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On $\leq k$-Edges, Crossings, and Halving Lines of Geometric Drawings of K_n

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Abstract Let P be a set of points in general position in the plane. Join all pairs of points in P with straight line segments. The number of segment-crossings in such a drawing, denoted by $\text{cr}(P)$, is the *rectilinear crossing number* of P. A *halving line* of P is a line passing through two points of P that divides the rest of the points of P in (almost) half. The number of halving lines of P is denoted by $h(P)$. Similarly, a *k-edge*, $0 \leq k \leq n/2-1$, is a line passing through two points of P and leaving exactly k points of P on one side. The number of $\leq k$-edges of P is denoted by $E_{\leq k}(P)$. Let $\overline{\text{cr}}(n)$, $h(n)$, and $E_{\leq k}(n)$ denote the minimum of $\text{cr}(P)$, the maximum of $h(P)$, and the minimum of $E_{\leq k}(P)$, respectively, over all sets P of n points in general position in the plane. We show that the previously best known lower bound on $E_{\leq k}(n)$ is tight for $k < \lceil (4n-2)/9 \rceil$ and improve it for all $k \geq \lceil (4n-2)/9 \rceil$. This in turn improves the lower bound on $\overline{\text{cr}}(n)$ from $0.37968\binom{n}{4} + \Theta(n^3)$ to $\frac{277}{729}\binom{n}{4} + \Theta(n^3) \geq 0.37997\binom{n}{4} + \Theta(n^3)$. We also give the exact values of $\overline{\text{cr}}(n)$ and $h(n)$ for all $n \leq 27$.

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Exact values were known only for $n \le 18$ and odd $n \le 21$ for the crossing number, and for $n \le 14$ and odd $n \le 21$ for halving lines.

Keywords k-Edges · k-Sets · Halving lines · Rectilinear crossing numbers · Allowable sequences · Geometric drawings

1 Introduction

We consider three important well-known problems in Combinatorial Geometry: the rectilinear crossing number, the maximum number of halving lines, and the minimum number of $(\le k)$-edges of complete geometric graphs on n vertices. All point sets in this paper are in the plane, finite, and in general position.

Let P be a finite set of points in general position in the plane. The *rectilinear crossing number* of P, denoted by $\mathrm{cr}(P)$, is the number of crossings obtained when all straight line segments joining pairs of points in P are drawn. (A *crossing* is the intersection of two segments in their interior.) The *rectilinear crossing number* of n is the minimum number of crossings determined by any set of n points, i.e., $\overline{\mathrm{cr}}(n) = \min\{\mathrm{cr}(P) : |P| = n\}$. The problem of determining $\overline{\mathrm{cr}}(n)$ for each n was posed by Erdős and Guy [18, 20] in the early 1970s. This is equivalent to finding the minimum number of convex quadrilaterals determined by n points, as every pair of crossing segments bijectively corresponds to the diagonals of a convex quadrilateral.

A *halving line* of P is a line passing through two points of P and dividing the rest in almost half. So when P has n points and n is even, a halving line of P leaves $n/2 - 1$ points of P on each side; whereas when n is odd, a halving line leaves $(n-3)/2$ points on one side and $(n-1)/2$ on the other. The number of halving lines of P is denoted by $h(P)$. Generalizing a halving line, a *k-edge* of P, with $0 \le k \le n/2 - 1$, is a line through two points of P leaving exactly k points on one side. The number of k-edges of P is denoted by $E_k(P)$. Since a halving line is a $(\lfloor n/2 \rfloor - 1)$-edge, then $E_{\lfloor n/2 \rfloor - 1}(P) = h(P)$. Similarly, for $0 \le k \le n/2 - 1$, $E_{\le k}(P)$ and $E_{\ge k}(P)$ denote the number of $\le k$-edges and $\ge k$-edges of P, respectively. That is, $E_{\le k}(P) = \sum_{j=0}^{k} E_j(P)$ and $E_{\ge k}(P) = \sum_{j=k}^{\lfloor n/2 \rfloor - 1} E_j(P) = \binom{n}{2} - \sum_{j=0}^{k-1} E_j(P)$. Let $h(n)$ and $E_{\le k}(n)$ be the maximum of $h(P)$ and the minimum of $E_{\le k}(P)$, respectively, over all sets P of n points. A concept closely related to k-edges is that of *k-sets*; a k-set of P is a set Q that can be separated from $P \setminus Q$ with a straight line. Rotating this separating line clockwise until it hits a point on each side yields a $(k-1)$-edge, and it turns out that this association is bijective. Thus the number of k-sets of P is equal to the number of $(k-1)$-edges of P. As a consequence, any of the results obtained here for k-edges can be directly translated into equivalent results for $(k+1)$-sets. Erdős, Lovász, Simmons, and Straus [17, 21] first introduced the concepts of halving lines, k-sets, and k-edges.

Since the introduction of these parameters back in the early 1970s, the determination (or estimation) of $\overline{\mathrm{cr}}(n)$, $h(n)$, and $E_{\le k}(n)$ have become classical problems in combinatorial geometry. General bounds are known but exact values have only been found for small n. The best known general bounds for the halving lines are $\Omega(ne^{c\sqrt{\log n}}) \le h(n) \le O(n^{4/3})$, due to Tóth [26] and Dey [16], respectively. Later on Nivasch [23] found a simpler construction with a smaller constant c. The previously best asymptotic bounds for the crossing number were

$$0.3792\binom{n}{4} + \Theta\left(n^3\right) \le \overline{\mathrm{cr}}(n) \le 0.380488\binom{n}{4} + \Theta\left(n^3\right). \tag{1}$$

The lower bound is due to Aichholzer et al. [11] and it follows from (2) as we indicate below. The upper bound follows from a recursive construction devised by Ábrego and Fernández-Merchant [6] using a suitable initial construction found by the authors in [2]. The best lower bound for the minimum number of $\le k$-edges is

$$E_{\le k}(n) \ge 3\binom{k+2}{2} + 3\binom{k+2-\lfloor n/3\rfloor}{2} - \max\left\{0, \left(k+1-\lfloor n/3\rfloor\right)\left(n-3\lfloor n/3\rfloor\right)\right\}, \tag{2}$$

due to Aichholzer et al. [11]. Further references and related problems can be found in [14].

The last two problems are naturally related, and their connection to the first problem is shown by the following identity, independently proved by Lovász et al. [22] and Ábrego and Fernández-Merchant [7]. For any set P of n points,

$$g(P) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2\rfloor-1} k(n-k-2)E_k(P), \quad \text{or equivalently}$$

$$g(P) = \sum_{k=0}^{\lfloor n/2\rfloor-2} (n-2k-3)E_{\le k}(P) - \frac{3}{4}\binom{n}{3} + \left(1+(-1)^{n+1}\right)\frac{1}{8}\binom{n}{2}. \tag{3}$$

Hence, lower bounds on $E_{\le k}(n)$ give lower bounds on $\overline{\mathrm{cr}}(n)$.

The majority of our results (all non-constructive parts) are proved in the more general context of generalized configurations of points, where the points in P are joined by pseudosegments rather than straight line segments. Goodman and Pollack [19] established a correspondence between the set of generalized configurations of points and what they called *allowable sequences.* In Sect. 2, we define allowable sequences, introduce the necessary notation to state the three problems above in the context of allowable sequences, and include a summary of results for these problems in both, the geometric and the allowable sequence context.

The main result in this paper is Theorem 1 in Sect. 3, which bounds $E_{\ge k}(P)$ by a function of $E_{k-1}(P)$. This result has the following important consequences.

1. In Sect. 4, we find exact values of $\overline{\mathrm{cr}}(n)$ and $h(n)$ for $n \le 27$. Exact values were only known for $n \le 18$ and odd $n \le 21$ in the case of $\overline{\mathrm{cr}}(n)$, and for $n \le 14$ and odd $n \le 21$ in the case of $h(n)$. (See Table 1.) We also show that the same values are achieved for the more general case of the pseudolinear crossing number $\widetilde{\mathrm{cr}}(n)$ and the maximum number of halving pseudolines $\widetilde{h}(n)$. (See Sect. 2 for the definitions.)

Table 1 New exact values. The * values were only known in the rectilinear case

	n									
	14	16	18	20	22	23	24	25	26	27
$h(n)=\widetilde{h}(n)$	22*	27	33	38	44	75	51	85	57	96
$\overline{\text{cr}}(n)=\widetilde{\text{cr}}(n)$	324*	603*	1029*	1657	2528	3077	3699	4430	5250	6180

2. Theorem 2 in Sect. 5 improves the lower bound in (2) for $k \geq \lceil (4n-11)/9 \rceil$. It gives a recursive lower bound whose asymptotic value is given by

$$E_{\leq k}(n) \geq \binom{n}{2} - \frac{1}{9}\sqrt{1-\frac{2k+2}{n}}\left(5n^2+19n-31\right),$$

as shown in Corollary 3.
3. Theorem 3 in Sect. 6 improves the lower bound in (1) to

$$\overline{\text{cr}}(n) \geq \frac{277}{729}\binom{n}{4} + \Theta\left(n^3\right) \geq 0.37997\binom{n}{4} + \Theta\left(n^3\right).$$

In Sect. 7, and to complement item 2 above, we show that (2) is tight for $k < \lceil (4n-11)/9 \rceil$. More precisely, we construct sets of points simultaneously achieving equality in (2) for all $k < \lceil (4n-11)/9 \rceil$.

Several results of this paper appeared (without proofs) in the conference proceedings of LAGOS'07 [3, 4].

2 Allowable Sequences and Generalized Configurations of Points

Any set P of n points in the plane can be encoded by a sequence of permutations of the set $[n]=\{1,2,\dots,n\}$ as follows. Consider a directed line l. Orthogonally project P onto l and label the points of P from 1 to n according to their order in l. In this order, the identity permutation $(1,2,\dots,n)$, is the first permutation of our sequence. Note that l can be chosen so that none of the projections overlap. Continuously rotate l counterclockwise. The order of the projections of P onto l changes every time two projections overlap, that is, every time a line through two points of P becomes perpendicular to l. Each time this happens, a new permutation is recorded as part of our sequence. After a 180°-rotation of l we obtain a sequence of $\binom{n}{2}+1$ permutations such that the first permutation $(1,2,\dots,n)$ is the identity, the last permutation $(n,n-1,\dots,2,1)$ is the reverse of the identity, any two consecutive permutations differ by a transposition of adjacent elements, and any pair of points (labels $1,\dots,n$) transpose exactly once. This sequence is known as a *halfperiod of the circular sequence* associated to P. The *circular sequence* of P is then a doubly infinite sequence of permutations obtained by rotating l indefinitely in both directions.

As an abstract generalization of a circular sequence, a *simple allowable sequence* on $[n]$ is a doubly infinite sequence $\Pi=(\dots,\pi_{-1},\pi_0,\pi_1,\dots)$ of permutations of $[n]$, such that any two consecutive permutations π_i and π_{i+1} differ by a transposition $\tau(\pi_i)$ of neighboring elements, and such that for every j, π_j is the reverse permutation of $\pi_{j+\binom{n}{2}}$. A *halfperiod* of Π is a sequence of $\binom{n}{2}+1$ consecutive permutations of $[n]$. As before, any halfperiod of Π uniquely determines Π and all properties for halfperiods mentioned above still hold. Moreover, the halfperiod $\pi=(\pi_i,\pi_{i+1},\dots,\pi_{i+\binom{n}{2}})$ is completely determined by the transpositions $\tau(\pi_i),\tau(\pi_{i+1}),\dots,\tau(\pi_{i+\binom{n}{2}-1})$. Note that the sequence $(\dots,\tau(\pi_{-1}),\tau(\pi_0),\tau(\pi_1),\dots)$ is $\binom{n}{2}$-periodic. Thus we indistinctly refer to π as a sequence of permutations or as a sequence of (suitable) transpositions. Allowable sequences that are the circular sequence of a set of points are called *stretchable*.

A *pseudoline* is a curve in $\mathbb{P}^2$, the projective plane, whose removal does not disconnect $\mathbb{P}^2$. Alternatively, a pseudoline is a simple curve in the plane that extends infinitely in both directions. A *simple generalized configuration of points* consists of a set of $\binom{n}{2}$ pseudolines and n points in the plane such that each pseudoline passes through exactly two points, and any two pseudolines intersect exactly once.

Circular and allowable sequences were first introduced by Goodman and Pollack [19]. They proved that not every allowable sequence is stretchable and established a correspondence between allowable sequences and generalized configurations of points.

The three problems at hand can be extended to generalized configurations of points, or equivalently, to simple allowable sequences. In this new setting, a transposition of two points in positions k and $k+1$, or $n-k$ and $n-k+1$ in a simple allowable sequence Π corresponds to a $(k-1)$-edge. We say that such transposition is a k-transposition, or respectively, an $(n-k)$-transposition, and if $1\le k\le n/2$ all these transpositions are called *k-critical*. Therefore $E_k(\Pi)$, $E_{\le k}(\Pi)$, and $E_{\ge k}(\Pi)$ correspond to the number of $(k+1)$-critical, $(\le k+1)$-critical, and $(\ge k+1)$-critical transpositions in any halfperiod of Π. A halving line of Π is a $\lfloor n/2\rfloor$-transposition, and thus $h(\Pi)=E_{\lfloor n/2\rfloor-1}(\Pi)$. Identity (3), which relates the number of k-edges to the crossing number, was originally proved for allowable sequences. In this setting, a *pseudosegment* is the segment of a pseudoline joining two points in a generalized configuration of points, and $\mathrm{cr}(\Pi)$ is the number of pseudosegment-crossings in the generalized configuration of points that corresponds to the allowable sequence Π. All these definitions and functions coincide with their original counterparts for P when Π is the circular sequence of P. However, when $\overline{\mathrm{cr}}(n)$, $h(n)$, and $E_{\le k}(n)$ are minimized or maximized over all allowable sequences on $[n]$ rather than over all sets of n points, the corresponding quantities may change and therefore we use the notation $\widetilde{\mathrm{cr}}(n)$, $\widetilde{h}(n)$, and $\widetilde{E}_{\le k}(n)$. Because n-point sets correspond to the stretchable simple allowable sequences on $[n]$, it follows that $\widetilde{\mathrm{cr}}(n)\le\overline{\mathrm{cr}}(n)$, $\widetilde{h}(n)\ge h(n)$, and $\widetilde{E}_{\le k}(n)\le E_{\le k}(n)$. Tamaki and Tokuyama [25] extended Dey's upper bound for allowable sequences to $\widetilde{h}(n)=O(n^{4/3})$. Ábrego et al. [1] proved that the lower bound for $E_{\le k}(n)$ in (2) is also a lower bound on $\widetilde{E}_{\le k}(n)$. They used this bound to extend (and even slightly improve) the corresponding lower bound on $\overline{\mathrm{cr}}(n)$ to $\widetilde{\mathrm{cr}}(n)$.

Our main result, Theorem 1 in Sect. 3, concentrates on the central behavior of allowable sequences. We bound $E_{\ge k}(\Pi)$ by a function of $E_{k-1}(\Pi)$. As a consequence, we improve (or match) the upper bounds on $\widetilde{h}(n)$ for $n\le 27$, and thus the lower bounds on $\widetilde{\mathrm{cr}}(n)$ in the same range. This is sufficient to match the corresponding best known geometric constructions [9] for $h(n)$ and $\overline{\mathrm{cr}}(n)$. This shows that for all $n \leq 27$, $\widetilde{h}(n) = h(n)$ and $\widetilde{\text{cr}}(n) = \overline{\text{cr}}(n)$ whose exact values are summarized in Table 1.

3 The Central Theorem

In this section, we present our main theorem. Given a halfperiod $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{\binom{n}{2}})$ of an allowable sequence and an integer $1 \leq k < n/2$, the *k-center* of the permutation π_j, denoted by $C(k, \pi_j)$, is the set of elements in the middle $n - 2k$ positions of π_j. Let L_0, $C_0 = C(k, \pi_0)$, and R_0 be the set of elements in the first k, middle $n - 2k$, and last k positions, respectively, of the permutation π_0. Define

$$s(k, \pi) = \min\left\{\left|C_0 \cap C(k, \pi_i)\right| : 0 \leq i \leq \binom{n}{2}\right\}.$$

Note that $s(k, \pi) \leq n - 2k - 1$ because at least one of the $n - 2k$ elements of C_0 must leave the k-center.

Theorem 1 *Let Π be an allowable sequence on $[n]$ and π any halfperiod of Π. If $s = s(k, \pi)$, then*

$$E_{\geq k}(\Pi) \leq (n - 2k - 1)E_{k-1}(\Pi) - \frac{s}{2}\big(E_{k-1}(\Pi) - n + 1\big).$$

Proof For presentation purposes, we divide this proof into subsections.

Let Π be an allowable sequence on $[n]$ and $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{\binom{n}{2}})$ any halfperiod of Π, $s = s(k, \pi)$, and $K = E_{k-1}(\pi)$.

Suppose that $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_K}$ is the subsequence of permutations in π obtained when the k-critical transpositions $\tau(\pi_{i_1}), \tau(\pi_{i_2}), \ldots, \tau(\pi_{i_K})$ of π occur (in this order). For simplicity we write τ_j instead of $\tau(\pi_{i_j})$. These permutations partition π into $K + 1$ parts $B_0(\pi), B_1(\pi), B_2(\pi), \ldots, B_K(\pi)$ called *blocks*, where $B_j(\pi) = \{\pi_l : i_j \leq l < i_{j+1}\}$ for $1 \leq j \leq K - 1$, $B_0(\pi) = \{\pi_l : 0 \leq l < i_1\}$, and $B_K(\pi) = \{\pi_l : i_K \leq l \leq \binom{n}{2}\}$. Denote by p_j the point that enters the k-center of π_{i_j} with τ_j. We say that a $(\geq k + 1)$-critical transposition in $B_j(\pi)$, $1 \leq j \leq K$, is an *essential* transposition if it involves p_j or if it occurs before τ_1, and a *nonessential* transposition otherwise.

Rearrangement of π We claim that, to bound $E_{\geq k}(\Pi)$, we can assume that all $(\geq k + 1)$-critical transpositions of π are essential transpositions. To show this, in case π has nonessential transpositions, we modify π so that the obtained halfperiod λ satisfies $E_j(\pi) = E_j(\lambda)$ for all $j < k$, and thus $E_{\geq k}(\pi) = E_{\geq k}(\lambda)$; and either λ has only essential transpositions or the last nonessential transposition of λ occurs in an earlier permutation than the last nonessential transposition of π. Applying this procedure sufficiently many times, we end with a halfperiod λ all of whose $(\geq k+1)$-critical transpositions are essential and such that $E_j(\pi) = E_j(\lambda)$ for all $j < k$, and thus $E_{\geq k}(\pi) = E_{\geq k}(\lambda)$.

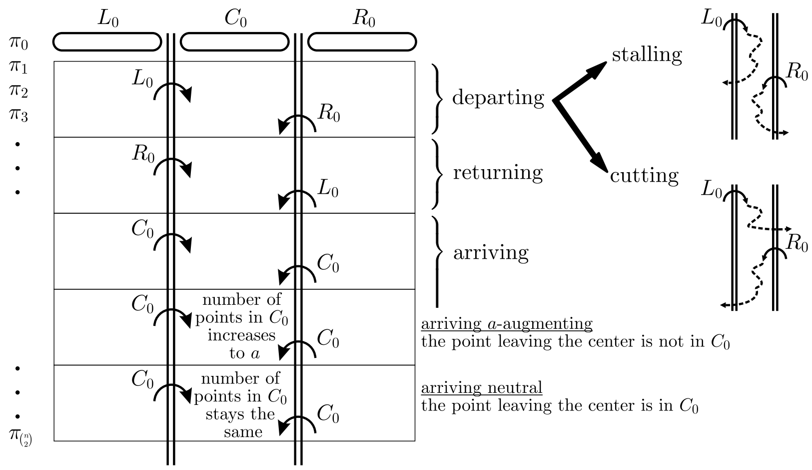

Fig. 1 Classification of essential k-critical transpositions

This is how λ is constructed. Suppose $B_j(\pi)$ is the last block of π that contains nonessential transpositions. Define λ as the halfperiod that coincides with π everywhere except for the $(\geq k+1)$-transpositions in $B_j(\pi)$. All nonessential transpositions in $B_j(\pi)$ take place right before τ_j in λ, and right after τ_j occurs, all essential transpositions in $B_j(\pi)$ occur consecutively in $B_j(\lambda)$ but probably in an order different from $B_j(\pi)$, so that the final position of p_j is the same in $B_j(\pi)$ and $B_j(\lambda)$. Note that in fact the last permutations of the blocks $B_j(\pi)$ and $B_j(\lambda)$ are equal.

Classification of k-Critical Transpositions From now on, we assume that π only has essential transpositions. We classify the k-critical transpositions as follows (see Fig. 1): τ_j is an *arriving* transposition if $p_j \in C_0$. An arriving transposition is *a-augmenting* if it increments the number of elements in C_0 in the k-center from $a-1$ to a, and it is *neutral* otherwise. We say that τ_j is a *returning* transposition if it is a k-transposition and $p_j \in R_0$, or if it is an $(n-k)$-transposition and $p_j \in L_0$. That is, p_i is "getting back" to its starting region. Similarly, τ_j is a *departing* transposition if it is a k-transposition and $p_j \in L_0$, or if it is an $(n-k)$-transposition and $p_j \in R_0$. That is, p_j is "getting away" from its original region. We say that a departing transposition τ_j is a *cutting* transposition, if τ_j is a k-transposition and the next k-critical transposition that involves p_j is an $(n-k)$-transposition; or if τ_i is an $(n-k)$-transposition and the next k-critical transposition that involves p_j is a k-transposition. All other departing transpositions are called *stalling*.

Finally, we define the *weight* of a k-critical transposition τ_j, denoted by $w(\tau_j)$, as the number of $(\geq k+1)$-critical transpositions in $B_j(\pi)$ that are not between two elements of C_0. Transpositions with weight at most $n-2k-1-s$ are called *light*. All other transpositions are *heavy*.

Let $A, N, R, C, S_{\text{light}}$, and S_{heavy} be the number of augmenting, neutral, returning, cutting, light stalling, and heavy stalling transpositions, respectively. Then $K = A + N + R + C + S_{\text{light}} + S_{\text{heavy}}$.

Bounding $E_{\geq k}(\Pi)$ Observe that the k-center of all permutations in $B_0(\pi)$ remains unchanged. It follows that all $(\geq k+1)$-critical transpositions of $B_0(\pi)$ are between elements of C_0. Thus $\sum_{j=1}^{K} w(\tau_j)$ counts all $(\geq k+1)$-critical transpositions except those between two elements of C_0. There are $\binom{n-2k}{2}$ transpositions between elements of C_0, but each neutral transposition corresponds to a k-critical (not $(\geq k+1)$-critical) transposition between two elements of C_0. Thus

$$E_{\geq k}(\Pi) \leq \binom{n-2k}{2} - N + \sum_{j=1}^{K} w(\tau_j). \tag{4}$$

Bounds for the Weight of a k-Critical Transposition We bound the weight of a transposition depending on its class (departing, returning, etc.), as well as the number of transpositions within a class, if necessary. For $j \geq 1$ all $(\geq k+1)$-critical transpositions in $B_j(\pi)$ involve p_j and thus $w(\tau_j) \leq n-2k-1$. However, since the weight of τ_j does not count transpositions between two elements of C_0, and there are always at least s elements of C_0 in the k-center, then $w(\tau_j) \leq n-2k-s$ whenever τ_j is arriving (because $p_j \in C_0$). Moreover, if τ_j is a-augmenting, then $w(\tau_j) \leq n-2k-a$. If τ_j is a returning transposition, then p_j has already been transposed with all the elements of C_0 that are in the k-center of π_{i_j}. Since there are at least s such elements, then $w(\tau_j) \leq n-2k-1-s$. Summarizing,

$$w(\tau_j) \leq \begin{cases} n-2k-1 & \text{for all } \tau_j, \\ n-2k-s, & \text{if } \tau_j \text{ is neutral,} \\ n-2k-a, & \text{if } \tau_j \text{ is } a\text{-augmenting,} \\ n-2k-1-s, & \text{if } \tau_j \text{ is light stalling or returning.} \end{cases} \tag{5}$$

Bounding C We bound the number of cutting transpositions. Since the first (last) k elements of π_0 are the last (first) elements of $\pi_{\binom{n}{2}}$, then the $2k$ elements not in C_0 must participate in at least one cutting transposition. That is, $C \geq 2k$. Note that, if $p \notin C_0$ participates in $c \geq 2$ cutting transpositions, then there must be at least $c-1$ returning transpositions of p. In other words, there must be at least $C-2k \geq 0$ returning transpositions. There are C cutting transpositions and at least $n-2k-s$ arriving transpositions (at least one a-augmenting arriving transposition for each $s+1 \leq a \leq n-2k$). Then $K-C-(n-2k-s)$ counts all other k-critical transpositions, including in particular all returning transpositions. Thus $K-C-(n-2k-s) \geq C-2k$, that is,

$$2C \leq 4k + K - n + s. \tag{6}$$

Augmenting and Heavy Stalling Transpositions We keep track of the augmenting and heavy stalling transpositions together. To do this, we consider the bipartite graph G whose vertices are the augmenting and the heavy stalling transpositions. The augmenting transposition τ_l is adjacent in G to the heavy stalling transposition τ_j if $j < l$, p_j is in the k-center of all permutations in blocks B_j to B_l, one transposition from τ_j and τ_l is a k-transposition and the other is an $(n-k)$-transposition, and p_l does not swap with p_j in $B_l(\pi)$. We bound the degree of a vertex in G.

Let τ_j be a heavy stalling transposition. If $p_j \in L_0$ (the case $p_j \in R_0$ is equivalent), then τ_j is a k-transposition. Because p_j moves to the right exactly $w(\tau_j) > n-2k-1-s$ positions within $B_j(\pi)$, it follows that the k-center right before τ_{j+1} occurs (i.e., the k-center of $\pi_{i_{j+1}-1}$) has at most $n-2k-1-w(\tau_j) < s$ points of C_0 to the right of p_j. Also, since τ_j is stalling, the next time that p_j leaves the k-center is by a k-transposition τ_{j+b}. This means that the k-center right before τ_{j+b} occurs (i.e., the k-center of $\pi_{i_{j+b}-1}$) has at least s points of C_0 to the right of p_j. Thus, between τ_j and τ_{j+b} there must be at least $s-(n-2k-1-w(\tau_j))$ arriving $(n-k)$-transpositions τ_l such that p_l remains to the right of p_j in $B_l(\pi)$, i.e., p_l does not swap with p_j in $B_l(\pi)$. These transpositions are adjacent to τ_j and thus the degree of τ_j in G is at least $w(\tau_j)-(n-2k-1-s)$. Hence,

$$\bigl|E(G)\bigr| \geq \sum_{\tau_j \text{ heavy stalling}} \bigl(w(\tau_j)-(n-2k-1-s)\bigr),$$

where $E(G)$ is the set of edges of G.

Let τ_l be an a-augmenting transposition. Since $p_l \in C_0$, and weights do not count transpositions between two elements of C_0, then at most $n-2k-a-w(\tau_l)$ points in $L_0 \cup R_0$ do not swap with p_l in $B_l(\pi)$. Only these points are possible p_js such that τ_j is adjacent to τ_l. Thus the degree of τ_l in G is at most $n-2k-a-w(\tau_l) \leq n-2k-1-s-w(\tau_l)$.

Note that there is at least one a-augmenting transposition for each $s+1 \leq a \leq n-2k$. This is because the k-center of at least one permutation of π contains exactly s elements of C_0 (by definition of s), and the k-center of $\pi_{\binom{n}{2}}$ contains exactly $n-2k$ elements of C_0 (since it coincides with C_0). Then the number of elements in the k-center must be eventually incremented from s to $n-2k$. For each $s+1 \leq a \leq n-2k$, we use $n-2k-a-w(\tau_l)$ to bound the degree of *one* a-augmenting transposition. For all other augmenting transpositions we use the bound $n-2k-1-s-w(\tau_l)$. Hence

$$\begin{aligned}\bigl|E(G)\bigr| &\leq \sum_{\tau_j \text{ augmenting}} \bigl((n-2k-1-s)-w(\tau_j)\bigr) - \sum_{a=s+1}^{n-2k} (a-s-1) \\ &= \sum_{\tau_j \text{ augmenting}} \bigl((n-2k-1-s)-w(\tau_j)\bigr) - \binom{n-2k-s}{2}.\end{aligned}$$

The previous two inequalities imply that

$$\begin{aligned}&\sum_{\tau_j \text{ augmenting}} w(\tau_j) + \sum_{\tau_j \text{ heavy stalling}} w(\tau_j) \\ &\quad \leq (n-2k-1-s)(A+S_{\text{heavy}}) - \binom{n-2k-s}{2}. \qquad (7)\end{aligned}$$

Final Calculations We use (5) and (7) to bound $\sum_{i=1}^{K} w(\tau_i) - N$.

$$\begin{aligned}\sum_{j=1}^{K} w(\tau_j) - N &= \sum_{\tau_j \text{ cutting}} w(\tau_j) + \sum_{\tau_j \text{ augmenting}} w(\tau_j) + \sum_{\tau_j \text{ heavy stalling}} w(\tau_j) \\ &\quad + \sum_{\tau_j \text{ light stalling}} w(\tau_j) + \sum_{\tau_j \text{ returning}} w(\tau_j) + \sum_{\tau_j \text{ neutral}} w(\tau_j) - N \\ &\le (n-2k-1)C + (n-2k-1-s)(A + S_{\text{heavy}}) - \binom{n-2k-s}{2} \\ &\quad + (n-2k-1-s)(S_{\text{light}} + R) + (n-2k-s)N - N \\ &\le sC + (n-2k-1-s)K - \binom{n-2k-s}{2}.\end{aligned}$$

By (4),

$$\begin{aligned}E_{\ge k}(\Pi) &\le \binom{n-2k}{2} - \binom{n-2k-s}{2} + sC + (n-2k-1-s)K \\ &= (n-2k-1)K - \frac{s}{2}(2K - 2n + 4k + 1 + s - 2C).\end{aligned}$$

Finally, by (6),

$$E_{\ge k}(\Pi) \le (n-2k-1)K - \frac{s}{2}(K - n + 1).$$ □

4 New Exact Values of $h(n)$, $\widetilde{h}(n)$, $\overline{\text{cr}}(n)$, and $\widetilde{\text{cr}}(n)$ for $n \le 27$

We start by stating a relaxed version of Theorem 1, which we use in the special case when $k = \lfloor n/2 \rfloor - 1$.

Corollary 1 *Let Π be a simple allowable sequence on $[n]$ and π any halfperiod of Π. If $s = s(k, \pi)$, then*

$$\begin{aligned}E_{\ge k}(\Pi) &\le (n-2k-1)E_{k-1}(\Pi) + \binom{s}{2} \\ &\le (n-2k-1)E_{k-1}(\Pi) + \binom{n-2k-1}{2}.\end{aligned}$$

Proof There are at least $n-2k-s$ elements of C_0 that leave the k-center, so there are at least $n-2k-s$ arriving transpositions. In addition, there are at least $2k$ departing transpositions, one per element not in C_0. It follows that $E_{k-1}(\Pi) \ge 2k + (n-2k-s) = n-s$. The first inequality now follows directly from Theorem 1. Finally, $s \le n-2k-1$ for all halfperiods of Π which yields the second inequality. Another consequence is that $E_{k-1}(\Pi) \ge n-s \ge 2k+1$, which is in fact the minimum possible value of E_{k-1} (cf. [22]). □

The previous corollary implies the following result for halving lines.

Corollary 2 *If Π is a simple allowable sequence on $[n]$ and $n \geq 6$, then*

$$h(\Pi) \leq \begin{cases} \left\lfloor \frac{1}{2}\binom{n}{2} - \frac{1}{2}E_{\leq n/2-3}(\Pi) \right\rfloor & \text{if } n \text{ is even}, \\ \left\lfloor \frac{2}{3}\binom{n}{2} - \frac{2}{3}E_{\leq (n-1)/2-3}(\Pi) + \frac{1}{3} \right\rfloor & \text{if } n \text{ is odd}. \end{cases}$$

Proof The inequality follows from Corollary 1 using $k = \lfloor n/2 \rfloor - 1$, and the identities $E_{\geq \lfloor n/2 \rfloor - 1}(\Pi) = h(\Pi)$ and $E_{\leq \lfloor n/2 \rfloor - 3}(\Pi) + E_{\lfloor n/2 \rfloor - 2}(\Pi) + h(\Pi) = \binom{n}{2}$. □

The exact values of $h(n)$ were previously known only for even $n \leq 14$ or odd $n \leq 21$ [8, 13]. The exact values of $\overline{\text{cr}}(n)$ were previously known only for even $n \leq 18$ or odd $n \leq 21$ [11]. Inequality (2) is also valid in the context of allowable sequences [1]. This bound for $k = \lfloor n/2 \rfloor - 3$ and Corollary 2 give the new upper bounds for $\widetilde{h}(n)$ in Table 1. We also obtained the new lower bounds for $\widetilde{\text{cr}}(n)$ reported in Table 1. To do this we used identity (3) together with the new values of $\widetilde{h}(n)$, the identity $E_{\leq \lfloor n/2 \rfloor - 2}(\Pi) = \binom{n}{2} - h(\Pi)$, and inequality (2) for $k \leq \lfloor n/2 \rfloor - 3$. For example, if $n = 24$ then $E_{\leq 10}(\Pi) = \binom{24}{2} - h(\Pi) \geq 276 - \widetilde{h}(24) = 225$ and by (2), the vector $(E_{\leq 0}(\Pi), E_{\leq 1}(\Pi), \ldots, E_{\leq 9}(\Pi))$ is bounded below entry-wise by $(3, 9, 18, 30, 45, 63, 84, 108, 138, 174)$, so (3) implies that $\widetilde{\text{cr}}(24) = \sum_{k=0}^{10}(21 - 2k)E_{\leq k}(\Pi) - \frac{3}{4}\binom{24}{3} \geq 3699$.

All the bounds shown in Table 1 are attained by Aichholzer's et al. constructions [9], and thus Table 1 actually shows the exact values of $\widetilde{h}(n)$, $h(n)$, $\widetilde{\text{cr}}(n)$, and $\overline{\text{cr}}(n)$ for n in the specified range.

The previous argument to obtain new upper bounds for $h(n)$ does not improve the general bound $h(n) \leq cn^{4/3}$. However, even with the current best constant $c = (31287/8192)^{1/3} < 1.5721$ [8, 24], our bound is better when n is even in the range $8 \leq n \leq 184$.

5 New Lower Bound for the Number of $\leq k$-Edges

In this section, we obtain a new lower bound for the number of $\leq k$-edges. Our emphasis is on finding the best possible asymptotic result as well as the best bounds that apply to the small values of n for which the exact value is unknown. Theorem 2 provides the exact result that can be applied to small values of n, whereas Corollary 3 gives the best known asymptotic behavior.

Let $m = \lceil (4n - 11)/9 \rceil$. For each n, define the following recursive sequence:

$$u_{m-1} = 3\binom{m+1}{2} + 3\binom{m+1-\lfloor n/3 \rfloor}{2} - 3\left(m - \left\lfloor \frac{n}{3} \right\rfloor\right)\left(\frac{n}{3} - \left\lfloor \frac{n}{3} \right\rfloor\right) \quad \text{and}$$

$$u_k = \left\lceil \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right) \right\rceil \quad \text{for } k \geq m.$$

The following is the new lower bound on $E_{\leq k}(n)$. It follows from Theorem 1.

Theorem 2 *For any n and k such that $m-1 \le k \le (n-3)/2$,*

$$E_{\le k}(n) \ge u_k.$$

Proof We need the following two lemmas to estimate the growth of the sequence u_k with respect to n and k. For presentation purposes, we defer their proofs to the end of the section.

Lemma 1 *For any k such that $m-1 \le k \le (n-5)/2$,*

$$3\sqrt{1-\frac{2k+9/2}{n}} < \frac{\binom{n}{2}-u_k}{\binom{n}{2}-u_{m-1}} \le 3\sqrt{1-\frac{2k+2}{n}}. \tag{8}$$

Lemma 2 *For any k such that $m \le k \le (n-5)/2$,*

$$3\sqrt{1-\frac{2k+9/2}{n}}\left(\binom{n}{2}-u_{m-1}\right) \ge (n-1)(n-2k-3).$$

We prove the stronger statement $\widetilde{E}_{\le k}(n) \ge u_k$. Let Π be an allowable sequence on $[n]$ and π any of its halfperiods. We proceed by induction on k. If $k = m-1$ the result holds by Inequality (2), proved in the more general context of allowable sequences [1]. Assume that $k \ge m$ and $E_{\le k-1}(\Pi) \ge u_{k-1}$. Let $s = s(k+1, \pi)$; by Theorem 1,

$$E_{\ge k+1}(\Pi) \le (n-2k-3)E_k(\Pi) - \frac{s}{2}\big(E_k(\Pi)-(n-1)\big).$$

If $s = 0$ or $E_k(\Pi) \ge n-1$, then $E_{\ge k+1}(\Pi) \le (n-2k-3)E_k(\Pi)$. Thus

$$\binom{n}{2} - E_{\le k}(\Pi) \le (n-2k-3)\big(E_{\le k}(\Pi) - E_{\le k-1}(\Pi)\big),$$

and by induction

$$\begin{aligned} E_{\le k}(\Pi) &\ge \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)E_{\le k-1}(\Pi)\right) \\ &\ge \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right), \end{aligned}$$

which implies that $E_{\le k}(\Pi) \ge u_k$ by definition of u_k. Now assume $s > 0$ and $E_k(\Pi) < n-1$. Because $E_k(\Pi) \ge 2k+3$ (see the proof of Corollary 1), it follows that $k \le (n-5)/2$. By Theorem 1,

$$\begin{aligned} E_{\ge k+1}(\Pi) &\le (n-2k-3)E_k(\Pi) - \frac{s}{2}\big(E_k(\Pi)-(n-1)\big) \\ &= \left(n-2k-3-\frac{s}{2}\right)E_k(\Pi) + \frac{s}{2}(n-1). \end{aligned}$$

Recall that $s = s(k+1,\pi) \le n-2k-3$. Because $E_k(\Pi) < n-1$, it follows that

$$E_{\ge k+1}(\Pi) \le \left(n-2k-3-\frac{s}{2}\right)(n-1)+\frac{s}{2}(n-1) = (n-1)(n-2k-3).$$

Therefore

$$E_{\le k}(\Pi) = \binom{n}{2} - E_{\ge k+1}(\Pi) \ge \binom{n}{2} - (n-1)(n-2k-3).$$

By Lemma 2,

$$E_{\le k}(\Pi) \ge \binom{n}{2} - 3\sqrt{1-\frac{2k+9/2}{n}}\left(\binom{n}{2} - u_{m-1}\right),$$

and by Lemma 1, $E_{\le k}(\Pi) \ge u_k$ for all allowable sequences Π on $[n]$. Therefore $E_{\le k}(n) \ge \widetilde{E}_{\le k}(n) \ge u_k$. □

Corollary 3 *For any n and k such that $m-1 \le k \le (n-2)/2$,*

$$E_{\le k}(n) \ge \binom{n}{2} - \frac{1}{9}\sqrt{1-\frac{2k+2}{n}}\bigl(5n^2+19n-31\bigr).$$

Proof Let Π be an allowable sequence on $[n]$. If $k = \lfloor n/2\rfloor - 1$, then $E_{\le \lfloor n/2\rfloor -1}(\Pi) = \binom{n}{2}$. For $k < \lfloor n/2\rfloor - 1$, it follows that $n \ge 3$ and from Theorem 2 and Lemma 1,

$$E_{\le k}(\Pi) \ge u_k \ge \binom{n}{2} - 3\sqrt{1-\frac{2k+2}{n}}\left(\binom{n}{2} - u_{m-1}\right).$$

Considering the possible residues of n modulo 9, it can be verified that for $n \ge 3$,

$$u_{m-1} \ge \frac{17}{54}n^2 - \frac{65}{54}n + \frac{31}{27} \quad \bigl(\text{equality if } n \equiv 3 \pmod 9\bigr).$$

Therefore $E_{\le k}(n) \ge \widetilde{E}_{\le k}(n) \ge \binom{n}{2} - \frac{1}{9}\sqrt{1-\frac{2k+2}{n}}(5n^2+19n-31)$. □

5.1 Proofs of Lemmas 1 and 2

Proof of Lemma 1 The integer range $[m-1,(n-5)/2]$ is empty for $n \le 5$. Assume $n \ge 6$ and proceed by induction on k. If $k = m-1$, then $3\sqrt{1-(2m+5/2)/n} \le 1 \le 3\sqrt{1-2m/n}$ is equivalent to $\lceil (4n-11)/9\rceil \le 4n/9 \le \lceil (4n-11)/9\rceil + 5/4$ which holds in general. Assume that $k \ge m$ and that (8) holds for $k-1$. From the definition of u_k and the induction hypothesis,

$$\binom{n}{2} - u_k \le \binom{n}{2} - \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right)$$

$$
\begin{aligned}
&= \frac{n-2k-3}{n-2k-2}\left(\binom{n}{2} - u_{k-1}\right) \\
&\le 3\left(\binom{n}{2} - u_{m-1}\right)\frac{n-2k-3}{n-2k-2}\sqrt{1-\frac{2k}{n}},
\end{aligned}
$$

and $(n-2k-3)\sqrt{1-2k/n}\,/(n-2k-2) \le \sqrt{1-(2k+2)/n}$ because $k \le (n-5)/2$, which proves the second inequality in (8). Similarly, from the definition of u_k and the induction hypothesis,

$$
\begin{aligned}
\binom{n}{2} - u_k &\ge \binom{n}{2} - \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u_{k-1}\right) - 1 \\
&= \frac{n-2k-3}{n-2k-2}\left(\binom{n}{2} - u_{k-1}\right) - 1 \\
&\ge 3\left(\binom{n}{2} - u_{m-1}\right)\frac{n-2k-3}{n-2k-2}\sqrt{1-\frac{2k+5/2}{n}} - 1.
\end{aligned}
$$

Hence, to prove the second inequality in (8), it is enough to show that $3(\binom{n}{2} - u_{m-1})d > 1$, where

$$
d = \frac{n-2k-3}{n-2k-2}\sqrt{1-\frac{2k+5/2}{n}} - \sqrt{1-\frac{2k+9/2}{n}} \tag{9}
$$

is always positive because $k \le (n-5)/2$. First note that

$$
\begin{aligned}
u_{m-1} &\le 3\binom{m+1}{2} + 3\binom{m+1-\lfloor n/3 \rfloor}{2} \\
&\le 3\binom{(4n+6)/9}{2} + 3\binom{(n+10)/9}{2},
\end{aligned}
$$

which implies that

$$
3\left(\binom{n}{2} - u_{m-1}\right) \ge \frac{1}{9}\left(5n^2 - 25n + 4\right). \tag{10}
$$

Multiplying the easily verified inequality

$$
1 > \frac{(n-2k-3)\sqrt{n-2k-5/2} + (n-2k-2)\sqrt{n-2k-9/2}}{(2n-4k-5)\sqrt{n-2k-5/2}}
$$

by (9), yields

$$
\begin{aligned}
d &> \frac{n-2k-9/4}{(n-2k-2)^2\sqrt{n(n-2k-5/2)}} \cdot \frac{2n-4k-4}{2n-4k-5} \\
&> \frac{n-2k-9/4}{(n-2k-2)^2\sqrt{n(n-2k-5/2)}}
\end{aligned}
$$

$$= \left(1 - \frac{1}{4(n-2k-2)}\right) \frac{1}{(n-2k-2)\sqrt{n(n-2k-2-1/2)}}.$$

Since $(4n-11)/9 \leq k \leq (n-5)/2$, then $3 \leq n-2k-2 \leq (n+4)/9$. Thus

$$d > \left(1 - \frac{1}{12}\right) \frac{27}{(n+4)\sqrt{n(n-1/2)}} = \frac{99}{4(n+4)\sqrt{n(n-1/2).}}$$

This inequality, together with (10), imply that for all $n \geq 6$,

$$3\left(\binom{n}{2} - u_{m-1}\right) d > \frac{11}{4}\left(\frac{5n^2-25n+4}{(n+4)\sqrt{n(n-1/2)}}\right) > 1. \qquad \square$$

Proof of Lemma 2 For each $n \leq 40$ the integer range $[m, (n-5)/2]$ is either empty or contains only $k = \lfloor (n-5)/2 \rfloor$. For these cases, the inequality can easily be verified. Assume $n \geq 41$, it follows from (10) that

$$9\left(1 - \frac{2k+9/2}{n}\right)\left(\binom{n}{2} - u_{m-1}\right)^2 \geq \frac{(n-2k-9/2)(5n^2-25n+4)^2}{81n}.$$

Since $k \leq (n-5)/2$, then

$$n-2k-9/2 \geq \frac{(n-2k-3)^2}{n-2k+3}.$$

Also $k \geq m \geq (4n-11)/9$ implies $n-2k+3 \leq (n+49)/9$ and thus

$$\frac{(n-2k-9/2)(5n^2-25n+4)^2}{81n} \geq \frac{(n-2k-3)^2(5n^2-25n+4)^2}{9n(n+49)}.$$

Finally, for $n \geq 41$,

$$\frac{(5n^2-25n+4)^2}{9n(n+49)} \geq (n-1)^2,$$

and consequently

$$9\left(1 - \frac{2k+9/2}{n}\right)\left(\binom{n}{2} - u_{m-1}\right)^2 \geq (n-1)^2(n-2k-3)^2. \qquad \square$$

6 New Lower Bound on $\overline{\mathrm{cr}}(n)$

In this section, we use Corollary 3 to get the following new lower bound on $\overline{\mathrm{cr}}(n)$.

Theorem 3 $\overline{\mathrm{cr}}(n) \geq \frac{277}{729}\binom{n}{4} + \Theta(n^3) > 0.379972\binom{n}{4} + \Theta(n^3)$.

Proof We actually prove that the right hand side is a lower bound on $\widetilde{\operatorname{cr}}(n)$. According to (3), if Π is an allowable sequence on $[n]$, then

$$\operatorname{cr}(\Pi)=\binom{n}{4}\left(24\sum_{k=0}^{\lfloor n/2\rfloor-1}\frac{1}{n}\left(1-\frac{2k}{n}\right)\frac{E_{\le k}(\Pi)}{n^2}\right)+\Theta\left(n^3\right). \tag{11}$$

Using inequality (2) for $0\le k\le m-1$ gives

$$\frac{E_{\le k}(\Pi)}{n^2}\ge\frac{3}{2}\left(\frac{k}{n}\right)^2+\frac{3}{2}\max\left(0,\frac{k}{n}-\frac{1}{3}\right)^2-\Theta\left(\frac{1}{n}\right).$$

Similarly, if $m\le k\le\lfloor n/2\rfloor-1$, then by Corollary 3,

$$\frac{E_{\le k}(\Pi)}{n^2}\ge\frac{1}{2}-\frac{5}{9}\sqrt{1-\frac{2k}{n}}+\Theta\left(\frac{1}{n}\right).$$

These two inequalities in their corresponding ranges applied to (11) give a Riemann sum (on the variable $x=k/n$) that can be estimated using the corresponding integral. Note that the error terms are uniformly bounded by $\Theta(n^3)$. Therefore,

$$\begin{aligned}\operatorname{cr}(\Pi)\ge&\binom{n}{4}\left(24\int_0^{4/9}\frac{3}{2}(1-2x)\left(x^2+\max\left(0,x-\frac{1}{3}\right)^2\right)dx\right)\\&+\binom{n}{4}\left(24\int_{4/9}^{1/2}(1-2x)\left(\frac{1}{2}-\frac{5}{9}\sqrt{1-2x}\right)dx\right)+\Theta\left(n^3\right)\\\ge&\binom{n}{4}\left(\frac{86}{243}+\frac{19}{729}\right)+\Theta\left(n^3\right)=\frac{277}{729}\binom{n}{4}+\Theta\left(n^3\right).\end{aligned}$$

□

Table 2 gives the best lower bounds for $\widetilde{\operatorname{cr}}(n)$ in the range $28\le n\le 99$ that follow from using (3) with the bound in either (2) or the new bound from Theorem 2, with the notable exception of $n=30$ for which $\widetilde{\operatorname{cr}}(30)=9726$ was recently determined by Cetina et al. [15].

7 A Point Set with Few $\le k$-Edges for Every $k\le 4n/9-1$

Combining (2) and Theorem 2, we obtain the best known lower bound for $E_{\le k}(n)$. If n is a multiple of 9 and $k\le(4n/9)-1$, then this bound reads

$$E_{\le k}(n)\ge\begin{cases}3\binom{k+2}{2} & \text{if } 0\le k\le n/3-1,\\ 3\binom{k+2}{2}+3\binom{k-n/3+2}{2} & \text{if } n/3\le k\le 4n/9-2,\\ 3\binom{(4n/9-1)+2}{2}+3\binom{(4n/9-1)-n/3+2}{2}+3 & \text{if } k=4n/9-1.\end{cases} \tag{12}$$

Our aim in this section is to show that this bound is tight for $n\ge 27$. This improves on the construction in [10], where tightness for (12) is proved for $k\le(5n/12)$.

We recursively construct, for each integer $r\ge 3$, a $9r$-point set S_r such that for every $k\le(4n/9)-1$, $E_{\le k}(S_r)$ equals the right-hand side of (12).

Table 2 New lower bounds for $\widetilde{\text{cr}}(n)$

n	$\widetilde{\text{cr}}(n) \geq$	n	$\widetilde{\text{cr}}(n) \geq$	n	$\widetilde{\text{cr}}(n) \geq$	n	$\widetilde{\text{cr}}(n) \geq$
28	7233	46	59410	64	234223	82	649190
29	8421	47	65015	65	249732	83	682308
30	**9726**	48	70948	66	265888	84	716507
31	11207	49	77362	67	282974	85	752217
32	12830	50	84146	68	300767	86	789077
33	14626	51	91374	69	319389	87	827289
34	16613	52	99073	70	338913	88	866947
35	18796	53	107251	71	359311	89	907990
36	21164	54	115878	72	380531	90	950372
37	23785	55	125087	73	402798	91	994394
38	26621	56	134798	74	425980	92	1039840
39	29691	57	145030	75	450078	93	1086725
40	33048	58	155900	76	475305	94	1135377
41	36674	59	167344	77	501531	95	1185551
42	40561	60	179354	78	528738	96	1237263
43	44796	61	192095	79	557191	97	1290844
44	49324	62	205437	80	586684	98	1346029
45	54181	63	219457	81	617310	99	1402932

7.1 Constructing the Sets S_r

If a and b are distinct points, then $\ell(ab)$ denotes the line spanned by a and b, and $\overline{ab}$ denotes the closed line segment with endpoints a and b, directed from a toward b. Let θ denote the clockwise rotation by an angle of $2\pi/3$ around the origin. At this point the reader may want to take a sneak preview at Fig. 2, where S_3 is sketched.

For each $r \geq 3$ the set S_r is naturally partitioned into nine sets of size r: $A_r = \{a_1, \dots, a_r\}$, $A'_r = \{a'_1, \dots, a'_r\}$, A''_r, and their respective $2\pi/3$ and $4\pi/3$ rotations around the origin. The elements of A''_r are not labeled because they change in each iteration. For $i = 1, \dots, r$, we let $b_i = \theta(a_i)$, $b'_i = \theta(a'_i)$, $c_i = \theta^2(a_i)$, and $c'_i = \theta^2(a_i)$. Thus if we let $B_r = \{b_1, \dots, b_r\}$, $B'_r = \{b'_1, \dots, b'_r\}$, $B''_r = \theta(A''_r)$, $C_r = \{c_1, \dots, c_r\}$, $C'_r = \{c'_1, \dots, c'_r\}$, and $C''_r = \theta^2(A''_r)$, then we obtain $B_r \cup B'_r \cup B''_r$ (respectively, $C_r \cup C'_r \cup C''_r$) by applying θ (respectively, θ^2) to $A_r \cup A'_r \cup A''_r$. We refer to this property as the 3-*symmetry* of S_r.

As we mentioned before, the construction of the sets S_r is recursive. For $r \geq 3$, we obtain A_{r+1} and A'_{r+1} by adding suitable points a_{r+1} to A_r and a'_{r+1} to A'_r. Keeping 3-symmetry, this determines B_{r+1}, B'_{r+1}, C_{r+1}, and C'_{r+1}. However, the set A''_{r+1} is *not* obtained by adding a point to A''_r, but instead is defined in terms of B_{r+1}, B'_{r+1}, C_{r+1}, and C'_{r+1}; this explains why we have not listed the elements in A''_r, B''_r, and C''_r.

Before moving on with the construction, we remark that the sets S_r contain subsets of more than two collinear points. As will become clear from the construction, the

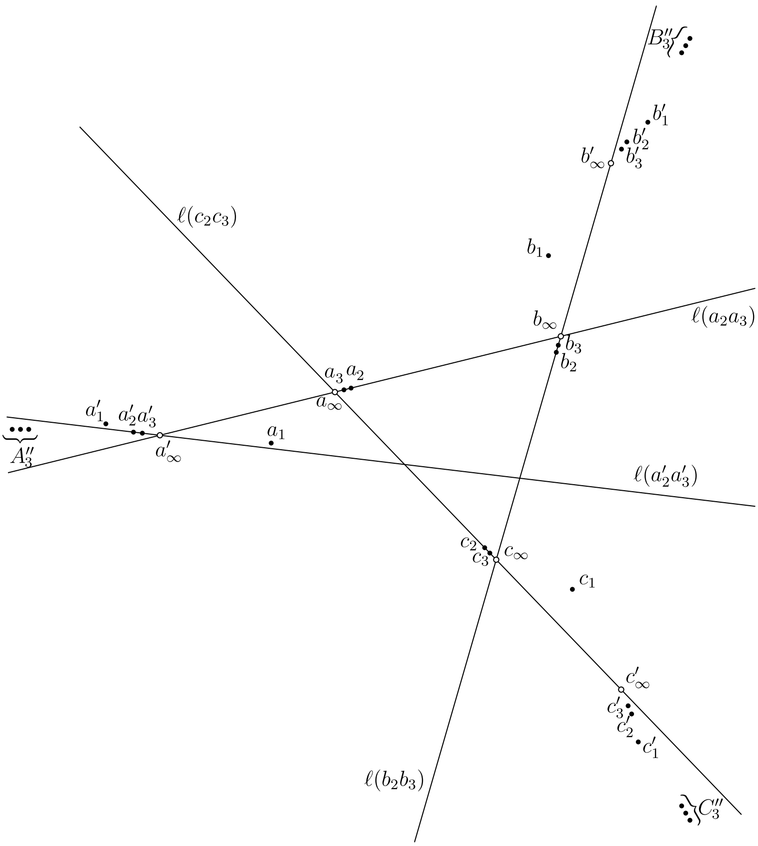

Fig. 2 The 27-point set S_3. The points $a_\infty, a'_\infty, b_\infty, b'_\infty, c_\infty$, and c'_∞ do not belong to S_3

points can be slightly perturbed to general position, so that the number of $\leq k$-edges remains unchanged for every $k \leq 4n/9 - 1$.

We start by describing S_3, see Fig. 2. First we explicitly fix A_3 and A'_3: $a_1 = (-700, -50)$, $a_2 = (-410, 150)$, $a_3 = (-436, 144)$, $a'_1 = (-1300, 20)$, $a'_2 = (-1200, -10)$, and $a'_3 = (-1170, -14)$. Thus B_3, B'_3, C_3, and C'_3 also get determined. For the points in A''_3 we do not give their exact coordinates, instead we simply ask that they satisfy the following: all the points in A''_3 lie on the x-axis, and are sufficiently far to the left of $A_3 \cup A'_3$ so that if a line ℓ_1 passes through a point in A''_3 and a point in $S_3 \setminus (B''_3 \cup C''_3)$, and a line ℓ_2 passes through two points in $S_3 \setminus A''_3$, then the slope of ℓ_1 is smaller in absolute value than the slope of ℓ_2, i.e., ℓ_1 is closer (in slope) to a horizontal line, than ℓ_2.

We need to define six auxiliary points not in S_r: $a_\infty = \ell(a_2a_3) \cap \ell(c_2c_3)$ and $a'_\infty = \ell(a'_2a'_3) \cap \ell(a_2a_3)$. As expected, let $b_\infty = \theta(a_\infty)$, $c_\infty = \theta^2(a_\infty)$, $b'_\infty = \theta(a'_\infty)$, and $c'_\infty = \theta^2(a'_\infty)$.

We now describe how to get S_{r+1} from S_r. The crucial step is to define the points b_{r+1} and a'_{r+1} to be added to B_r and A'_r to obtain B_{r+1} and A'_{r+1}, respectively. Then

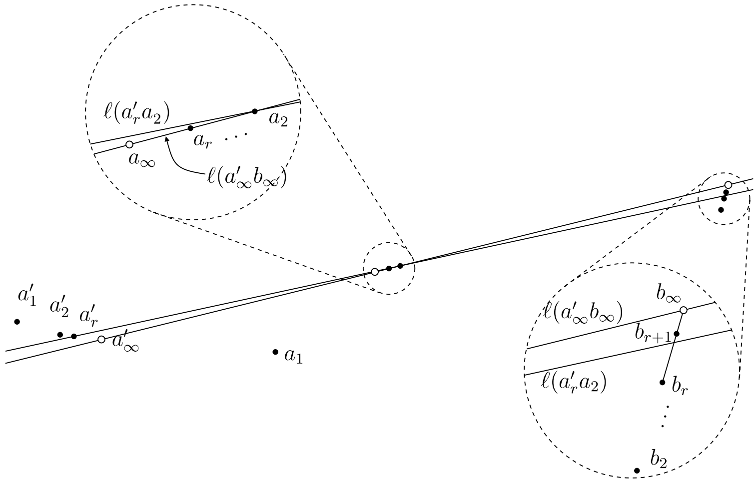

Fig. 3 b_{r+1} is placed in between b_r and b_∞, above the line $\ell(a'_r a_2)$

we construct A''_{r+1} and applying θ and θ^2 to B_{r+1}, A'_{r+1}, and A''_{r+1}, we obtain the rest of S_{r+1}.

Suppose that for some $r \geq 3$, the set S_r has been constructed so that the following properties hold for $t = r$ (this is clearly true for the base case $r = 3$):

(I) The points $a_2, \dots, a_t$ appear in this order along $\overline{a_2 a_\infty}$.
(II) The points $a'_2, \dots, a'_t$ appear in this order along $\overline{a'_2 a'_\infty}$.
(III) For all $i = 2, \dots, t-1$ and $j = 2, \dots, t$, $\ell(a'_i a_j)$ intersects the interior of $\overline{b_i b_{i+1}}$.
(IV) For all $j = 2, \dots, t$, $\ell(a'_t a_j)$ intersects the interior of $\overline{b_t b_\infty}$.

Now we add b_{r+1} and a'_{r+1}. Place b_{r+1} anywhere on the open line segment determined by b_∞ and the intersection point of $\ell(a'_r a_2)$ with $\overline{b_r b_\infty}$. (The existence of this intersection point is guaranteed by (IV), see Fig. 3.) Place a'_{r+1} anywhere on the open line segment determined by a'_∞ and the intersection point of $\ell(b_{r+1} a_\infty)$ with $\overline{a'_r a'_\infty}$. (This intersection exists because a'_∞, a_∞, a_2, and b_∞ are collinear and appear in this order along $\ell(a'_\infty b_\infty)$, the line $\ell(a'_\infty b_\infty)$ separates b_{r+1} from a'_r, and the line $\ell(a'_r a_2)$ separates b_{r+1} from a_∞, see Fig. 4.) Thus B_{r+1} and A'_{r+1} and consequently A_{r+1}, C_{r+1}, B'_{r+1}, and C'_{r+1}, are defined. It is straightforward to check that (I)–(IV) hold for $t = r + 1$.

It only remains to describe how to construct A''_{r+1}. As we mentioned above, this set is not a superset of A''_r, instead it gets defined analogously to A''_3: we let the points in A''_{r+1} lie on the x-axis, and sufficiently far to the left of $A_{r+1} \cup A'_{r+1}$, so that if ℓ_1 passes through a point in A''_{r+1} and through a point in $S_{r+1} \setminus (B''_{r+1} \cup C''_{r+1})$, and ℓ_2 spans two points in $S_{r+1} \setminus A''_{r+1}$, then the slope of ℓ_1 is smaller in absolute value than the slope of ℓ_2.

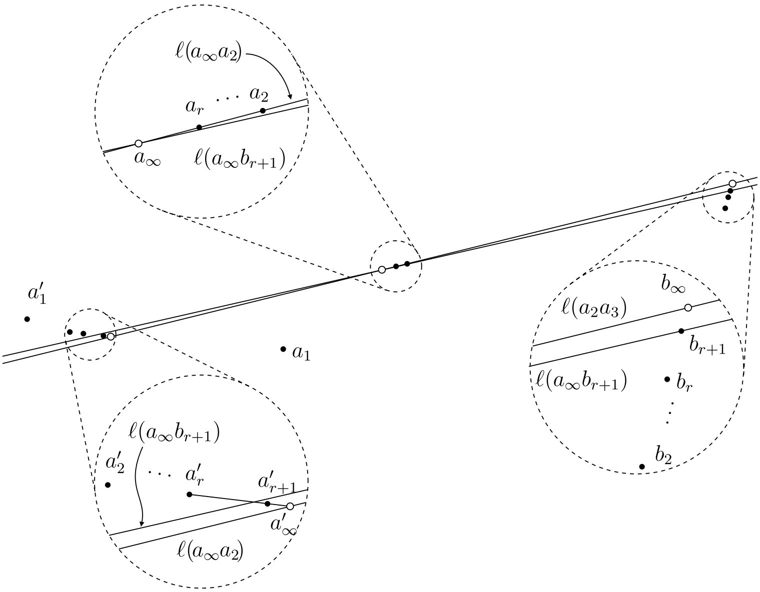

Fig. 4 a'_{r+1} is placed in between a'_r and a'_∞, below the line $\ell(a_\infty b_{r+1})$

7.2 Calculating $E_{\leq k}(S_r)$

We fix $r \geq 3$, and proceed to determine $E_{\leq k}(S_r)$ for each k, $0 \leq k \leq 4r-1$. It is now convenient to label the elements of A''_r, B''_r, and C''_r. Let $a''_1, a''_2, \ldots, a''_r$ be the elements of A''_r, ordered as they appear from left to right along the negative x-axis. As expected, let $b''_i = \theta(a''_i)$ and $c''_i = \theta^2(a''_i)$, for $i = 1, \ldots, r$.

We call a k-edge *bichromatic* if it joins two points with different label letters (i.e., if it is of the form ab, bc, or ac); otherwise, a k-edge is *monochromatic*. A monochromatic edge is *of type aa* if it is of the form $\ell(a_i a_j)$ for some integers i, j; edges of types $aa', aa'', a'a', a'a'', a''a''$ (and their counterparts for b and c) are similarly defined. Finally, we say that an edge of any of the types $aa, aa', aa'', a'a', a'a''$, or $a''a''$ is *of type* **A**; edges of types **B** and **C** are similarly defined. We let $E^{\text{bic}}_{\leq k}$ (respectively, $E^{\text{mono}}_{\leq k}$) stand for the number of bichromatic (respectively, monochromatic) $\leq k$-edges, so that $E_{\leq k}(S_r) = E^{\text{bic}}_{\leq k}(S_r) + E^{\text{mono}}_{\leq k}(S_r)$.

We say that a finite point set P is 3-*decomposable* if it can be partitioned into three equal-size sets $\overline{A}$, $\overline{B}$, and $\overline{C}$ satisfying the following: there is a triangle T enclosing P such that the orthogonal projections of P onto the three sides of T show $\overline{A}$ between $\overline{B}$ and $\overline{C}$ on one side, $\overline{B}$ between $\overline{A}$ and $\overline{C}$ on another side, and $\overline{C}$ between $\overline{A}$ and $\overline{B}$ on the third side (see [2]). We say that $\{\overline{A}, \overline{B}, \overline{C}\}$ is a 3-*decomposition* of P. It is easy to see that if we let $\overline{A} := A_r \cup A'_r \cup A''_r$, $\overline{B} := B_r \cup B'_r \cup B''_r$, and $\overline{C} := C_r \cup C'_r \cup C''_r$, then $\{\overline{A}, \overline{B}, \overline{C}\}$ is a 3-decomposition of S_r: indeed, it suffices to take an enclosing triangle of S_r with one side orthogonal to the line spanned by the points in A'', one side orthogonal to the line spanned by the points in B'', and one side orthogonal to the line spanned by the points in C''. Thus, it follows from Claim 1 in [2] (where it is proved in the more general setting of allowable sequences) that

$$E^{\text{bic}}_{\le k}(S_r) = \begin{cases} 3\binom{k+2}{2}, & \text{if } 0 \le k \le 3r-1; \\ 3\binom{3r+1}{2} + (k-3r+1)9r, & \text{if } 3r \le k \le 4r-1. \end{cases} \tag{13}$$

We now count the monochromatic $\le k$-edges. By 3-symmetry, it suffices to focus on those of type **A**.

It is readily checked that for all i and j distinct integers, $\ell(a_i a_j)$, $\ell(a'_i a'_j)$, and $\ell(a''_i a''_j)$ are k-edges for some $k > 4r-1$. The same is true for $\ell(a_i a'_j)$ whenever i and j are not both equal to 1 (when $i \neq 1$ and $j \neq 1$ this follows from (III) and (IV)), while $\ell(a_1 a'_1)$ is a $(4r-1)$-edge. Now, for $1 \le i \le r$ and $2 \le j \le r$, the line $\ell(a''_i a'_j)$ separates $4r+i-j$ points of S_r from the rest. That is, $\ell(a''_i a'_j)$ is a $(4r+i-j)$-edge if $i-j \le r/2-1$ or a $(5r-2-i+j)$-edge if $i-j > r/2-1$. Similarly, $\ell(a''_i a'_1)$, $\ell(a''_i a_j)$, and $\ell(a''_i a_1)$ separate $4r+i-2$, $3r+i+j-3$, and $3r+i-1$ points of S_r from the rest, respectively. In conclusion,

(i) for $1 \le s \le r$, the number of $(3r-1+s)$-edges of types $a'a''$ or aa'' is $2s$;
(ii) there is exactly one $(4r-1)$-edge of type aa'; and
(iii) all other edges of type **A** are k-edges for some $k > 4r-1$.

It follows that the number of $\le k$-edges of type A is

(a) 0, for $k \le 3r-1$;
(b) $2\sum_{s=1}^{k-(3r-1)} s = 2\binom{k-3r+2}{2}$, for $3r \le k \le 4r-2$;
(c) $1 + 2\sum_{s=1}^{(4r-1)-(3r-1)} s = 2\binom{r+1}{2} + 1$, for $k = 4r-1$.

By 3-symmetry, for each integer k there are exactly as many $(\le k)$-edges of type **A** as there are of type **B**, and of type **C**. Therefore

$$E^{\text{mono}}_{\le k}(S_r) = \begin{cases} 0 & \text{if } 0 \le k \le 3r-1, \\ 6\binom{k-(3r-2)}{2} & \text{if } 3r \le k \le 4r-2, \\ 6\binom{r+1}{2} + 3 & \text{if } k = 4r-1. \end{cases} \tag{14}$$

Because $E_{\le k}(S_r) = E^{\text{bic}}_{\le k}(S_r) + E^{\text{mono}}_{\le k}(S_r)$, it follows by (13) and (14) that $E_{\le k}(S_r)$ equals the right-hand side of (12).

8 Concluding Remarks

The inequality in Theorem 1 is best possible. That is, there are n-point sets P whose simple allowable sequence Π gives equality in the inequality of Corollary 1:

$$E_{\ge k}(\Pi) = (n-2k-1)E_{k-1}(\Pi) + \binom{s}{2}.$$

We present two constructions. The first has $s = n-2k-1$ and consists of $2k+1$ points which are the vertices of a regular polygon and $n-2k-1$ central points very close to the center of the polygon. This construction was given in [22] to show that $E_{k-1} \geq 2k+1$ is best possible. Indeed, note that the $(k-1)$-edges of P correspond to the larger diagonals of the polygon, and so $E_{k-1}(\Pi) = 2k+1$; moreover, any edge formed by two points in the central part or one point in the central part and a vertex of the polygon determine a $\geq k$-edge. Thus $E_{\geq k}(\Pi) = \binom{n-2k-1}{2} + (2k+1)(n-2k-1)$, which achieves the desired equality.

The second construction has $s = 0$ and thus it can only be achieved when $k \geq n/3$. Consider a $(2t+1)$-regular polygon where each vertex is replaced by a set of m points on a small segment pointing in the direction of the center of the polygon. Let Π be the allowable sequence corresponding to this point set, $n = (2t+1)m$, and $k = tm$. It is straightforward to verify that $E_{k-1}(\Pi) = (2t+1)m$ and $E_{\geq k}(\Pi) = 2(2t+1)\binom{m}{2}$. Thus $E_{\geq k}(\Pi) = (m-1)E_{k-1}(\Pi) = (n-2k-1)E_{k-1}(\Pi)$.

Prior to this work, there were two results that provided a lower bound for $E_{\leq k}(P)$ based on the behavior of values of k close to $n/2$. First, Welzl [27] as a particular case of a more general result proved that $E_{\leq k}(P) \geq F_1(k,n)$, where

$$F_1(k,n) = \binom{n}{2} - 2n\left(\sum_{j=k+1}^{n/2} k\right)^{1/2} < \binom{n}{2} - \frac{\sqrt{2}}{2} n^{3/2}\sqrt{n-2k}.$$

Second, Balogh and Salazar [12] proved that $E_{\leq k}(P) \geq F_2(k,n)$, where $F_2(k,n)$ is a function that, for $n/3 \leq k \leq n/2$, satisfies

$$F_2(k,n) < \binom{n}{2} - \frac{13\sqrt{3}}{36} n^{3/2}\sqrt{n-2k} + o\bigl(n^2\bigr).$$

By direct comparison, it follows that both $F_1(k,n)$ and $F_2(k,n)$ are smaller than the bound in Corollary 3. Thus our bound is better than these two previous bounds.

A nice feature of Theorem 1 is that it can give better bounds for $E_{\leq k}(n)$ and k large enough, and for $\overline{\mathrm{cr}}(n)$, provided someone finds a better bound than inequality (2) for $E_{\leq k}(n)$ when $4n/9 < k < n/2$. For example, Ábrego et al. [5] considered 3-regular point sets P. These are point sets with the property that for $1 \leq j \leq n/3$, the jth depth layer of P has exactly 3 points of P. A point $p \in P$ is in the jth depth layer if p belongs to a $(j-1)$-edge but not to a $(\leq j-2)$-edge of P. If n is a multiple of 18, they proved the following lower bound:

$$E_{\leq k}(P) \geq 3\binom{k+2}{2} + 3\binom{k+2-n/3}{2} + 18\binom{k+2-4n/9}{2}. \tag{15}$$

This is better than the bound in Theorem 2 for $k > 4n/9$, however using Theorem 1 it is possible to find an even better lower bound when $k \geq 17n/36$. We construct a new recursive sequence u' starting at $m = 17n/36$ given by

$$u'_{m-1} = 3\binom{m+1}{2} + 3\binom{m+1-\lfloor n/3\rfloor}{2} + 18\binom{m+1-\lfloor 4n/9\rfloor}{2} \quad \text{and}$$

$$u'_k = \left\lceil \frac{1}{n-2k-2}\left(\binom{n}{2} + (n-2k-3)u'_{k-1}\right)\right\rceil \quad \text{for } k \geq m.$$

The value of $m = 17n/36$ is the smallest possible for which u'_m is greater than the right-hand side of (15). Following the proof of Theorem 2 it is possible to show that $E_{\leq k}(P) \geq u'_k$ for $17n/36 \leq k < n/2$. Thus, if we could show that (15) holds for arbitrary point sets P, then we know that bound will no longer be tight for $k \geq 17n/36$. From equivalent statements to Lemmas 1 and 2, it follows that $u'_k \sim \binom{n}{2} - (7\sqrt{2}n^2/18)\sqrt{1-2k/n}$. This in turn improves the crossing number of 3-regular point sets P to $\overline{\text{cr}}(P) \geq 0.380024\binom{n}{4} + \Theta(n^3)$.

In [2] we considered other class of point sets called 3-decomposable. These are point sets P for which there is a triangle T enclosing P and a balanced partition A, B, and C of P, such that the orthogonal projections of P onto the sides of T show A between B and C on one side, B between A and C on another side, and C between A and B on the third side. For 3-decomposable sets P we were able to prove a lower bound consisting of an infinite series of binomial coefficients:

$$E_{\leq k}(P) \geq 3\binom{k+2}{2} + 3\binom{k+2-n/3}{2} + 3\sum_{j=2}^{\infty} j(j+1)\binom{k+2-c_j n}{2}, \qquad (16)$$

where $c_j = 1/2 - 1/(3j(j+1))$.

Our main result does not improve this lower bound, however it gives an interesting heuristic that provides some evidence about the potential truth of this inequality for unrestricted point sets P. If we assume that the sum of the first $t+1$ terms in the right-hand side of (16) is a lower bound for $E_{\leq k}(P)$, then, just as we outlined in the previous paragraph for $t = 2$, Theorem 1 gives a better bound when k is big enough. This happens to be precisely when $k \geq c_{t+1}n$, which is also the value of k for which the next term in the sum of (16) gives a nonzero contribution.

It was also shown in [2] that (16) implies the following bound for 3-decomposable sets P:

$$\overline{\text{cr}}(P) \geq \frac{2}{27}\left(15 - \pi^2\right)\binom{n}{4} + \Theta\left(n^3\right) > 0.380029\binom{n}{4} + \Theta\left(n^3\right). \qquad (17)$$

Theorem 1 does not improve the $\binom{n}{4}$ coefficient, but it improves the speed of convergence. For instance, using Theorem 1 together with the first 30 terms of (16) gives a better bound than the one obtained solely from the first 101 terms of (16).

Finally, we reiterate our conjectures from [2] that inequalities (16) and (17) are true for unrestricted point sets P. We in fact conjecture that for every k and n, the class of 3-decomposable sets contains optimal sets for both $E_{\leq k}(n)$ and $\overline{\text{cr}}(n)$.

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