Two Upper Bounds for the Erdős–Szekeres Number with Conditions

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Abstract We use a projective transformation method of Tóth and Valtr to show that a certain number of points in general position in the plane contain the vertex set of a convex n -gon if their convex hull is an $(n-1)$ -gon.

Keywords Erdős–Szekeres problem · Discrete geometry · Combinatorial convexity

In 1933 the following question was asked, inspired by Esther Klein.

Let $n \geq 3$ be an integer. Is there a finite number, of which the smallest is denoted by $ES(n)$ in the following, such that every set of $ES(n)$ points in general position in the plane contains the vertex set of a convex n -gon?

Erdős and Szekeres answered this question positively in their joint paper [2]. More precisely, they proved in [2, 3] that for all integers $n \geq 3$ one has

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1.$$

The lower bound is sharp for $n \in \{3, 4, 5, 6\}$ where the case $n = 6$ was treated recently in [6] with the help of computers. Erdős and Szekeres conjectured that the lower bound is tight in general which is known as the *Erdős–Szekeres conjecture*. The actual best upper bound has been given by Tóth and Valtr [8]. They proved that for all $n \geq 5$

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Fig. 1 A point set in the plane with $s = 4$ points inside a triangle determined by three consecutive vertices of its convex hull

$$ES(n) \leq \binom{2n-5}{n-2} + 1.$$

This is approximately an improvement by the factor two in relation to the upper bound of Erdős and Szekeres. The result of Tóth and Valtr combines two previous methods. One method has been applied by Chung and Graham [1] to lower the upper bound of Erdős and Szekeres by one. The second method was used by Tóth and Valtr themselves in [7] where they were able to show that $ES(n) \leq \binom{2n-5}{n-2} + 2$ by using a projective transformation method. In the same year Kleitman and Pachter [4] proved that $ES(n) \leq \binom{2n-4}{n-2} - 2n + 7$ for all $n \geq 4$. Unfortunately their method seems to resist a combination with the projective transformation idea of Tóth and Valtr in general. A survey of results concerning the Erdős–Szekeres conjecture can be found in [5].

In this paper we use the projective transformation method of Tóth and Valtr to show the following statements.

Theorem 1 For $n \geq 4$, every set of $\binom{2n-5}{n-2} - 2n + 9$ points in general position in the plane with the property that there is no point inside a triangle determined by three consecutive vertices of its convex hull contains a convex n -gon.

Theorem 2 For $n \geq 4$, every set of $\binom{2n-5}{n-2} - n - (\sum_{i=1}^{n-4} \min\{i, s\}) + 6$ points in general position in the plane with an $(n-1)$ -gon as its convex hull contains a convex n -gon where $s \geq 0$ denotes the number of points inside a triangle determined by three consecutive vertices of its convex hull (Fig. 1).

Corollary 3 For $n \geq 4$, every set of $\binom{2n-5}{n-2} - 2n + 10$ points in general position in the plane with an $(n-1)$ -gon as its convex hull contains a convex n -gon.

Definition 4 A tuple $(p_1, p_2, \dots, p_n)$ of n points in the plane is called a *convex n -gon* if none of its points is contained in the convex hull of the others and the counterclockwise radial order around p_1 starting outside the convex hull is $(p_2, p_3, \dots, p_n)$.

By abuse of notation, we identify a convex n -gon $(p_1, p_2, \dots, p_n)$ with its cyclic permutations and the convex hull of a finite set of points in the plane with the n -gon given by its vertices.

The above mentioned results all make use of special convex n -gons, the n -caps and n -cups.

Definition 5 Let $p_1 = (x_1, y_1), p_2 = (x_2, y_2), \dots, p_n = (x_n, y_n)$ be points in the plane with $x_1 < x_2 < \dots < x_n$. The tuple $(p_1, p_2, \dots, p_n)$ is called an *n -cap* if

$$\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \dots > \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

Fig. 2 Eight points in general position with a 4-cup above, a 4-cap below and containing no convex 5-gon

Analogously $(p_1, p_2, \dots, p_n)$ is called an n -cup if

$$\frac{y_2 - y_1}{x_2 - x_1} < \frac{y_3 - y_2}{x_3 - x_2} < \dots < \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

For $a, b \geq 2$ let $f(a, b)$ denote the least number such that every set of $f(a, b)$ points in general position in the plane contains an a -cap or a b -cup (Fig. 2). One may always fix a coordinate system such that no x -coordinates of the relevant points are equal. Since every n -cap and every n -cup is in particular a convex n -gon, one has $\text{ES}(n) \leq f(n, n)$. An advantage of considering n -caps and n -cups instead of general convex n -gons is the following: Suppose the rightmost point of an a -cap is simultaneously the leftmost point of a b -cup, then one may extend either the cap to an $(a+1)$ -cap or the cup to a $(b+1)$ -cup. This observation led to the following lemma of Erdős and Szekeres providing their upper bound to the number $\text{ES}(n)$.

Lemma 6 ([2,3]) For $a, b \geq 2$ one has

$$f(a, b) = \binom{a+b-4}{a-2} + 1.$$

A key idea of Tóth and Valtr was to transform the considered set of points into a special one without affecting the convexity relations between the points.

Definition 7 A *special point set* is a finite set M of points in the plane together with one of its elements a_1 , convex hull $(a_1, a_2, \dots, a_k)$ and $|M| \geq 3$ such that

- (1) no three of its points are on a common line,
- (2) the point a_1 has strictly the biggest y -coordinate,
- (3) only a_1 and a_3 have the same x -coordinate which is zero,
- (4) no line determined by any two points of $M \setminus \{a_1, a_3\}$ crosses the ray above a_1 on the line a_1a_3 .

The point a_1 is called the *upmost point* of the special point set and the set $M \setminus \{a_1\}$ is called the associated *reduced special point set* (Fig. 3).

The crucial advantage of considering a special point set is property (4). It guarantees that every $(n-1)$ -cup of the associated reduced special point set can be extended by the upmost point a_1 to a convex n -gon. Properties (2) and (3) are only added for convenience and were not used by Tóth and Valtr in their proof.

Fig. 3 A special point set with convex hull $(a_1, a_2, \dots, a_6)$

Lemma 8 Let M be a finite set of points in the plane with no three points on a common line, let $|M| \geq 3$ and let a_1 be a point of its convex hull. Then there exists a bijection $\varphi : M \rightarrow M'$ where $(M', \varphi(a_1))$ is a special point set and φ respects the convexity relations, i. e. a point p is in the convex hull of the points $p_1, p_2, \dots, p_n$ if and only if $\varphi(p)$ is in the convex hull of $\varphi(p_1), \varphi(p_2), \dots, \varphi(p_n)$.

Proof Since scaling and rotation of a point set do not change the convexity relations between the points, one may assume that M has properties (2) and (3). Now one proceeds as in [7]. Move the set without changing the x -coordinates such that a_1 has a strictly negative y -coordinate and no line determined by any two points of $M \setminus \{a_1, a_3\}$ crosses the segment between a_1 and $(0, 0)$. The projective transformation mapping the x -axis to the line at infinity along the y -axis gives the desired special point set $(M', \varphi(a_1))$. The transformation on the lower halfplane is explicitly given by $\varphi(x, y) = (x/y, 1/y)$. Hence, property (2) still holds for the transformed set. Moreover, $\varphi(a_1)$ and $\varphi(a_3)$ have the same x -coordinate and by slightly moving the point set M upwards before the transformation, property (3) holds and $(M', \varphi(a_1))$ is a special point set fulfilling the convexity condition. $\square$

Definition 9 Let (M, a_1) be a special point set with convex hull $(a_1, a_2, \dots, a_k)$. A tuple $(p, p_1, \dots, p_{a-1})$ is called an a -ecap if it is either an a -cap or if it is a convex a -gon, $p = a_3$ and $(p_1, \dots, p_{a-1})$ is an $(a-1)$ -cap.

Proof of Theorem 1 Let M be a set of $\binom{2n-5}{n-2} - 2n + 9$ points in general position in the plane with the property that there is no point inside the triangle determined by the consecutive vertices a_1, a_2, a_3 of its convex hull $(a_1, a_2, \dots, a_k)$. Using Lemma 8, we may assume that (M, a_1) is a special point set with only a_2 having a negative x -coordinate.

For $a, b \geq 3$ let $g(a, b)$ be the least number such that every reduced special point set with $g(a, b)$ points, associated to a special point set with upmost point a_1 and only a_2 having a negative x -coordinate, contains an a -ecap or a b -cup (without a_1). The statement of the Theorem follows from the relation

$$g(a, b) \leq \binom{a+b-4}{a-2} - (a-3) - (b-3) + 1$$

for the special case $a = n$ and $b = (n-1)$ as every n -ecap is a convex n -gon and every $(n-1)$ -cup (without a_1) can be extended to a convex n -gon by a_1 . Since $g(3, b) = 3$ for any $b \geq 3$, this is implied by an induction over $a \geq 3$ by the relation

$$g(a, b) \leq g(a-1, b) + f(a, b-1) - 2,$$

which we are now going to prove.

Let $a \geq 4$, $b \geq 3$, and M' be a reduced special point set with cardinality at least $g(a-1, b) + f(a, b-1) - 2$ associated to a special point set with upmost point a_1 and only a_2 having a negative x -coordinate. Let R be the set of rightmost points of $(a-1)$ -ecaps in M' . Since $a \geq 4$, the point a_3 does not belong to R and the set $M' \setminus R$ is a reduced special point set with only a_2 having a negative x -coordinate. If $M' \setminus R$ contains a b -cup we are done. Moreover, $M' \setminus R$ contains no $(a-1)$ -ecap. Therefore we have

$$|M' \setminus R| \leq g(a-1, b) - 1$$

and hence $|R| \geq f(a, b-1) - 1$. Set $R' = R \cup \{a_3\}$. If R' contains an a -cap we are done. Hence, assume that R' contains a $(b-1)$ -cup. If the leftmost point of this $(b-1)$ -cup is in R , it is the rightmost point of an $(a-1)$ -ecap. By the usual argument we could then either extend the ecap or the cup and we are done. If the leftmost point of this cup is a_3 then it can be extended by a_2 . $\square$

Proof of Theorem 2 Let M be a set of $\binom{2n-5}{n-2} - n - (\sum_{i=1}^{n-4} \min\{i, s\}) + 6$ points in general position in the plane with an $(n-1)$ -gon as its convex hull $(a_1, a_2, \dots, a_{n-1})$ and with a set S of s points inside the triangle determined by the points a_1, a_2, a_3 (see Fig. 1). Using Lemma 8 we may assume that (M, a_1) is a special point set.

For $a, b \geq 3$ let $h(a, b)$ be the least number such that every reduced special point set with $h(a, b)$ points, associated to a special point set with upmost point a_1 and with the property that there are at least $\min\{b-3, s\}$ points inside the triangle determined by a_1, a_2, a_3 , contains an a -ecap or a b -cup (without a_1). The statement follows from the relation

$$h(a, b) \leq \binom{a+b-4}{a-2} - (b-3) - \left(\sum_{i=1}^{b-3} \min\{i, s\} \right) + 1$$

for the special case $a = n$ and $b = (n-1)$. Since $h(a, 3) = a$ for any $a \geq 3$, this is implied by an induction over $b \geq 3$ by the relation

$$h(a, b) \leq h(a, b-1) + f(a-1, b) - 2 - \min\{b-3, s\}.$$

Let $a \geq 3$, $b \geq 4$, and M' be a reduced special point set with cardinality at least $h(a, b-1) + f(a-1, b) - 2 - \min\{b-3, s\}$, associated to a special point set with upmost point a_1 , convex hull $(a_1, \dots, a_k)$ where $k \geq b$ and with the property that there are at least $\min\{b-3, s\}$ points inside the triangle determined by a_1, a_2, a_3 . If $k > b$ we are done. Hence we may assume that M has convex hull $(a_1, \dots, a_b)$. Let R be the set of rightmost points of $(b-1)$ -cups in M' . The points $a_1, \dots, a_{b-1}$ do not belong to R because otherwise we could extend the associated $(b-1)$ -cup. Since $b \geq 4$ the set $M' \setminus R$ is a reduced special point set associated to $M \setminus R$ which has convex hull $(a_1, \dots, a_{b-1}, a'_1, \dots, a'_l)$ with $l \geq (b-1)$ and which has at least $\min\{b-4, s\}$

Fig. 4 A special point set as in the proof of Theorem 2

points inside the triangle determined by a_1, a_2, a_3 as it is guaranteed that at least the $\min\{b-3, s\}$ leftmost points of S are not rightmost points of some $(b-1)$ -cup in M' . If $M' \setminus R$ contains an a -ecap we are done. Since $M' \setminus R$ contains no $(b-1)$ -cup either, one has

$$|M' \setminus R| \leq h(a, b-1) - 1$$

and therefore $|R| \geq f(a-1, b) - 1 - \min\{b-3, s\}$. Set $R' = R \cup \{a_2\} \cup S$. Written as a disjoint union, this is $R' = R \cup \{a_2\} \cup (S \setminus (S \cap R))$ and therefore $|R'| \geq f(a-1, b)$. If R' contains a b -cup we are done. Hence assume that R' contains an $(a-1)$ -cap. If the leftmost point of this $(a-1)$ -cap is in R , it is the rightmost point of a $(b-1)$ -cup (without a_1). By the usual argument we could then either extend the cap or the cup and we are done. If the leftmost point of this $(a-1)$ -cap $(p_1, \dots, p_{a-1})$ is in $\{a_2\} \cup S$, then the line $p_{a-2}p_{a-1}$ either intersects the segment a_2a_3 or it does not. In the first case $(p_{a-2}, p_{a-1}, a_3, \dots, a_b)$ is a b -cup in M' (see Fig. 3). In the second case the tuple $(a_3, p_1, \dots, p_{a-1})$ is an a -ecap and we are done (see Fig. 4). $\square$

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