined vertices $(v_{S_q})_{d_q}$ for $q \in S$ and the two points $(v_S)_0$ and $(v_S)_{\delta_{k-2}}$. In the general case, by construction, $(L_\sigma)_0$ contains n previously defined vertices $(v_{\sigma_q})_{d_q}$ for each $q \in \sigma$, the pair of points $(v_\sigma)_0$ and $(v_\sigma)_{\delta_n}$, and the $(k - 2) - n$ subsequently defined vertices $(v_{\sigma_p})_0$ for each $p \notin \sigma$, for a total of $n + 2 + (k - 2) - n = k$ points lying on each line.

In total, we have constructed $\binom{k-2}{|\sigma|}$ symmetry classes of points and lines for each $\sigma \subset \{1, 2, \dots, k - 2\}$, for a total of 2^{k-2} symmetry classes of points and lines. Each symmetry class contains m elements for a total of $m2^{k-2}$ points and lines.

By construction, the configurations that have been constructed have m -fold rotational symmetry. Given appropriate choices of discrete parameters, the configurations, such as the configuration shown in Fig. 1, may in fact have dihedral symmetry and fewer symmetry classes; classifying which parameters yield configurations with dihedral symmetry is beyond the scope of this paper and will be addressed in a future work.

This completes the proof of Theorem 6.

6 Valid Parameters and Further Directions

Theorem 6 shows that if valid parameters are chosen for $\mathcal{A}(m; a, b; d_1, \dots, d_{k-2})$, then following the construction techniques outlined in Sect. 2, a k -configuration is produced. However, the theorem is silent as to which parameters are valid. Unfortunately, it turns out that completely characterizing precisely which parameters are valid is fairly complex and beyond the scope of this paper. In this section, we will identify large sets of valid parameters. A more complete characterization of valid discrete parameters will be discussed in a subsequent work.

In this work we present the following theorems.

Theorem 10 *In the configuration $\mathcal{A}(m; a, b; d)$, if $d = 0$, $a, b, a + b$, then the construction method produces a degenerate configuration: either a $(2, 4)$ - or $(4, 2)$ -configuration.*

Proof Consider circumcircle $\mathcal{C} = \mathcal{C}(v_d, \mathcal{O}, v_{d-b})$. By construction, it is a span b circle.

Suppose that $d = b$; then $\mathcal{C}$ intersects $(L)_0$ at $(v)_{b-b} = (v)_0$ and at some other point. If we call the second intersection point w_0 and use this as the point $(v_1)_0$ in the construction algorithm, the algorithm has us construct a second line $(L_1)_0$ which passes through $(v_1)_0$, $(v_1)_b$ and $(v)_d = (v)_b$ in this case (since $d = b$). On the other hand, the line $(L)_b$ passes through the points $(v)_b$, $(v)_{b+a}$, and $(v_1)_b$. That is, both lines $(L_1)_0$ and $(L)_b$ pass through the two points $(v)_b$ and $(v_1)_b$, so they are the same line. Hence the result of the construction is an incidence structure in which every point has two lines passing through it. Namely, $(v)_i$ has lines $(L)_i$ and $(L)_{i-a}$ through it, and $(v_1)_i$ has lines $(L)_{i+b}$ and $(L)_{i+b-b} = (L)_i$ passing through it. Each line contains four points. Namely, $(L)_i$ contains points $(v)_i$, $(v)_{i+a}$, $(v_1)_i$ and $(v_1)_{i+b}$. Thus, the incidence structure is a $(4, 2)$ -configuration.

On the other hand, if we let $(v_1)_0 = (v)_0$ (choosing the other possible intersection of $\mathcal{C}$ with line $(L)_0$), then the line $(L_1)_0$ is distinct from $(L)_0$ (as long as $a \neq b$), and each line has only two points on it. Namely, $(L)_i$ contains $(v)_i$ and $(v)_{i+a}$, while $(L_1)_i$ contains $(v)_i$ and $(v)_{i+b}$. Each point has four lines passing through it. Namely, $(v)_i$ contains lines $(L)_i$, $(L)_{i-a}$, $(L_1)_i$ and $(L_1)_{i-b}$. Thus the algorithm produces a $(2, 4)$ -configuration.

In the other possible cases for d , if $d = 0$ then $\mathcal{C}$ passes through $(v)_d = (v)_0$, which is on $(L)_0$; if $d = a$ then $\mathcal{C}$ passes through $(v)_d = (v)_a$, which is on $(L)_0$; and if $d = a + b$ then $\mathcal{C}$ passes through $(v)_{d-b} = (v)_a$, which is on $(L)_0$. In each case, arguing as above, we achieve a $(4, 2)$ -configuration by choosing the “other” point on $(L)_0$ as our point $(v_1)_0$, while choosing the point $(v)_a$ or $(v)_0$ as the point $(v_1)_0$ yields a $(2, 4)$ -configuration. $\square$

Lemma 11 *Given a circle $\mathcal{C}$ centered at $\mathcal{O}$ and four points on $\mathcal{C}$ labelled v_0, v_a, v_d , and v_{d-b} , let $L_0 = v_0 \vee v_a$ and $\mathcal{C}_d = \mathcal{C}(\mathcal{O}, v_d, v_{d-b})$. Let m_a be the midpoint of chord v_0v_a and let w_0 and w'_0 be points of intersection between L_0 and $\mathcal{C}_d$. Let $2\alpha = \angle v_0\mathcal{O}v_a$, $2\beta = \angle v_d\mathcal{O}v_{d-b}$, $\tau_1 = \angle m_a\mathcal{O}w_0$, $\tau_2 = \angle m_a\mathcal{O}w'_0$, and $2\delta = \angle v_d\mathcal{O}v_0$. Then*

- (i) $2\delta = \alpha + \beta - \tau_1 - \tau_2$;
- (ii) $\cos(\alpha)\cos(\beta) = \cos(\tau_1)\cos(\tau_2)$.

Proof Construct $\mathcal{C}$, $v_0, v_a, v_d, v_{d-b}, L_0, \mathcal{C}_d, m_a, w_0, w'_0, \tau_1, \tau_2$, and δ as described (see Fig. 12). Construct point w_{-b} by rotating w_0 clockwise by $\angle v_d\mathcal{O}v_{d-b} = 2\beta$ and construct m_b , the midpoint of segment w_0w_{-b} . From the Configuration Construction Lemma, line $L_1 = w_0 \vee w_{-b}$ also intersects v_{d-b} . Observe that v_{d-b} and w'_0 both lie on $\mathcal{C}_d$ and both subtend segment $\mathcal{O}w_0$. Thus, $\angle \mathcal{O}v_{d-b}w_0 = \angle \mathcal{O}w'_0w_0$ and right triangles $\triangle \mathcal{O}m_bv_{d-b}$ and $\triangle \mathcal{O}m_aw'_0$ are congruent. Thus $\angle w_{-b}\mathcal{O}v_{d-b} = \tau_2 - \beta$ and by rotational symmetry, $\angle w_0\mathcal{O}v_d = \tau_2 - \beta$. Finally, $2\delta = \angle v_d\mathcal{O}v_0 = (\angle m_a\mathcal{O}v_0 - \angle m_a\mathcal{O}w_0) - (\angle w_0\mathcal{O}v_d) = (\alpha - \tau_1) - (\tau_2 - \beta) = \alpha + \beta - \tau_1 - \tau_2$.

For part (ii), we observe that by elementary right-triangle trigonometry, $\cos \alpha = \mathcal{O}m_a/\mathcal{O}v_a$, since m_a was constructed so that $\mathcal{O}m_a \perp v_0v_a$. Thus, $\mathcal{O}m_a = (\cos \alpha)(\mathcal{O}v_a)$. Now, triangle $\triangle \mathcal{O}m_aw_0$ is also a right triangle, with angle $\angle m_a\mathcal{O}w_0 = \tau_1$, adjacent

side $\mathcal{O}m_a$, and hypotenuse $\mathcal{O}w_0$. Thus,

$$\cos(\tau_1) = \frac{\mathcal{O}m_a}{\mathcal{O}w_0} = \frac{(\cos \alpha)(\mathcal{O}v_a)}{\mathcal{O}w_0}$$

or, equivalently,

$$\mathcal{O}w_0 = \frac{(\cos \alpha)(\mathcal{O}v_a)}{\cos(\tau_1)}.$$

Similarly, since $\mathcal{O}m_b \perp w_0v_{d-b}$, triangle $\Delta m_b\mathcal{O}v_{d-b}$ is also a right triangle. From part (i), angle $\angle m_b\mathcal{O}v_{d-b} = \tau_2$. Thus, $\cos(\tau_2) = \mathcal{O}m_b/\mathcal{O}v_{d-b}$ or equivalently $\mathcal{O}m_b = (\cos \tau_2)(\mathcal{O}v_{d-b})$. Now triangle $\Delta \mathcal{O}m_bw_{-b}$ is also a right triangle, with angle $\angle m_b\mathcal{O}w_{-b} = \beta$, adjacent side $\mathcal{O}m_b$, and hypotenuse $\mathcal{O}w_{-b}$. Thus,

$$\cos(\beta) = \frac{\mathcal{O}m_b}{\mathcal{O}w_{-b}} = \frac{(\cos \tau_2)(\mathcal{O}v_{d-b})}{\mathcal{O}w_{-b}}$$

or, equivalently,

$$\mathcal{O}w_{-b} = \frac{(\cos \tau_2)(\mathcal{O}v_{d-b})}{\cos(\beta)}.$$

Since triangle $\Delta w_0\mathcal{O}w_{-b}$ is isosceles by construction, $\mathcal{O}w_0 = \mathcal{O}w_{-b}$. Hence,

$$\frac{\cos(\alpha)\mathcal{O}v_a}{\cos(\tau_1)} = \frac{\cos(\tau_2)\mathcal{O}v_{d-b}}{\cos(\beta)}.$$

Since both v_a and v_{d-b} lie on $\mathcal{C}$, we know $\mathcal{O}v_a = \mathcal{O}v_{d-b}$, so

$$\frac{\cos(\alpha)}{\cos(\tau_1)} = \frac{\cos(\tau_2)}{\cos(\beta)}$$

and part (ii) is proved. $\square$

As a consequence of Lemma 11, if any four of the angles $\alpha, \beta, \tau_1, \tau_2$ or δ are multiples of a particular angle, the fifth must be also. In particular, in the configuration construction algorithm, α, β , and δ are always multiples of $2\pi/m$. Furthermore, for some positive integer $r < m/2$, $\tau_1 = r\pi/m$ if and only if $w_0 = L_0 \wedge L_{-r}$, where L_{-r} is the rotation of L_0 by $-2r\pi/m$. Lemma 11 implies that if a circumcircle intersects L_0 in two points, either both of those points of intersection correspond to intersections of the form $L_0 \wedge L_r$ for some r or neither do. That is:

Theorem 12 (Extra Incidences) *In a configuration $\mathcal{A}(m; a, b; d)$, circumcircle $c = \mathcal{C}(v_d, \mathcal{O}, v_{d-b})$ intersects line L_0 at the point of intersection of L_0 and L_{-t} precisely when either*

- (i) $d \in \{0, a, b, a + b\}$ and the configuration produced is degenerate; or

Fig. 12 Diagram for the proof of Theorem 12, extra intersections (Color figure online)

(ii) the discrete parameters a, b, d, t satisfy

$$\cos\left(\frac{a\pi}{m}\right)\cos\left(\frac{b\pi}{m}\right) = \cos\left(\frac{t\pi}{m}\right)\cos\left((2d - a - b + t)\frac{\pi}{m}\right). \quad (1)$$

Proof Suppose $c = \mathcal{C}(v_d, \mathcal{O}, v_{d-b})$ intersects $(L)_0$ at two points, w_0 and w'_0 . We may apply Lemma 11 to the geometric construction, where

$$\alpha = \frac{a\pi}{m}, \beta = \frac{b\pi}{m} \quad \text{and} \quad \delta = \frac{d\pi}{m}$$

to conclude that if τ_1 and τ_2 are defined as in the proof of Lemma 11 that $\tau_1 + \tau_2$ is also an integer multiple of π/m , since

$$\frac{2d\pi}{m} = \frac{a\pi}{m} + \frac{b\pi}{m} - (\tau_1 + \tau_2).$$

If w_0 lies at the intersection of L_0 and L_{-t} , then $\tau_1 = \frac{t\pi}{m}$, so it follows that τ_2 must also be an integer multiple of $\frac{\pi}{m}$, say $\tau_2 = \frac{t_2\pi}{m}$. Thus, if a circumcircle intersects L_0 in two points, either both of those points of intersection correspond to intersections of the form $L_0 \cap L_r$ for some r or neither do, and this happens precisely when

$$\cos\left(\frac{\pi}{m}a\right)\cos\left(\frac{\pi}{m}b\right) = \cos\left(\frac{\pi}{m}t_1\right)\cos\left(\frac{\pi}{m}t_2\right). \quad (2)$$

We showed previously that $2\delta = \alpha + \beta - \tau_1 - \tau_2$, or in this case,

$$2d\frac{\pi}{m} = a\frac{\pi}{m} + b\frac{\pi}{m} - t\frac{\pi}{m} - t_2\frac{\pi}{m}$$

so $2d = a + b - t - t_2$. Solving for t_2 , we see that $t_2 = 2d - a - b + t$, and substituting into Eq. (2) yields Eq. (1).

If $d = 0$, $a, b, a + b$, then by the proof techniques in Theorem 10, we can show that $t = b, b, -b, -b$ respectively, and Eq. (1) is trivially satisfied. For example, if $d = a$, then Eq. (1) becomes

$$\begin{aligned}\cos\left(a\frac{\pi}{m}\right)\cos\left(b\frac{\pi}{m}\right) &= \cos\left(t\frac{\pi}{m}\right)\cos\left((2a - a - b + t)\frac{\pi}{m}\right) \\ &= \cos\left(b\frac{\pi}{m}\right)\cos\left((a - b + b)\frac{\pi}{m}\right) \\ &= \cos\left(b\frac{\pi}{m}\right)\cos\left(a\frac{\pi}{m}\right).\end{aligned}$$

However, Lemma 10 showed that if $d = 0$, $a, b, a + b$, then the configurations produced are degenerate. $\square$

Equation (2) is a familiar equation in the study of configurations: it is the hard axiom to fulfill in the construction of a certain class of highly symmetric 4-configurations, the *astral 4-configurations*; see [2, 10] and [12, Sect. 3.6]. These astral 4-configurations have symbol $m\#(a, b; c, d)$ and the parameters are required to satisfy certain axioms. Myerson in [13] determined all rational solutions to

$$\sin(x_1\pi)\sin(x_2\pi) = \sin(x_3\pi)\sin(x_4\pi),$$

and elementary trigonometry converts these solutions into integer solutions to (2). One interesting result is that all solutions require $6 \mid m$. A consequence of Theorem 12 is that the circumcircle construction outlined in this paper provides a novel construction for producing astral 4-configurations. For example, applying the construction to $\mathcal{A}(12; 4, 4; 1)$, where $d_1 = 1$ is a well-defined but not valid parameter choice, produces a (24_4) astral 4-configuration, rather than the expected 3-configuration, because if $m = 12$, $a = b = 4$, $t = 1$, $t_2 = 5$, then it is known (and easy to show) that (2) is satisfied, and $2d = a + b - t_1 - t_2 = 8 - 6 = 2$, so $d = 1$. That is:

Corollary 13 *Given parameters m, a, b, d, t that satisfy (1), the configuration $\mathcal{A}(m; a, b; d)$ is the astral 4-configuration $m\#(a, t; b, 2d - a - b + t)$.*

In addition, if we apply the construction algorithm using the symbol $\mathcal{A}(12; 4, 4; 1, 3, 5)$, which would be expected to produce a 5-configuration, because of the extra incidences caused by the fact that $d_1 = 1$ a (96_6) configuration is generated which has only 8 symmetry classes of points and lines. This is now the smallest known 6-configuration, shown in Fig. 13.

Lemma 14 *Given m, a, b , if $d \in \{1, 2, \dots, a + b - 1\} \setminus \{a, b\}$, then circle $c = \mathcal{C}((v)_d, \mathcal{O}, (v)_{d-b})$ intersects $(L)_0$ in two distinct points; in particular, if $d \in \{1, 2, \dots, a + b - 1\} \setminus \{a, b\}$ then d is a well-defined parameter.*

Fig. 13 The smallest known 6-configuration, a (96_6) configuration constructed as $\mathcal{A}(12; 4, 4; 1, 3, 5)$, which has unexpected additional incidences (Color figure online)

Proof Under these conditions, at least one of $(v)_d$ and $(v)_{d-b}$ is separated from $\mathcal{O}$ by $(L)_0$: specifically, if d is strictly between 0 and a , then $(L)_0$ separates $\mathcal{O}$ and $(v)_d$, while if d is strictly between a and $a + b$, $(v)_{d-b}$ is separated from $\mathcal{O}$ by $(L)_0$. In either case, the circle c intersects $(L)_0$ twice. $\square$

For certain values of m, a, b there are other well-defined parameters d ; these will be discussed in a subsequent work.

Theorem 15 *If $m \geq k + 1$ when k is even or if $m \geq k + 2$ for k odd, and if $6 \nmid m$, then it is possible to construct a k -configuration with 2^{k-2} symmetry classes of points and lines, by choosing $a = b = \lfloor \frac{m-1}{2} \rfloor$ and any subset of size $k - 2$ from the set $S = \{1, 2, \dots, a - 1, a + 1, \dots, 2a - 1\}$ to be the sequence $d_1, \dots, d_{k-2}$ in the configuration $\mathcal{A}(ma, b; d_1, \dots, d_{k-2})$.*

Proof Suppose k is even. Then

$$a = \lfloor \frac{m-1}{2} \rfloor \quad \text{and} \quad m \geq k+1,$$

so $a \geq \lfloor k/2 \rfloor = k/2$. Hence $|S| = 2a - 2 \geq k - 2$.

On the other hand, if k is odd, then $a = \lfloor (m-1)/2 \rfloor$ and $m \geq k+2$, so $a \geq \lfloor (k+1)/2 \rfloor$; therefore

$$|S| = 2a - 2 \geq 2 \lfloor (k+1)/2 \rfloor - 2 = k - 1 > k - 2.$$

In either case, it is possible to choose a subset of size $k - 2$ from S to serve as a sequence of d_j in the construction algorithm. By Lemma 14, each of these d_j is well-defined. So it is possible to implement the construction algorithm.

It remains to show that each d_j is valid; that is, there are no additional intersections. By Theorem 12, the only way that a circle c_j can intersect a point that is the point of intersection of $(L)_0$ and $(L)_{-t}$ for some t is either when $d_j = 0, a, b, a+b$, which we have disallowed, or when a, b, d, t satisfy (1). However, solutions to (1), in the form of (2), have been completely characterized by Meyerson in [13] and all require that $6 \mid m$, so if $6 \nmid m$, there are no extra intersections. $\square$

We conjecture that if k is even, the configuration

$$\mathcal{A}(k+1; \frac{k}{2}, \frac{k}{2}; 1, 2, \dots, \frac{k}{2} - 1, \frac{k}{2} + 1, k - 1)$$

is minimal for this construction, while if k is odd, the configuration

$$\mathcal{A}(k+2; \frac{k+1}{2}, \frac{k+1}{2}; 1, 2, \dots, \frac{k+1}{2} - 1, \frac{k+1}{2} + 1, k - 1)$$

is minimal (although any other subset of size $k - 2$ from

$$\{1, 2, \dots, \frac{k+1}{2} - 1, \frac{k+1}{2} + 1, k\}$$

would also work), unless there are extra incidences. Note that the configuration of this type for $k = 4$ is the configuration $\mathcal{A}(5; 2, 2; 1, 3)$, a (204) configuration. The configuration $\mathcal{A}(7; 3, 3; 1, 2, 4)$ produces a (56_5) configuration, but the configuration $\mathcal{A}(12; 4, 4; 1, 3)$, which has extra incidences, produces a (48_5) configuration (also discussed in [8]). The configuration $\mathcal{A}(7; 3, 3; 1, 2, 4, 5)$ is a $(112)_6$ configuration, slightly larger than the previously known record-holder, a (110_6) configuration constructed in [4]; both configurations have dihedral symmetry. The configuration $\mathcal{A}(9; 4, 4; 1, 2, 3, 5, 6)$ produces a chiral $(9 \cdot 2^5_7) = (288_7)$ configuration, the smallest known. Unfortunately, neither $\mathcal{A}(7; 3, 3; 1, 2, 4, 5)$ nor $\mathcal{A}(9; 4, 4; 1, 2, 3, 5, 6)$ can be drawn intelligibly at a scale appropriate to this paper, because many of the point classes cluster too close to the center of the configuration; for example, if the radius of the points (v) is set to 1, then the configuration $\mathcal{A}(7; 3, 3; 1, 2, 4, 5)$ has point classes with

smallest radius approximately 0.0629012 and largest radius approximately 15.8979 (and the next largest radius is 4.32402).

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