Allowable Interval Sequences and Line Transversals in the Plane

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Abstract Given a family F of n pairwise disjoint compact convex sets in the plane with non-empty interiors, let $T(k)$ denote the property that every subfamily of F of size k has a line transversal, and T the property that the entire family has a line transversal. We illustrate the applicability of allowable interval sequences to problems involving line transversals in the plane by proving two new results and generalizing three old ones. Two of the old results are Klee's assertion that if F is totally separated then $T(3)$ implies T , and the following variation of Hadwiger's Transversal Theorem proved by Wenger and (independently) Tverberg: If F is ordered and each four sets of F have some transversal which respects the order on F , then there is a transversal for all of F which respects this order. The third old result (a consequence of an observation made by Kramer) and the first of the new results (which partially settles a 2008 conjecture of Eckhoff) deal with fractional transversals and share the following general form: If F has property $T(k)$ and meets certain other conditions, then there exists a transversal of some m sets in F , with $k < m < n$. The second new result establishes a link between transversal properties and separation properties of certain families of convex sets.

1 Introduction

One of the central questions of transversal theory in the plane is the following: If a family F of compact convex sets in the plane with non-empty interiors has the property $T(k)$ for some $k < |F|$ (that is, for every k sets of F there exists a line transversal), what can be said about line transversals for all of F ? It has been known for a long time that there is no Helly-type theorem of this sort; that is, $T(k)$ does not in general imply T (the existence of a transversal for the entire family) for any k .

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However, many results have been proven which assert that under appropriate conditions $T(k)$ indeed implies T for some small value of k . For instance, Tverberg [15] showed that if F is a family of disjoint translates then $T(5)$ implies T . Another such result is a theorem of Klee:

Theorem 1 (See [12]) *In a totally separated family F , not necessarily finite, $T(3)$ implies T .*

Another is a variation of Hadwiger's Transversal Theorem proved by Wenger and independently by Tverberg:

Theorem 2 (See [16, 17]) *Given an ordered family F of n pairwise disjoint compact convex sets in the plane ($n \geq 4$), if every four sets can be intersected by some directed line consistent with the ordering, then there exists a transversal to all of F consistent with the ordering.*

We will provide new proofs of these two results, at the same time generalizing them.

In 1980, Katchalski and Liu [11] initiated a new direction of study when they proved the following “fractional Helly theorem” for line transversals: For any $0 < \alpha < 1$, there exists $k = k(\alpha)$ such that if F (defined as above) has property $T(k)$ then there exists a transversal for some $\alpha|F|$ sets of F (this property is denoted αT , and we refer to the intersecting line as an α -transversal). Or, stated dually: For any k (sufficiently large, but in fact true even for $k = 3$) there exists $\alpha = \alpha(k)$ such that $T(k)$ implies αT . Katchalski (in an oral communication reported by Eckhoff [2]) conjectured that $\alpha(3) = \frac{2}{3}$; this was subsequently disproved by Holmsen [9] who showed that $\frac{1}{3} \leq \alpha(3) \leq \frac{1}{2}$. In a recent paper, Eckhoff [2] has generalized this approach by defining, for all integers $m > k \geq 3$, the number $N(m, k)$ to be the smallest integer satisfying the following statement: If F has property $T(k)$ and $|F| = N(m, k)$, then some m members of F have a common transversal (the existence of $N(m, k)$ follows from the result of Katchalski and Liu). He states without proof that $N(4, 3) = 6$ and conjectures that $N(k + 1, k) = k + 2$ for all $k \geq 4$.

We will prove three results relating to fractional transversals for families of pairwise disjoint convex sets. In the first we introduce the concept of an α -balanced family and prove using interval sequences a consequence of an observation attributed to Kramer in [3]:

Theorem 3 *Every α -balanced family possesses an α -transversal.*

We make progress on Eckhoff's conjecture:

Theorem 4 *For all $k \geq 8$, we have $N(k + 1, k) = k + 2$.*

In the process of proving Theorem 4 we discover the following interesting connection between transversal properties and separation properties:

Theorem 5 *If F is a family of $k + 2$ pairwise disjoint convex sets in which every k sets have a common transversal and no $k + 1$ sets do ($k \geq 3$), then no set of F can be separated from four sets of F by a line. Furthermore, no two sets of F can be separated from three sets of F by a line.*

Our method of proof for all of these results is that of allowable interval sequences. These sequences were introduced by Goodman and Pollack [6] in 2005 as a method of encoding the combinatorial properties of a family of pairwise disjoint convex sets in the plane. As such they represent a generalization of allowable permutation sequences, introduced 25 years earlier by the same authors [5] as an encoding of point configurations in the plane. Allowable permutation sequences have proven over the decades to be a valuable tool in yielding many results about point configurations, and allowable interval sequences have recently been used [13] to significantly improve a result of Hope and Katchalski [10] on the $(1, k)$ -separation question originally investigated by Tverberg [14]. The broad goal of this paper is to illustrate the applicability of allowable interval sequences to questions related to line transversals in the plane.

2 Interval Sequences and Line Transversals

We review here the essential properties of allowable interval sequences (see [1, 6] for more details). Let F be a family of n pairwise disjoint compact convex sets in the plane labeled $1, \dots, n$, and let L be a line which lies below F . The projection of each set on L is an interval. For each $1 \leq i \leq n$, we denote the left and right endpoints of the projection of set i on L by i and i' , respectively. Thus the projection of F on L is encoded by a *double permutation* of $[n] := \{1, \dots, n\}$, consisting of one set of unprimed indices and one set of primed indices. In any such double permutation, i precedes i' for every $1 \leq i \leq n$. When we speak of the *type* of an index we refer to its status as primed or unprimed, so that, e.g., i and j' are indices of different types but i and j are of the same type. For any $1 \leq i \leq n$, we call i' the *complement* of i and vice versa.

Let L move counter-clockwise around F . The double permutation encoding of F changes every time L crosses orthogonally a line tangent to two sets of F . The change which occurs at every such crossing is a transposition of two consecutive indices, or a *switch*. We denote a switch by the corresponding pair of transposed indices in the order they appear after the switch (see Fig. 1). A switch between two indices of the same type (primed or unprimed) is called a *supporting switch* because it occurs when L crosses a supporting tangent to two sets. A switch between two indices of different types is called a *separating switch* because it occurs when L crosses a separating tangent. A separating switch in which the unprimed index precedes the primed one is called an *opening switch*, and one in which the primed index comes first is called *closing* (the reasons for these terms will be clarified shortly).

Continuing the movement of L around F , we obtain an infinite *sequence of double permutations*, or *interval sequence* $I(F)$, possessing the following properties:

- (1) $I(F)$ is periodic, and each period consists of two half-periods. Each term in the second half-period is obtained by reversing the corresponding term in the first

half-period, and changing all primed indices to unprimed and vice versa (we henceforth refer to this term as the *reverse* of the original term, and denote the reverse of $\tau \in I(F)$ by $r(\tau)$).

- Note that yx is not the reverse of xy ; these two switches may in fact occur within a single half-period. We call yx the *inverse* of xy and refer to the pair $\{xy, yx\}$ as an *inverse pair* of switches.

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from the preceding term in exactly one switch. By the second property, the period of the sequence in this case is exactly $8\binom{n}{2}$, corresponding to the fact that an arrangement of n convex sets in general position determines $4\binom{n}{2}$ tangents (each of which is crossed by L in a half-period), $2\binom{n}{2}$ each of the supporting and separating kinds.

If an element is not contained in any separating switch in a given half-period P of $I(F)$, then we say it is *unseparated in P* (all other elements are *separated*). The following lemma was proven in [13] and will be of use here as well; the proof is short enough that we reproduce it in full.

Lemma 2.1 *In any half-period P of $I(F)$, at most one unprimed (resp., primed) element is unseparated in P .*

Proof Suppose some unprimed element i is unseparated in P . By the second property of allowable interval sequences, every unprimed element $j \neq i$ must switch with i' in P (in both directions, in fact: $i'j$ and ji'). Thus i is the only unprimed element unseparated in P . $\square$

As a convention, we will use lower-case Latin letters to refer to labels (or *indices*) of sets in F . We use a lower-case Greek letter to refer to three related items (taking care to avoid confusion): a term in $I(F)$, the pair of elements in a switch (in which case the term of $I(F)$ yielded by that switch will be denoted by the same letter with a bar over it), and a direction in the plane. When we speak of a term of $I(F)$ corresponding to some direction θ , we mean the term in which the projection is in the direction θ , and hence L is normal to θ .

We have assumed in this introduction that F is finite. If F is infinite (and hence countably infinite) then we may still define an interval sequence $I(F)$, with an infinite period and in which each term is an infinite double permutation of $\mathbb{N} \cup \mathbb{N}'$. The rules governing switches between indices remain identical to the finite case. Unless otherwise specified, we shall continue to assume in general that F is finite.

F possesses a line transversal iff there exists a term in the sequence $I(F)$ in which all unprimed indices precede all primed indices (corresponding to a position of L orthogonal to a line transversal); we call this a *transversal term*. Similarly, given a subset $S \subset [n]$, $|S| \geq 2$, the subfamily of F with indices in S possesses a line transversal iff there exists a term in $I(F)$, known as a partial transversal term, whose restriction to S is a transversal term. Note that the reverse of a (partial) transversal term is again a (partial) transversal term, corresponding to the fact that transversals are undirected. Thus if there exists a transversal for (some subfamily of) F , there exists a corresponding (partial) transversal term in *every* half-period of $I(F)$.

The opening switch ij' creates a partial transversal term for the subset $S = \{i, j\}$. It may also create a partial transversal for other subsets of $[n]$, all of which contain S . Every subsequent term in $I(F)$ is a partial transversal for S until the closing switch $i'j$. From that point no partial transversal of S appears until the opening switch ji' , a half-period after ij' (see Fig. 2). Thus an opening switch initiates a string of partial transversal terms for (some subset of $[n]$ containing) the indices of the switch, while a closing switch terminates this string (this explains the choice of “opening” and “closing”).

Fig. 2 The positions of switches for two sets labeled i and j

Throughout this work we will assume that the sets in a given family of convex sets are in general position, and hence that the associated interval sequence is simple. We are justified in doing so because the properties of these families which we assume and which we attempt to prove—namely, the existence of certain types of transversal—exist in a general family F if and only if they exist in some family F' in general position obtained from F by perturbing some or all of its sets very slightly. Thus every switch referred to in this work consists of exactly two elements.

We complete this background section by recalling that allowable interval sequences encode not only families of convex sets as described above but also families of compact connected sets together with a pseudoline arrangement of supporting and separating tangents (see [6]). In fact, it has recently been shown (see [7], Sect. 6) that there are natural geometric objects which realize every allowable interval sequence, namely families of pairwise disjoint convex bodies in a topological plane. Hence even the proofs we provide of already established results regarding convex sets serve to generalize these results to these broader classes of families. Although some of our new results may be amenable to more direct geometric proofs using the properties of topological planes, this does not detract from the significance of the combinatorial proofs presented here.

3 Results of Klee and Wenger

Using the basics of interval sequences outlined in the previous section, we can prove Theorem 1 quite simply. A family F of convex sets is called *totally separated* if there exist mutually parallel lines which partition the plane into strips such that each strip contains exactly one set of F .

Theorem 1 (Klee, [12]) *In a totally separated family F (not necessarily finite), $T(3)$ implies T .*

Proof We assume first that F is finite, and let $|F| = n$. The sets of F are obviously pairwise disjoint, so consider a term $\sigma \in I(F)$ corresponding to the direction of L orthogonal to some parallel lines which separate the sets in F . Up to a permutation of $[n]$ (which we may assume without loss of generality to be the identity permutation), this term is $11'22' \dots nn'$ (see Fig. 3).

Fig. 3 The term $\sigma \in I(F)$, with $n = 4$

We make two important observations regarding σ ; first, it is not a partial transversal term for any subset of $[n]$; second, every pair of (non-complementary) indices appears in reverse orders in σ and in $r(\sigma) = nn' \dots 22' 11'$ (e.g., 2 appears before n' in σ , but 2 follows n' in $r(\sigma)$). The last observation implies that every non-complementary pair of indices switches exactly once in the half-period P of $I(F)$ from σ to $r(\sigma)$; there can be no inverse pairs in P . Specifically, there occurs in P the opening switch ji' and subsequently the closing switch $j'i$ for every $1 \leq i < j \leq n$, and no other separating switches.

Consider the first closing switch $\tau = b'a$ in P . For the remainder of P after τ , there can be no partial transversal for any subset of $[n]$ containing the subset $S = \{a, b\}$. But the $T(3)$ condition implies that in P there exists a partial transversal for each set of the form $S \cup \{i\}$ ($i \in [n] \setminus \{a, b\}$). Thus before τ there must occur in P an opening switch for every pair of indices $\{b, i\}$ and $\{a, i\}$, and no closing switch (as τ is assumed to be the first closing switch in P). Specifically, just before τ there cannot be any unprimed index after b' , nor any primed index before a . Thus the term of $I(F)$ just before τ is a transversal term for F , proving the desired property of F .

If F is infinite, then as described in the last section we can still define the sequence $I(F)$, and the proof remains essentially identical. The major difference is technical—one must define an order on $\mathbb{N}$ corresponding to the left-to-right order of the sets of F in σ (the natural ordering will not necessarily work because there may not be a minimal, i.e., leftmost, element). The requirement that every non-complementary pair of indices switch exactly once in the half-period from σ to $r(\sigma)$ remains true, and thus so does the discussion in the last paragraph. $\square$

For any transversal of a family F of n pairwise disjoint convex sets, the pair of reverse orderings in which the sets of F are intersected is known as the *geometric permutation* of F associated with the transversal. As pointed out by Goodman and Pollack [6] in their proof that F has no more than $2n - 2$ geometric permutations (a result proved originally by Edelsbrunner and Sharir [4]), the interval sequence $I(F)$ reveals not only whether a transversal term exists but also, assuming such a term exists, the geometric permutation associated with it. Let a directed transversal, pointing upwards from L , meet the set j and then the set i . Then the switches $j'i$ and ij' occur in the following half-period, while the switches $i'j$ and ji' occur in the

preceding half-period (see Fig. 2). This observation will help to prove the following lemma:

Lemma 3.1 *The following two statements about F are equivalent:*

- (a) *There exists a transversal intersecting the sets of F in the order $1, 2, \dots, n$.*
- (b) *In some period of $I(F)$, one half-period contains all switches of the form $i'j$ and the other half-period contains all switches of the form ij' ($1 \leq i < j \leq n$).*

Proof The first statement implies the second by the observation made in the last paragraph. For the reverse direction, note that if θ is a direction in the plane in which there does not exist a transversal to the sets indexed i and j , then there is a line in this direction which strictly separates these two sets; hence the switches ij' and $i'j$ both occur either in the half-period following, or the one preceding, the term in $I(F)$ corresponding to θ . Thus, if μ is a direction such that the half-periods of $I(F)$ preceding and following a term corresponding to μ satisfy statement (b) then μ is a transversal direction for every pair of sets in F and hence a transversal direction for all of F . The order in which the sets are intersected by a transversal in this direction follows again from the observation made before the lemma. $\square$

With this lemma in hand we can prove Theorem 2, due to Wenger [17] and, independently, Tverberg [16]. Hadwiger's theorem states that in an ordered family of pairwise disjoint compact convex sets in the plane, if every three sets can be intersected by a directed line consistent with the ordering on the sets, then there exists a transversal to the entire family. Interestingly enough, the theorem does not claim that the common transversal intersects the family in a manner consistent with the ordering, and there exists a simple counterexample consisting of four line segments (see [17], Fig. 2). This anomaly can be rectified by replacing the word “three” with “four” in the statement of the theorem:

Theorem 2 *Given an ordered family F of n pairwise disjoint compact convex sets in the plane ($n \geq 4$), if every four sets can be intersected by some directed line consistent with the ordering, then there exists a transversal to all of F consistent with the ordering.*

To prove this theorem we need a lemma from elementary geometry:

Lemma 3.2 *Let P , Q , and R be three closed arcs of a circle C , each smaller than a semicircle, such that $P \cup Q \cup R = C$, and let a, b, c be points of C satisfying $a \in P \cap Q$, $b \in Q \cap R$, $c \in P \cap R$. Then the points a, b , and c do not lie in one closed semicircle of C .*

Proof We first note that the points a, b , and c must be distinct, for otherwise $P \cap Q \cap R \neq \emptyset$; if (without loss of generality) $a \in P \cap Q \cap R$ then the point opposite a on C cannot be in any of the three arcs, contradicting the assumption that $P \cup Q \cup R = C$. We assume that the three points lie in some semicircle and derive a contradiction. Let S be the smallest closed arc of C containing the three points (no larger than a

semicircle, by assumption), with $T = \overline{C \setminus S}$. Suppose without loss of generality that the endpoints of S are a and c ; then b is in the interior of S , and let $d \in \text{int}(T)$ be the point opposite b on C . If $d \in Q$ then Q must contain the two antipodal points d and b , an impossibility (and similarly for R if we assume $d \in R$). If $d \in P$ then P contains both endpoints of T and an interior point of T , hence $T \subset P$, a contradiction because T is no smaller than a semicircle. We conclude that $d \notin P \cup Q \cup R = C$, a contradiction. $\square$

We are now ready to prove Theorem 2:

Proof By Lemma 3.1, it suffices to prove the following about F : If for every four indices $1 \leq i < j < k < l \leq n$ there exists one half-period of $I(F)$ containing the switches $i'j, i'k, i'l, j'k, j'l$, and $k'l$ whose complementary half-period contains the switches ij', ik', il', jk', jl' , and kl' (this is the $T(4)$ condition), then there exists a half-period of $I(F)$ containing all switches of the form $a'b$ whose complementary half-period contains all switches of the form ab' ($a < b$).

On a circle C , label $4\binom{n}{2}$ points corresponding radially to the directions of the $4\binom{n}{2}$ separating switches in a period of $I(F)$ (each point labeled by the appropriate switch). For $1 \leq j \leq n$ let A_j denote the smallest closed arc of C containing the $n-1$ points labeled $i'j$ and $j'k$ ($1 \leq i < j < k \leq n$). The assumption about F implies that A_j is smaller than a semicircle for each j . Clearly for every $i < j$ we have $A_i \cap A_j \neq \emptyset$, as both arcs contain the point $i'j$.

No three arcs of the form A_i, A_j, A_k span all of C , for if they did, then by Lemma 3.2 we would conclude that $i'j, i'k$, and $j'k$ do not lie in one semicircle, a contradiction. Thus (by Helly's Theorem in one dimension) these three arcs have a common point. By a well-known and basic result (see, e.g., [8], p. 7, #17), in any family of circular arcs, each smaller than a semicircle, if every three have a common point then the entire family has a common point; thus we have $\bigcap_{1 \leq i \leq n} A_i \neq \emptyset$, and hence $A = \bigcup_{1 \leq i \leq n} A_i \neq C$. We claim that A is smaller than a semicircle. Suppose the endpoints of A are $i'j$ and $k'l$. If either of i, j (say i , without loss of generality) is the same index as either of k or l then these endpoints both lie on A_i , along with other points of A ; hence $A = A_i$ and thus is smaller than a semicircle. If the indices i, j, k, l are all distinct (and assuming without loss of generality that $i < k$) then let $D = \overline{C \setminus A}$; if D is smaller than a semicircle then we may apply Lemma 3.2 to the arcs D, A_i and A_k to yield the conclusion that the points $i'j, i'k$, and $k'l$ do not lie in one semicircle, a contradiction. Thus we conclude that D is at least a semicircle, so that A is at most a semicircle, as claimed. By our assumption that $I(F)$ is simple, A is smaller than a semicircle.

Now define B_j ($1 \leq j \leq n$) to be the smallest closed arc of C containing the points ij' and jk' ($1 \leq i < j < k \leq n$). By the same arguments as for the arcs A_j , we conclude that $B = \bigcup_{1 \leq i \leq n} B_i$ is smaller than a semicircle. We claim that $A \cap B = \emptyset$. Establishing this claim clearly completes the proof, for if A and B (both smaller than a semicircle) are disjoint then there exists some semicircle containing A whose complement contains B .

Suppose that A and B are not disjoint, so that $p_0 = ij' \in A$ for some $i < j$. Let the endpoints of A be $p_1 = k'l$ and $p_2 = r's$ for some $k < l, r < s$ (we assume that

Fig. 4 A balancing pair for an α -balanced family

the indices i, j, k, l, r, s are all distinct, but to assume otherwise only simplifies what follows). The $T(4)$ condition applied to the indices i, j, k, l implies that the point $i'j$ is located on A between the endpoint p_1 and p_0 , but the same condition applied to the indices i, j, r, s implies that this point is located on A between p_0 and the endpoint p_2 . This contradiction refutes the assumption that A and B intersect. $\square$

4 Some Results on Fractional Transversals

The results of the last section addressed situations in which it can be shown that a transversal exists for an entire family of convex sets in the plane. In this section we provide weaker conditions under which some subfamily of a given family must possess a common transversal.

Definition 4.1 A pair of elements k, l in a family F of convex sets is an α -balancing pair for F ($0 < \alpha < \frac{1}{2}$) if the separating tangents to k and l divide the plane into quadrants such that each of the two quadrants whose interiors do not meet k or l contains at least $\alpha|F|$ sets of F in its interior. A family F containing an α -balancing pair is said to be α -balanced.

In proving the next theorem we use the following basic geometric fact (see, e.g., [3]): There does not exist a line transversal to three convex sets iff each of the three sets can be strictly separated from the other two by a line.

Theorem 3 An α -balanced family F with property $T(3)$ has an α -transversal.

Proof Let $|F| = n$, and let s, t be a balancing pair for F with separating tangents l_1, l_2 . Let Q_1 denote the quadrant formed by these tangents and traversed as one moves counter-clockwise from s to t (if we imagine l_1 to be the x -axis and l_2 the y -axis, then Q_1 is the third quadrant), and let Q_2 denote the quadrant opposite Q_1 (see Fig. 4).

Let $A \subset F$ denote the sets in the interior of Q_1 and let $B \subset F$ denote the sets in the interior of Q_2 . By assumption, $|A|, |B| \geq \alpha n$. Note that given any two sets $a \in A$ and $b \in B$, the line l_1 separates sets s and b from a , while l_2 separates s and a from

b . Thus by the $T(3)$ condition, there cannot exist a line which separates a and b from s . The same clearly holds with t replacing s .

Consider the terms in $I(F)$ corresponding to the directions of l_1 and l_2 at the boundary of Q_1 , i.e., the terms after the separating switches $s't$ and ts' (henceforth we restrict all terms of $I(F)$ to the elements s, s', t, t' , and the sets of A and B , as the other sets of F do not concern us further). The first is of the form $B_1s'tA_1$, where B_1 is a string consisting of the indices of the sets in B (unprimed and primed) along with the unprimed element s , and A_1 is a string consisting of the indices of the sets in A (unprimed and primed) along with the primed element t' . The second is of the form $A_2ts'B_2$, where A_2 is a string consisting of the indices of the sets in A (unprimed and primed) along with the unprimed element s , and B_2 is a string consisting of the indices of the sets in B (unprimed and primed) along with the primed element t' . Thus, in the subsequence of (a period of) $I(F)$ corresponding to the directions of quadrant Q_1 , the terms change such that all indices of sets in A shift backward while all indices of sets in B shift forward. That is, for any $a \in A, b \in B$, the switches $ab, ab', a'b$, and $a'b'$ occur somewhere in this quadrant. Throughout the terms of $I(F)$ in this quadrant (until the last one), the indices of sets s and t appear in the order $s \dots s' \dots t \dots t'$.

Consider the term $\sigma \in I(F)$ corresponding to the first closing switch in this quadrant of the form $a'b$ ($a \in A, b \in B$). We claim that the separating tangent l to sets a and b in this direction is a transversal either for all the sets of F in A or for all those in B ; hence this line is the desired α -transversal. Note that since σ corresponds to the first closing switch of Q_1 , all primed indices of sets in A must appear after a' , and all unprimed indices of sets in B must appear before b . Hence, if the separating tangent to a and b is not a transversal for A nor for B then in σ there must appear a primed index $\tilde{b}'$ before b and some unprimed index $\tilde{a}$ after a' ; i.e., σ must have the form $\dots \tilde{b}' \dots a'b \dots \tilde{a} \dots$. It is impossible for the indices s, s', t , and t' to appear (in that order) in σ unless σ corresponds to a separating direction of sets a and b from t or sets $\tilde{a}$ and b from s , contradicting the $T(3)$ condition on F . $\square$

Our second result addresses Eckhoff's conjecture that $N(k+1, k) = k+2$ for all $k \geq 4$. We prove this conjecture for a slightly smaller range of k :

Theorem 4 $N(k+1, k) \leq k+3$; if $k \geq 8$, then $N(k+1, k) = k+2$.

Proof Consider a family F of $k+3$ convex sets with property $T(k)$. We establish a natural mapping Φ (introduced in [9]) from the set X of k -tuples of sets in F to the set Y of closing separating tangents to pairs of sets of F as follows: Given any k -tuple $x \in X$ of sets of F , the corresponding tangent $\Phi(x)$ is the unique one obtained by rotating any transversal l of x counter-clockwise until it is about to cease to be a transversal for all k sets of x . Note that if Φ is not injective, then some separating tangent is a transversal for two distinct k -tuples of sets in F , and hence some $(k+1)$ -tuple of sets in F possesses a transversal. Thus, if F lacks a $(k+1)$ -transversal, then Φ must be injective, and hence $|X| \leq |Y|$. But $|X| = \binom{k+3}{k} = \binom{k+3}{3}$ and $|Y| \leq \binom{k+3}{2}$ (with equality iff the sets of F are pairwise disjoint). Since $\binom{k+3}{3} > \binom{k+3}{2}$ for $k > 2$, we conclude that some $k+1$ sets of F possess a transversal, and hence $N(k+1, k) \leq$

Fig. 5 A family which illustrates $N(4, 3) > 5$. Every three sets possess a common transversal but no four sets do

$k + 3$. Specifically, since there exists a family of five convex sets in the plane such that every three sets have a common transversal but no four sets do (see, e.g., Fig. 5), we conclude that $N(4, 3) = 6$, verifying Eckhoff's claim.

Now suppose that $|F| = k + 2$ and that no $(k + 1)$ -tuple of sets in F possesses a transversal. In this case $|X| = \binom{k+2}{k} = \binom{k+2}{2}$ and $|Y| \leq \binom{k+2}{2}$, so Φ can be injective only if the last inequality is actually an equality, i.e., if the sets of F are pairwise disjoint. This immediately shows that $N(k + 1, k) \leq k + 2$ for families with some non-disjoint pair of sets, so any counterexample to Eckhoff's conjecture must consist of pairwise disjoint sets. We thus assume henceforth that F is such a family and we consider the corresponding interval sequence $I(F)$.

Any given half-period P of $I(F)$ contains, for every k sets of F , a transversal term for these sets. Since no $k + 1$ sets have a transversal, each k -transversal corresponds to a distinct string of terms in $I(F)$, initiated by its own opening switch and terminated by its own closing switch (if P begins in a transversal direction for a given k -tuple of sets, then the closing switch of that k -tuple will occur before the opening switch). Since the number of opening switches in P (and similarly for closing switches) is equal to the number of k -tuples of sets in F (i.e., $\binom{k+2}{k} = \binom{k+2}{2}$), we conclude the following: *Every* opening switch in P initiates a string of terms in $I(F)$ corresponding to a transversal for a distinct k -tuple of sets in F , and *every* closing switch in P terminates such a string, breaking a k -transversal into two $(k - 1)$ -transversals. Note that the elements in every such separating switch (opening or closing) must be preceded, within the term of P in which the switch occurs, by at least $k - 1$ unprimed elements (corresponding to the sets in the transversal opened or closed by the switch) and similarly must be followed by at least $k - 1$ primed elements. Specifically, no separating switches can occur in P (and hence in $I(F)$) among the first $k - 1$ elements in a term nor within the last $k - 1$ elements in a term. It is clear that no primed element can appear among the first $k - 1$ in any term of P . For if some primed element did, it would not be in any separating switches and thus would not appear after k unprimed elements in any term of P ; hence the corresponding set would not be part of any k -transversal, contrary to assumption. So every term of P begins with $k - 1$ unprimed elements and ends with $k - 1$ primed elements (the *uniform regions*), leaving only six positions in the middle (the *separating zone*) in which separating switches can occur. It is easy to see (and also explained in [13]) that this is the exact characterization of a family in which no set can be separated from four others. Applying the result from [13] bounding the size of such a family, we have proven all but the last sentence of Theorem 5:

Theorem 5 *If F is a family of $k + 2$ pairwise disjoint convex sets in which every k sets have a common transversal and no $k + 1$ sets do ($k \geq 3$), then no set of F can be separated from four sets of F by a line; hence, $k \leq 16$. Furthermore, no two sets of F can be separated from three sets of F by a line.*

To prove the last sentence, note first that if there exists a line l such that two sets of F lie to the left of l and three sets of F lie to its right, then the corresponding term σ of $I(F)$ must contain two primed elements appearing before three unprimed ones. But given the structure of each term of $I(F)$ as determined in the last paragraph (two uniform regions and a six-element separating zone), this can only happen if the first two elements in the separating zone are both primed (call them a' and b' , from left to right). But this cannot happen because of the condition on the opening and closing separating switches. In some subsequent term $\tau \in I(F)$ an unprimed element from the left uniform region must enter the separating zone by switching with another unprimed element c leaving the separating zone; assume without loss of generality that σ is the last term before τ in which the first two elements of the separating zone are primed. Between σ and τ the switches cb' and ca' need to occur (without a closing switch containing c between them). Clearly, if the first of these to occur initiates a k -transversal then the second initiates a $(k + 1)$ -transversal, contrary to assumption.

To improve the upper bound on k and prove Theorem 4, we adopt a strategy similar to that used in [13]. We will establish a correspondence between separating switches and certain supporting switches in P such that every separating switch corresponds to some supporting switch. This correspondence will have two important properties: Many supporting switches will not be associated to any separating switch at all, and none will be associated to more than two separating switches. These properties, combined with the fact that P must contain exactly $2\binom{k+2}{2}$ each of supporting and separating switches, will provide us with an upper bound on k .

The first step in establishing our correspondence is the following lemma:

Lemma 4.1 *Let F be a family as described in Theorem 5, and let γ be a separating switch in $I(F)$. For at least one of the two elements in γ (say a) it must be true that the last switch before γ in $I(F)$ containing a is a supporting switch.*

We introduce the proof of Lemma 4.1 by defining, for an index i and a term $\sigma \in I(F)$, the number $m(i, \sigma)$ to be the largest number of sets met by a transversal that meets the set i and has the direction corresponding to σ . That is, $m(i, \sigma)$ is the size of the largest set $S \subseteq [n]$ such that $i \in S$ and such that σ can be split down the middle into subwords σ_1, σ_2 with $a \in \sigma_1, a' \in \sigma_2$ for all $a \in S$. By the discussion preceding Theorem 5, in our case we must have $m(i, \sigma) \in \{k, k - 1\}$ for all $i \in [n]$ and all $\sigma \in I(F)$. We also define, for $i \in [n] \cup [n]'$ and $\sigma \in I(F)$, the switch $\rho(i, \sigma)$ to be the last switch before σ containing i .

The proof of the lemma will proceed as follows: Suppose that $\gamma = ab'$ is an opening switch. We first show that neither $\rho(a, \gamma)$ nor $\rho(b', \gamma)$ is opening, and then we show that these two switches cannot both be closing. The only possibility remaining is that one of these switches is supporting, as the lemma claims. Similar arguments

yield the same result if γ is closing; we mention the main points without repeating all the details.

If $\gamma = ab'$ is opening then we show that neither $\rho(a, \gamma)$ nor $\rho(b', \gamma)$ is an opening switch. For suppose that $\rho(a, \gamma)$ is opening (a nearly identical argument follows if $\rho(b', \gamma)$ is opening). Then since every opening switch in $I(F)$ opens a k -transversal, we have $m(a, \rho(a, \gamma)) = k$ (recall that $\rho(a, \gamma)$ denotes the term of $I(F)$ yielded by the switch $\rho(a, \gamma)$). If $S \subset [n]$ denotes the set of indices in the k -transversal opened by $\rho(a, \gamma)$, then a follows every other unprimed index of S in $\rho(a, \gamma)$ and precedes every primed index of S in $\rho(a, \gamma)$. Since a does not switch with any element between $\rho(a, \gamma)$ and γ , in $\overline{\gamma}$ as well a follows every other unprimed index of S and precedes every primed index of S . But in $\overline{\gamma}$, a also follows b and precedes b' , so $m(a, \overline{\gamma}) = k + 1$ (with $S' = S \cup \{b\}$ the set of transversal indices), contrary to the assumption that F has no $(k + 1)$ -transversal.

Similarly, if $\gamma = a'b$ is closing, then it cannot be that either $\rho(a', \gamma)$ or $\rho(b, \gamma)$ is an closing switch; an argument parallel to that of the last paragraph shows that if, say, $\rho(a', \gamma)$ is closing, then $m(a', \overline{\gamma}) = k - 2$, contrary to a property we have deduced regarding F .

Now we show that $\rho(a, \gamma)$ and $\rho(b', \gamma)$ cannot both be closing. If these switches are distinct (and hence the latter of them is not $b'a$), then this is a simple matter. Assume without loss of generality that $\rho(a, \gamma)$ is the latter. Then just after $\rho(a, \gamma)$, a must immediately follow b' ; since $\rho(a, \gamma) \neq b'a$, we conclude that $\rho(a, \gamma)$ is not a closing switch.

The only remaining obstacle to proving the lemma is ruling out the case that $\rho(a, \gamma) = \rho(b', \gamma) = b'a$. To do so, we suppose this equation holds and derive a contradiction. Let $S \subset [n]$ denote the subfamily of F of size k whose transversal is opened by γ . Since both a and b' are not contained in any switches between $b'a$ and γ , it must be that $b'a$ closes a transversal of S . Let δ be the switch which opens the S -transversal closed by $b'a$, and consider the half-period P of $I(F)$ starting with δ . There are (altogether in $I(F)$) $k(k - 1) = 2\binom{k}{2}$ closing switches with both indices in S , only one of which (namely $b'a$) occurs between after δ and before ab' . This leaves no more than $2\binom{k+2}{2} - 2\binom{k}{2} = 4k + 2$ closing switches which can occur after δ and before ab' ; since every half-period of $I(F)$ contains $\binom{k+2}{2} > 4k + 2$ closing switches (the inequality holds for $k > 5$), we conclude that $\overline{\gamma} \in P$, and hence P contains two switches which open a transversal for S . This contradicts what we know must be true of $I(F)$, namely that every opening switch in a half-period opens a distinct k -transversal. A similar argument shows that if $\gamma = a'b$ is closing then we cannot have $\rho(a', \gamma) = \rho(b, \gamma) = ba'$.

Thus we have proven that neither of $\rho(a, \gamma)$ and $\rho(b', \gamma)$ is an opening switch, and at most one of them is closing; we conclude that at least one is supporting, concluding the proof of Lemma 4.1.

Using this lemma we can establish the desired correspondence between separating and supporting switches in P . To a separating switch γ containing the indices a and b' we associate one of the switches $\rho(a, \gamma)$ and $\rho(b', \gamma)$. If only one of these is supporting then that is the one we associate to γ ; if both are supporting, then it makes no difference which one we associate to γ . In the event that these switches (or the one of them which is supporting) are not in P , we associate to γ one of the reverse

switches $r(\rho(a, \gamma))$ and $r(\rho(b', \gamma))$, which by the symmetry of $I(F)$ are in P and are not associated to any other separating switches in P .

We note two important facts about this correspondence. First, a supporting switch associated to any separating switch must occur in the separating zone or at a “door” to the separating zone (i.e., an element in the zone trades places with one outside the zone); any supporting switch occurring within the uniform regions in any term of $I(F)$ is not associated to a separating switch. Second, a supporting switch ab can only be associated to a separating switch with which it shares one element in common and only to one separating switch for each such element; hence, each supporting switch is associated to no more than two separating switches. A door switch can only be associated to at most one separating switch, as one of the switched elements is outside the separating zone following a door switch.

We wish to find the minimum number of supporting switches in P which occur in the uniform regions; as these switches are not associated to any separating switches, this minimum will allow us to bound from above the number of separating switches in P and hence the size of k . By Lemma 2.1, every element in the uniform regions at the start of P must migrate to the separating zone with the possible exception of one primed and one unprimed element. As shown in [13], we actually minimize the number of supporting switches in the uniform regions by assuming that *all* these elements migrate to the separating zone in P . This accounts for $2(1 + 2 + \cdots + (k - 1)) = k(k - 1)$ supporting switches, of which $2k - 2$ are door switches and the rest occur within the uniform regions (and hence do not correspond to any separating switches). As P contains $2\binom{k+2}{2} = (k + 2)(k + 1)$ supporting switches altogether, this leaves at most $(k + 2)(k + 1) - k(k - 1) = 4k + 2$ supporting switches which can occur within the separating zone in P . Thus, by our correspondence, the number of separating switches in P is at most $(2k - 2) + 2(4k + 2) = 10k + 2$; yet we know that the number of separating switches in P is exactly $2\binom{k+2}{2} = (k + 2)(k + 1)$. Solving the inequality $(k + 2)(k + 1) \leq 10k + 2$ yields $k \leq 7$. Hence, for $k \geq 8$ there cannot exist a family of $k + 2$ pairwise disjoint convex sets in the plane such that each k have a common transversal but no $k + 1$ do.

5 Conclusion

While we have demonstrated several results on line transversals which can be proven or generalized using allowable interval sequences, a number of questions remain. First, can Theorem 4 be proven for the range $4 \leq k \leq 7$? Even a slight tightening of the correspondence used in our proof should extend this result to $k = 7$. Second, the doubly restrictive separation property proved in Theorem 5 raises the following general question: For arbitrary k and l (with $l < k$), how large can a family of pairwise disjoint convex sets be if it lacks both $(1, k)$ -separation and $(2, l)$ -separation? It is possible that the proof of an upper bound in [13] for families lacking $(1, k)$ -separation can be modified to address this question. Finally, the following question on partial transversals seems very suited to the approach used in this paper: Given a family F of convex sets satisfying the $T(3)$ property, what is the smallest k such that we can always find k lines whose union intersects all sets in F ? Eckhoff [3] proved almost

two decades ago that $k \in \{3, 4\}$, and conjectured that $k = 3$. Dhandapani et al. [1] asked whether allowable interval sequences can be used to prove Eckhoff's result, and we can ask additionally whether they can be used to prove his conjecture for families of pairwise disjoint sets. Our demonstration in this paper of the usefulness of allowable interval sequences in proving results on transversals encourages us to hope for a positive answer to both questions.

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