A Face of a Projective Triangulation Removed for Its Geometric Realizability

Atsuhiko Nakamoto · Shoichi Tsuchiya

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Abstract Let M be a map on a surface F^2 . A *geometric realization* of M is an embedding of F^2 into a Euclidian 3-space $\mathbb{R}^3$ with no self-intersection such that each face of M is a flat polygon. In Bonnington and Nakamoto (Discrete Comput. Geom. 40:141–157, 2008), it has been proved that every triangulation G on the projective plane has a face f such that the triangulation $G - f$ on the Möbius band obtained from G by removing the interior of f has a geometric realization. In this paper, we shall characterize such a face f of G .

Keywords Geometric realization · Triangulation · Projective plane · Möbius band

1 Introduction

A *map* M means a fixed embedding of a simple graph on a surface F^2 . We only deal with *2-cell embedding*, that is, one all of whose faces are homeomorphic to 2-cells. A *triangulation* on a surface F^2 is a map on F^2 such that each face is bounded by a 3-cycle, where a *k-cycle* means a cycle of length k . For a map M on a surface, a cycle C is said to be *essential* if C does not bound a 2-cell on the surface, and it is *trivial* otherwise. (Similarly, we define *essential* and *trivial* closed curves on the surface.) For a face or a 2-cell region R of a map M , let ∂R denote the boundary

A. Nakamoto

Department of Mathematics, Faculty of Education and Human Sciences, Yokohama National University, 79-2 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
e-mail: nakamoto@ynu.ac.jp

S. Tsuchiya (✉)

Department of Information Media and Environment Sciences, Graduate School of Environment and Information Sciences, Yokohama National University, 79-7 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
e-mail: s-s-t-b@mail.goo.ne.jp

Fig. 1 A Möbius triangulation with no geometric realization constructed by Brehm

closed walk of R . For a cycle C of M bounding a 2-cell, let $\text{Int } C$ denote the subgraph of G consisting of all vertices and edges in the interior and the boundary of C . On the other hand, let $\text{int } C = \text{Int } C - V(C)$. We say that a vertex and an edge of $\text{int } C$ are *inner*. Let C and D be trivial cycles of M . We say that C *surrounds* D if $\text{Int } C$ contains D . In particular, if $\text{int } C$ contains D , then C is a *nesting cycle* of D , which is an important definition in our paper. We often denote vertices and edges by symbols with subscripts, in which we take suitable modulus.

Let M be a map on a surface F^2 . A *geometric realization* of M is an embedding of F^2 into a Euclidian 3-space $\mathbb{R}^3$ with no self-intersection such that each face of M is a flat polygon. Steinitz [11] has proved that a spherical map has a geometric realization as a convex polyhedron if and only if its graph is 3-connected. (The 3-connectedness is necessary to make a geometric realization as a convex polygon. However, our definition of a geometric realization allows two adjacent faces to lie on the same plane in $\mathbb{R}^3$.) Moreover, Archdeacon, Bonnington, and Ellis-Monaghan [1] have proved that every toroidal triangulation has a geometric realization. In general, Grünbaum has conjectured that every triangulation on any orientable closed surface has a geometric realization [6]. However, Bokowski and Oliveira [3] have shown that there exists a triangulation on the orientable closed surface of genus six which has no geometric realization. Moreover, Schewe [10] has shown that there exists a triangulation on the orientable closed surface of genus five which has no geometric realization. Hence Grünbaum's conjecture is no longer true now, but it is still open for the orientable closed surfaces of genus from two to four.

Let us consider nonorientable surfaces, in particular, the projective plane. Since the projective plane itself is not embeddable in $\mathbb{R}^3$, no map on the projective plane has a geometric realization. However, the surface obtained from the projective plane by removing a disk (i.e., a *Möbius band*) is embeddable in $\mathbb{R}^3$, and hence a triangulation on the Möbius band might have a geometric realization. For simple notation, we call a triangulation on the projective plane and that on the Möbius band a *projective triangulation* and a *Möbius triangulation*, respectively, throughout the paper.

In this paper, we discuss geometric realizations of Möbius triangulations. However, Brehm [4] found a Möbius triangulation with no geometric realization, which is shown in Fig. 1. (In Fig. 1, identify vertices with the same label.) Brehm essentially proved in [4] that for each of its spatial embeddings, the two disjoint 3-cycles 123 and 456 have a linking number at least two. (See [9] for the definition of the linking number.) However, two 3-cycles with straight segments in $\mathbb{R}^3$ have a linking number at most one. Hence we have the following:

Fact 1 *If a Möbius triangulation G has a boundary 3-cycle C and a 3-cycle C' disjoint from C which forms an annular region with C , then G has no geometric realization.*

Bonnington and Nakamoto have proved the following theorem. Let G be a projective triangulation, and let f be a face of G . Let $G - f$ denote the Möbius triangulation obtained from G by removing the interior of f .

Theorem 2 (Bonnington and Nakamoto [2]) *Every projective triangulation G has a face f such that $G - f$ has a geometric realization.*

In this paper, we would like to consider which faces f of a given projective triangulation G can be chosen so that $G - f$ is geometrically realizable. Suppose that G has a face f such that the boundary 3-cycle of f has a nesting 3-cycle. In this case, we simply say that “ f has a nesting 3-cycle.” Then, by Fact 1, $G - f$ has no geometric realization.

Our main theorem is as follows.

Theorem 3 *Let G be a projective triangulation, and let f be a face of G . Then $G - f$ has a geometric realization if and only if f has no nesting 3-cycle in G .*

This theorem immediately implies Theorem 2, since we can always choose a face f in a projective triangulation G with no nesting 3-cycle. (Suppose that a face g of G has nesting 3-cycles. Choosing C among them to *outermost*, i.e., C has no nesting 3-cycle, we can take a face f in the 2-cell region bounded by C and sharing an edge with C ; clearly, f has no nesting 3-cycle.)

Moreover, Theorem 3 implies the following since, for any 3-cycle C of a 4-connected projective triangulation, $\text{Int } C$ contains no 3-cycle other than C .

Corollary 4 *Let G be a 4-connected projective triangulation. Then, for any face f of G , $G - f$ has a geometric realization.*

A weaker version of Corollary 4 has already been proved by Chávez, Fijavž, Márquez, Nakamoto, and Suárez [5].

Theorem 5 (Chávez, Fijavž, Márquez, Nakamoto, and Suárez [5]) *If G is a 5-connected projective triangulation, then for any face f of G , $G - f$ has a geometric realization.*

Corollary 4 can be strengthened by introducing a notion of the cyclical k -connectivity. A graph H is said to be *cyclically k -connected* if H has no k -cut S such that at least two components of $H - S$ have a cycle. We have the following, which is conjectured in [5].

Corollary 6 *Let G be a projective triangulation, and let f be a face of G . Then, for any face f of G , $G - f$ has a geometric realization if and only if G is cyclically 4-connected.*

2 Outline of the Proof

In this section, we give an outline for the proof of Theorem 3. We prove our main result by using methods similar to those in [2, 5]. The proof is split into two parts; the first part is graph-theoretical, and the second part is geometrical. It is the notion of exhibitions of maps that connects two different methods.

A plane map R with a boundary walk C of length $m \geq 3$ is said to be a *near triangulation* if C is a cycle and if each inner face of $\text{Int } C$ is triangular. Let R be a near triangulation, and let C be the boundary cycle of R . Suppose that we are given an embedding of C in $\mathbb{R}^3$ so that each edge is a straight-line segment, and that there exists a plane $P \subset \mathbb{R}^3$ such that the image of C in $\mathbb{R}^3$ is projected to the plane P as a convex polygon. Such an embedding of C in $\mathbb{R}^3$ is called an *exhibition*.

Lemma 7 [1, 2] *Let R be a near triangulation, and let C be the boundary of R . Suppose that an exhibition of C in $\mathbb{R}^3$ is given. Then the exhibition of C extends to a geometric realization of R in the convex-hull of C in $\mathbb{R}^3$.*

Let X be a map on a surface with each face bounded by a cycle. An embedding ψ of X into $\mathbb{R}^3$ is called an *exhibition* if

- (i) for any face boundary C of X , $\psi(C)$ is an exhibition,
- (ii) for any two distinct faces f and f' of X , the convex hulls of $\psi(f)$ and $\psi(f')$ intersect only at their common vertices and edges in the map.

An exhibition is a relaxed notion of a geometric realization since a geometric realization is an exhibition. In particular, if G is a triangulation, then an exhibition of G is equivalent to a geometric realization of G . The following is an important lemma for constructing a geometric realization of a map, and its proof can easily be obtained from the definition of exhibitions of near triangulations and maps.

Lemma 8 *Let G be a triangulation on a surface F^2 , and let X be a submap of G which is a 2-cell embedding of F^2 . If X has an exhibition, then G has a geometric realization.*

In order to prove Theorem 3, we first prove the following, and Sect. 3 is devoted to it.

Lemma 9 *Let G be a projective triangulation, and let f be a face of G with no nesting 3-cycle. Then G has a 4-cycle $C = v_1 v_2 v_3 v_4$ separating G into a Möbius triangulation G_M and a near triangulation G_D both of whose boundaries are C such that*

- (i) G_M has a submap X with boundary cycle C which is isomorphic to a subdivision of one of the nine maps shown in Fig. 2,
- (ii) G_D has a diagonal $v_i v_{i+2}$ for some i , and
- (iii) G_D contains f such that $f = v_i v_{i+1} v_{i+2}$, or $v_i v_{i+2} \notin E(f)$ but $v_i \in V(f)$.

Fig. 2 Nine maps on the projective plane

In Sect. 4, we shall prove that G_M in Lemma 9 has a geometric realization, and so does $G_D - f$ in the lemma to complete a geometric realization of $G - f$. In order prove that, we use computer programming.

In Sect. 5, we give a proof of Theorem 3 and concluding remarks.

3 Contraction in Möbius 4-Triangulations

Let G be a triangulation on a surface F^2 , and let e be an edge of G . *Contraction* of e is removing e , identifying the two endpoints of e , and replacing two pairs of parallel edges by two single edges. (For an edge e contained in ∂G , the contraction of e can be defined similarly except replacing one pair of parallel edges with a single edge.) Let G/e denote the graph obtained from G by contracting e . If G/e is simple, then e is said to be *contractible*. We always contract only a contractible edge. The inverse operation of the contraction of e is a *splitting* of v , where v is a vertex obtained from e by its contraction.

Lemma 10 [8] *Let G be a triangulation on a surface F^2 , and let Δ be a triangular region which is not a face. Then there is a contractible edge in the interior of Δ .*

Lemma 10 immediately implies the following:

Lemma 11 *Let G be a triangulation on a nonspherical surface. Then G has no contractible edge if and only if each edge of G is contained in an essential 3-cycle.*

Lemma 12 *Let G be a triangulation on a nonspherical surface, and let $C = v_1v_2v_3v_4$ be a trivial 4-cycle of G . If G has no contractible edge, then the interior of C contains either a single diagonal or a single vertex. Furthermore, the latter happens if and only if the exterior of C has two edges v_1v_3 and v_2v_4 .*

Proof If $\text{Int } C$ has a diagonal $v_i v_{i+2}$, C satisfies Lemma 12 by Lemma 11. So we suppose that C has no diagonal in the interior of C . Then it is easy to see that the subgraph of G induced by the inner vertices of C is connected. So, if the interior of C contains at least two vertices, then we can find an edge joining two inner vertices w_1 and w_2 . By Lemma 11, $w_1 w_2$ is contained in an essential 3-cycle $D = w_1 w_2 x$ since G has no contractible edge. However, since w_1 and w_2 are inner vertices of the interior of C , D must be contained in $\text{Int } C$. This contradicts that D is essential. If the interior of C contains a single vertex v , then v is adjacent to v_1, v_2, v_3, v_4 . Since vv_i is not contractible for $i = 1, 2, 3, 4$, v_i must be adjacent to v_{i+2} in the exterior of C . $\square$

Let R be a near triangulation with a boundary cycle C , and let $x, y \in V(C)$. A path jointing x and y and intersecting C only at x and y is called an *inner* (x, y) -*path* or an *inner path*. We often use the following lemma proved in [2] to find a suitable subgraph in a near triangulation.

Lemma 13 [2] *Let R be a near triangulation with the boundary cycle C , and let $x, y \in V(C)$ with $xy \notin E(C)$. There is an inner (x, y) -path in R if and only if there is no chord pq for any $p, q \in V(C) - \{x, y\}$ such that x, p, y , and q appear on C in this cyclic order.*

The Möbius band admits two types of essential simple closed curves: One is homotopic to the boundary of the Möbius band, and the other is homotopic to a center line of the Möbius band. The former separates the Möbius band, and the latter is nonseparating.

In this section, we deal with a Möbius triangulation M whose boundary cycle has length exactly 4, which is called a *Möbius 4-triangulation*. (We sometimes let a Möbius 4-triangulation express a map on the projective plane with only one quadrilateral face, and all other faces are triangular.) Let ∂M denote the boundary 4-cycle of M . Let e be an *inner* edge of M , that is, an edge not lying on ∂M . We always consider the contractions of contractible inner edges. A Möbius 4-triangulation G is said to be *contractible* to another Möbius 4-triangulation G' if G is transformed in G' by a sequence of contractions of inner edges.

Let X be the map on the Möbius band with six vertices and the boundary 4-cycle $v_1v_2v_3v_4$ such that each of two inner vertices is adjacent to all of v_1, v_2, v_3 and v_4 (see Fig. 3). A Möbius 4-triangulation G with at least six vertices is said to be *X-framed* if G has two inner vertices each of which is adjacent to all four vertices in the boundary.

Fig. 3 The map X

Lemma 14 *Let M be a Möbius 4-triangulation with the boundary cycle C . If M has no 3-cycle homotopic to C , M is contractible to a X -framed Möbius 4-triangulation with the boundary cycle C .*

Proof Let $C = v_1v_2v_3v_4$. Suppose that M has a trivial 3-cycle D which is not a face boundary. Then $\text{int } D$ has a contractible edge e by Lemma 10. If M/e has a 3-cycle homotopic to C , e is contained in a 4-cycle homotopic to C in M , contrary to that M has no 3-cycle homotopic to C or M is simple. Therefore M/e has no 3-cycle homotopic to C , and we may suppose that M has no trivial separating 3-cycle. Suppose that M has a 4-cycle $C' = v'_1v'_2v'_3v'_4$ homotopic to C but $C' \neq C$. If M has at least two such cycles, we take *outermost* one (i.e., no 4-cycle homotopic to C in the region bounded by C and C'). By Menger's theorem [7], M has a path P_i from v_i to v'_i for $i = 1, 2, 3, 4$ such that each P_i is disjoint to P_j for $j = 1, 2, 3, 4$, but $j \neq i$, since M has no 3-cycle homotopic to C . There exists an edge $e = v'_ip$ on P_i adjacent to v'_i for some i since $C \neq C'$. If e is a noncontractible edge, e is contained in an essential 3-cycle v'_ipq since M has no trivial separating 3-cycle. Moreover, since M has no 3-cycle homotopic to C , v'_ipq is nonseparating. So q must be contained in C' , and $q \in V(P_{i-1})$ or $V(P_{i+1})$, contrary to the simpleness of M . Therefore we may suppose that e is a contractible edge. If M/e has a 3-cycle homotopic to C , e is contained in a 4-cycle v'_ipqr homotopic to C in M . Let M' be a triangulation on the Möbius band bounded by C' . Since C' is outermost in M , r must be an inner vertex of M' , contrary to that M has no 3-cycle homotopic to C or M is simple. Therefore if M has a 4-cycle C' homotopic to C , we can contract edges of M so that $C' = C$ without 3-cycles homotopic to C . So we may suppose that M has no 4-cycle homotopic to C except for C , and we shall prove that if M has no contractible edge, M is an X -framed Möbius 4-triangulation.

Observe that $\deg_M(v_i) \geq 3$ for $i = 1, 2, 3, 4$. (For otherwise, i.e., if $\deg_M(v_2) = 2$ for example, then M has a face $v_1v_2v_3$, and hence $v_1v_3v_4$ is a 3-cycle of M homotopic to C , a contradiction.) If $\deg_M(v_i) = 3$ for some i , say $N_M(v_2) = \{v_1, x, v_3\}$, then we have $x \notin \{v_2, v_4\}$. Hence we take $C' = v_1xv_3v_4$ as a 4-cycle homotopic to C in M , a contradiction. Therefore we may suppose that $\deg_M(v_i) \geq 4$ for $i = 1, 2, 3, 4$. So, if we let $v_iv_{i+1}l_i$ be the face of M incident to the edge v_iv_{i+1} for each i , then l_i and l_{i+1} are distinct. We shall prove that $l_1 = l_3$ and $l_2 = l_4$. So we may suppose that $l_1 \neq l_3$ in M by symmetry. Since v_1l_1 is not contractible, it is contained in an essential 3-cycle, say $C_1 = v_1l_1x$, by Lemma 11. Moreover, since M has no 3-cycle homotopic to C , C_1 is nonseparating. Similarly, since v_3l_3 is not contractible by symmetry, v_3l_3 is contained in an essential nonseparating 3-cycle $C_2 = v_3l_3y$. Since C_1 and C_2 are nonseparating cycles on the Möbius band and since $l_1 \neq l_3$, $v_1 \neq l_3$ ($l_1 \neq v_3$), $v_1 \neq v_3$, $x \neq v_3$ ($v_1 \neq y$), and $x \neq l_3$ ($l_1 \neq y$), we have $x = y$. See the left

Fig. 4 The structure of M

Fig. 5 The structure of M (left) and an inner trivial cycle of H (right)

in Fig. 4. By Lemma 12, each of the two quadrilateral regions $v_1x l_3 v_4$ and $l_1 v_2 v_3 y$ has a diagonal, which must be an edge $x v_4$ and $y v_2$, respectively. (For otherwise, we would have $\deg_M(v_2) = 3$ or $\deg_M(v_4) = 3$, contrary to that $\deg_M(v_i) \geq 4$ for $i = 1, 2, 3, 4$.) Moreover, we have $x = y = l_2 = l_4$. See the right in Fig. 4.

Now we focus on the edges $v_2 l_1$ and $v_4 l_3$. Similarly to the edge $v_1 l_1$, if they are not contractible, they are contained in essential nonseparating 3-cycles $v_2 l_1 x'$ and $v_4 l_3 y'$, respectively. Since v_2 and l_1 have only one common neighbor other than v_1 and l_2 , we have $x' = v_3$. By the same argument, we have $y' = v_1$. This is a contradiction since two 3-cycles $v_2 l_1 v_3$ and $v_4 l_3 v_1$ cannot exist simultaneously. So, we can contract inner edges until $l_1 = l_3$ and $l_2 = l_4$. See the left of Fig. 5. Hence M is an X -framed Möbius 4-triangulation. $\square$

Let M_0 be an X -framed Möbius 4-triangulation with the boundary cycle $C = v_1 v_2 v_3 v_4$ and inner vertices v_5, v_6 adjacent to each vertex on C . Let H_0 be an X -frame of M_0 . Consider a splitting of v_i of H_0 into two vertices v_i and v'_i of degree 3. We always suppose that v_i , for $i = 1, 2, 3, 4$, lies on C . Let H , called a *split X -frame*, be a map with the boundary 4-cycle C on the Möbius band obtained from H_0 by applying a sequence of either splitting of each v_i or subdividing each edge. We call a vertex v of H whose degree is greater than 2 a *node* and a path in H containing only two nodes as its endpoints a *segment*. Let $P(a, b)$ denote a path on a segment of H with two endpoints a, b . If $a = b$, we suppose that $P(a, b) = \{a\}$. Moreover, $\{v_i, v'_i\}$ is called a *boundary split pair* or *boundary nodes* if they arose by a splitting of v_i on the boundary; otherwise it is called an *inner split pair* or *inner nodes*. Let *d-segment* be a segment of a split X -frame whose endpoints are a boundary node and an inner node. A trivial cycle of the split X -frame which contains four *d*-segments without segments on the boundary is called an *inner trivial cycle*. (For example, see the right of Fig. 5.) Observe that a split X -frame has two inner trivial cycles.

Proof of Lemma 9 Let D be a 3-cycle surrounding f . If G has at least two such 3-cycles, we choose the maximal one. Let $\tilde{f}$ denote a triangular region bounded by D . Observe that G has no 3-cycle surrounding $\tilde{f}$ except for $\partial\tilde{f}$ since G has no nesting 3-cycle of f . Since G is a triangulation, G has three faces neighboring to $\tilde{f}$, among which we choose a face f' as follows. If f has an edge e such that $e \in E(\partial\tilde{f})$, we choose f' so that e lies on $\partial(\tilde{f} \cup f')$. So we consider the case that $E(f) \cap E(\partial\tilde{f}) = \emptyset$. Let u_1, u_2, u_3 be vertices lying on $\partial\tilde{f}$. Since f has no nesting 3-cycle, there exists a vertex v of f such that $v \in V(\partial\tilde{f})$. We may suppose that $v = u_1$. In this case, we choose f' whose boundary contains u_1u_3 . Let C be $\partial(\tilde{f} \cup f')$, and let G_M (resp., G_D) be the Möbius triangulation (resp., the near triangulation) with boundary cycle C . Observe that C satisfies the conditions (ii) and (iii) of Lemma 9.

We shall prove that C satisfies the condition (i) of Lemma 9. Observe that, G_M has no 3-cycle homotopic to C by the definition of C . So G_M is contractible to an X -framed 4-triangulation G_{M0} with the boundary cycle $v_1v_2v_3v_4$ by Lemma 14. Therefore G_M has a split X -frame.

Claim 1 *There exists a split X -frame of G_M with the boundary cycle $v_1v_2v_3v_4$ and no inner split pair.*

Proof Let H_0 be an X -frame of G_{M0} , and let $V(H_0) - V(C) = \{v_5, v_6\}$. Let H be a split X -frame of G_M with the boundary cycle $C = v_1v_2v_3v_4$ obtained from H_0 by a sequence of splitting of a vertex v_i of H_0 into v_i, v'_i of degree 3 and subdividing edges. We may suppose that each segment of H has no chord in G_M . (Otherwise we can take a shorter segment in G_M .) By the simpleness of G_{M0} , we may suppose that the interior of $v_5v_4v_6v_2$ contains no chord from v_5 to v_6 in G_{M0} . (Otherwise the interior of $v_5v_3v_6v_1$ does not contain it.) Therefore we may suppose that the interior of one inner trivial cycle of H has no chord xy such that $x \in V(P(v_5, v'_5))$ and $y \in V(P(v_6, v'_6))$ in G_M . Let H' be a split X -frame of G_M with the boundary cycle $v_1v_2v_3v_4$. Let u_i, u'_i for $i = 1, 2, 3, 4$ be boundary nodes of H' , and u_i, u'_i for $i = 5, 6$ be inner nodes of H' . We always take H' in G_M so that the interior of one inner trivial cycle has no chord from $x \in V(P(u_5, u'_5))$ to $y \in V(P(u'_6, u_6))$ and $P(u_i, u'_i)$ for $i = 5, 6$ is as short as possible.

Suppose that each of u_i for $i = 1, 2, 3, 4$ is a boundary split pair. (Otherwise we consider $u'_i = u_i$ for $i = 1, 2, 3, 4$.) We shall prove that neither u_5 nor u_6 is a split pair of H' . Suppose that u_5 is a split pair of H' . Then, there are two cases of the splitting of u_5 as shown in Fig. 6.

Fig. 6 A splitting of u_5 in H'

Fig. 7 Other split X -frames

Fig. 8 Other split X -frames

First, we consider the case in the left-hand side of Fig. 6. There exist two inner trivial cycles D_1, D_2 in H' . By the definition of H' , we may suppose that $D_1 = P(u_5, u'_5)P(u'_5, u'_4)P(u'_4, u'_6)P(u'_6, u_6)P(u_6, u'_2)P(u'_2, u_5)$ and $\text{Int } D_1$ has no chord from $x \in V(P(u_5, u'_5))$ to $y \in V(P(u'_6, u_6))$ in G_M . Let us consider the neighbor of u_5 in $\text{Int } D_1$ on G_M . If $\text{Int } D_1$ has a chord from u_5 to l , where $l \in V(P(u'_4, u'_5)) - \{u'_5\}$ or $l \in V(P(u'_4, u_6)) - \{u_6\}$, we can take other split X -frames, contrary to the assumption of H' . (See Fig. 7.)

If $\text{Int } D_1$ has a chord from u_5 to l , where $l \in V(P(u'_2, u_6)) - \{u_6\}$, then we can take an inner path from l to l' , where l' is adjacent to u_5 on $P(u_5, u'_5)$, through the neighbor of u_5 . If we regard l' as new u_5 , we can take another split X -frame, contrary to the assumption of H' . (See the left of Fig. 8.) If $\text{Int } D_1$ does not have such chords, we can take a path from u'_5 to u'_2 through the neighbor of u_5 . So we can take another split X -frame, contrary to the assumption of H' . (See the right of Fig. 8.)

In the right-hand side of Fig. 6, we consider the neighbor of u_5 in the interior of $P(u_5, u'_5)P(u'_5, u'_2)P(u'_2, u_2)u_2u_1P(u_1, u'_1)P(u'_1, u_5)$. By the same argument, we can obtain a contradiction. So u_5 of H' is not a split node. We can deal with u_6 similarly. $\square$

By Claim 1, G_M has a split X -frame with no inner split pair. Next, we shall prove that there exists a split X -frame with no inner split pair such that an inner trivial cycle has an inner path whose endpoints are two boundary nodes of it.

Claim 2 Let D_j for $j = 1, 2$ be an inner trivial cycle of a split X -frame in G_M . There exists a split X -frame with the boundary $v_1v_2v_3v_4$ and no inner split pair such that one of $\text{Int } D_j$ has an inner path from a boundary node to another boundary node in G_M .

Fig. 9 Other split X -frames

Proof Let H be a split X -frame of G_M with the boundary $v_1v_2v_3v_4$ and no inner split pair. We may suppose that each segment of H has no chord in G_M . (Otherwise we can take a shorter segment in G_M .) Let v_5, v_6 be inner nodes of H . We may suppose that $D_1 = P(v'_1, v_5)P(v_5, v'_3)P(v'_3, v_6)P(v_6, v'_1)$. (If v_i is not a split node, we regard v_i as a node on D_j for $j = 1, 2$.) We shall prove that one of $\text{Int } D_j$ contains an inner (v'_j, v'_{j+2}) -path in G_M . By the symmetry and simpleness of G_M , we may suppose that $\text{Int } D_1$ contains no chord v_5v_6 . We take H so that $\text{Int } D_1$ contains few chords in G_M as possible. By Lemma 13 and symmetry, if $\text{Int } D_1$ contains no inner (v'_1, v'_3) -path, we may suppose that $\text{Int } D_1$ contains a chord pq such that $p \in V(P(v_5, v'_1)) - \{v'_1, v_5\}$, $q \in V(P(v'_1, v_6)) - \{v'_1\}$ or $q \in V(P(v'_3, v_6)) - \{v'_3, v_6\}$.

If $q \in V(P(v'_1, v_6)) - \{v'_1\}$, we can regard p as a new boundary node v'_1 , and we can take another split X -frame with no chord v_5v_6 in $\text{Int } D_1$ so that the number of chords contained in $\text{Int } D_1$ is fewer, a contradiction. (See the left of Fig. 9.) Therefore we may suppose that $\text{Int } D_1$ contains no chord xy such that $x \in V(P(v_5, v'_1))$, $y \in V(P(v_6, v'_1))$ or $x \in V(P(v_5, v'_3))$, $y \in V(P(v_6, v'_3))$.

If $q \in V(P(v'_3, v_6)) - \{v'_3, v_6\}$, let $P(v_3, v'_3)P(v'_3, v_6)P(v_6, v'_4)P(v'_4, v_4)v_4v_3 = D$. In this case, $\text{Int } D$ has no chord rs such that $r \in V(P(v_6, q)) - \{q\}$, $s \in V(P(q, v'_3)) - \{q\}$ or $s \in V(P(v_3, v'_3))$ in G_M . (Otherwise the segment from v_6 to v'_3 has a chord, or we can take another split X -frame with no chord v_5v_6 in $\text{Int } D_1$ so that the number of chords in $\text{Int } D_1$ is fewer, a contradiction.) Therefore $\text{Int } D$ has a path from q to v_4 which does not intersect $V(P(v_3, v'_3) \cup P(v'_3, v_6) - \{q\})$. If we regard q as a v_6 , we can take another split X -frame with no chord v_5v_6 in $\text{Int } D_1$ so that the number of chords in $\text{Int } D_1$ is fewer, a contradiction. (See the right of Fig. 9.) Hence $\text{Int } D_1$ has no chord pq , and we can take an inner (v'_1, v'_3) -path in $\text{Int } D_1$. $\square$

By Claims 1 and 2, G_M has a split X -frame H with no inner split nodes and an inner (v_i, v_{i+2}) -path P such that $P \cap H = P(v_i, v'_i) \cup P(v_{i+2}, v'_{i+2})$. Then G_M has one of sixteen subgraphs since for each boundary node, there exist two possibilities depending on whether it has a split pair or not. By the symmetry of $H \cup P$, G_M has a subdivision of one of the nine maps shown in Fig. 2 as a subgraph. Therefore C satisfies the condition (i) of the lemma. $\square$

4 Constructions of Geometric Realizations

In this section, in order to prove Theorem 3, we shall construct geometric realizations and exhibitions of given maps. We first construct an exhibition of the nine maps $X_0, X_1, \dots, X_8$ shown in Fig. 2.

Lemma 15 *For $i = 0, \dots, 8$, each X_i has an exhibition.*

Let $f_1, \dots, f_n$ be the faces of X_i , and let D_j be the boundary cycle of f_j for each j . To prove Lemma 15, by the definition of exhibitions, we have to arrange the vertices of X_i in $\mathbb{R}^3$ so that

- (i) the embedding of each D_j is projected to some plane as a convex polygon, and
- (ii) for any distinct $j, k \in \{1, \dots, n\}$, two convex hulls $\langle D_j \rangle$ and $\langle D_k \rangle$ are disjoint except for their common points in X_i .

The following two computer programs, Programs 1 and 2, check that the coordinates of the vertices given really satisfy the above two conditions (i) and (ii), respectively.

Program 1 Let $A = a_1 a_2 \dots a_n$ be a cycle embedded in $\mathbb{R}^3$ so that each edge $a_i a_{i+1}$ is a straight segment, where $a_i \in \mathbb{R}^3$ for each i .

Input: $\mathbb{R}^3$ -coordinates of point sets $a_1, a_2, \dots, a_n$ and a plane F .

Output: *True* if the projection of A to F is a convex polygon, and *false* otherwise.

The following is the procedures in Program 1.

Step 1. Let $A' = a'_1 a'_2 \dots a'_n$ be a cycle on F obtained by an orthogonal projection of $A = a_1 a_2 \dots a_n$ to F , where a'_i corresponds to a_i for each i . For $i = 1, 2, \dots, n$, let l_i be the straight line on F through a'_i and a'_{i+1} . If there exists a'_{i+2} lying on l_i for some i , then return *false*. Otherwise go to Step 2.

Step 2. Let F_i^L and F_i^R be the two half-planes on F separated by l_i , where we suppose that F_i^L contains a'_{i+2} . For each i , if there exists no a'_j with $j \neq i$ contained in F_i^R , then return *true*. Otherwise, return *false*.

Program 2 Let A and B be two point sets in $\mathbb{R}^3$. We check whether or not there exists a plane F containing all points of $A \cap B$ and separating $A' = A - (A \cap B) \neq \emptyset$ and $B' = B - (A \cap B) \neq \emptyset$ in distinct sides. If there exists such F , then we say that F is *admissible*.

Input: $\mathbb{R}^3$ -coordinates of point sets $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$, and a plane F containing all points of $A \cap B$.

Output: *True* if F is admissible, and *false* otherwise.

The following is the procedures in Program 2.

Step 1. Set F^+ and F^- to be the two half-spaces in $\mathbb{R}^3$ separated by F , where some point $a_i \in A'$ is contained in F^+ .

Step 2. If there is some $a_i \in F^-$, then return *false*. Otherwise go to Step 3.

Step 3. If there is some $b_i \in F^+$, then return *false*. Otherwise return *true*.

The following is an easy observation.

Proposition 16 *Let X be a map on a surface, and let $f_1, \dots, f_n$ be the faces of X . Suppose that the coordinates of all vertices of X are given. If any two of f_i and f_j share at most one edge, then Programs 1 and 2 check whether the coordinates give an exhibition of X .*

In Proposition 16, suppose that two faces f_1 and f_2 of X share two vertices x and y , but x and y are not adjacent in the boundary of f_1 . Then, to start Program 2, we are given a plane π containing x and y such that all vertices of f_1 and those of f_2 except x and y are located in two half-spaces separated by π . Though f_1 and f_2 share only x and y , the convex polygon, say P_1 , corresponding to f_1 and constructed by Program 1 shares a segment xy on π with the convex polygon, say P_2 , corresponding to f_2 . We can see that P_1 and P_2 share a segment xy but f_1 and f_2 do not share an edge xy . Hence this is never an exhibition of X .

Proof of Lemma 15 For each map X_i , since any two faces share at most one edge, we can apply Proposition 16. We construct an exhibition of only X_2 since the remaining cases can be dealt similarly. (The cases for X_0 and X_1 are easier than that of X_2 , since they have many triangular disks.)

Give the $\mathbb{R}^3$ -coordinates of the vertices $v_1, v_2, v'_2, v_3, v_4, v'_4, v_5, v_6$ of X_2 as follows:

$$\begin{array}{lll} v_1 = (12, -10, 10), & v_2 = (-10, 10, -6), & v'_2 = (-9, 10, -4), \\ v_3 = (-16, -10, 8), & v_4 = (5, 10, -12), & v'_4 = (4, 10, -10), \\ v_5 = (0, 1, 0), & v_6 = (-1, 20, 0). \end{array}$$

See Fig. 10. We put A, B, C, D, E, F, G to be the faces of X_2 as shown in the left. Then we can verify our coordinates of X_2 really gives an exhibition, as follows. We

Fig. 10 An exhibition of X_2

Table 1 Planes distinguishing two point sets

The plane	Convex-hulls
$11x - 128y - 154z = -128$	A and B
$3x + 16y + 28z = 16$	A and C
$38x + 2y - 162z = 2$	A and D
$11x - 128y - 154z = -128$	A and E
$190x + 10y - 217z = 10$	A and F
$11x - 128y - 154z = -128$	A and G
$18x + 62y + 63z = 62$	B and C
$18x + 14y - 9z = 14$	B and D
$10x - 6y - 5z = -130$	B and E
$95x + 65y + 19z = 585$	B and F
$90x - 101y - 420z = -2110$	B and G
$54x + 106y + 117z = 106$	C and D
$28x - 48y - 28z = -192$	C and E
$8x + 7y + 4z = 62$	C and F
$30x - 127y - 420z = -257$	C and G
$11x - 6y - 5z = -130$	D and E
$30x + 2y + 13z = 10$	D and F
$30x - 127y - 840z = -2570$	D and G
$10x - 6y - 5z = -130$	E and F
$90x + 179y + 840z = 3490$	E and G
$90x - 101y - 420z = -2110$	F and G

first see that by Program 1, each of the boundary cycles of A, B, C, D, E, F, G can be projected to some plane as a convex polygon:

The projection B to the plane $y = 0$ is a convex polygon,

The projection of C to the plane $z = 0$ is a convex polygon,

The projection of D to the plane $z = 0$ is a convex polygon,

The projection of E to the plane $-\frac{1}{1000}x - \frac{5017}{1000}y + \frac{8675}{1000}z = 0$ is a convex polygon,

The projection of F to the plane $y = 0$ is a convex polygon.

Secondly, by taking some suitable plane F in Program 2, we can see that for any choice of two distinct faces, the convex hulls corresponding to them do not collide except their common points as in Table 1.

Therefore, our coordinates for X_2 can be verified to give an exhibition. The center of Fig. 10 shows three polygons corresponding to the convex-hulls of B, C, D determined by our coordinates, and the right shows the body obtained from the convex hulls corresponding to A, B, C, D, E, F . Finally, to the body, we can easily add the disk corresponding to G to get an exhibition of X_2 .

For other X_i 's, we can check that the following coordinates give an exhibition similarly to those for X_2 : We put movies of each exhibition of X_i in the web site [12]. $\square$

For X_0 ,	$v_1 = (12, -10, 10),$	$v_2 = (-10, 10, -6),$
$v_3 = (-16, -10, 8),$	$v_4 = (5, 10, -12),$	$v_5 = (0, 1, 0),$
$v_6 = (-1, 20, 0).$		
For X_1 ,	$v_1 = (12, -10, 10),$	$v_2 = (-10, 10, -6),$
$v'_2 = (-9, 10, -4),$	$v_3 = (-16, -10, 8),$	$v_4 = (5, 10, -12),$
$v_5 = (0, 1, 0),$	$v_6 = (-1, 20, 0).$	
For X_3 ,	$v_1 = (12, -10, 10),$	$v'_1 = (11, -10, 20),$
$v_2 = (-10, 10, -6),$	$v_3 = (-16, -10, 8),$	$v_4 = (5, 10, -12),$
$v_5 = (0, 1, 0),$	$v_6 = (-1, 20, 0).$	
For X_4 ,	$v_1 = (12, -10, 10),$	$v'_1 = (11, -10, 20),$
$v_2 = (-10, 10, -6),$	$v'_2 = (-9, 10, -4),$	$v_3 = (-16, -10, 8),$
$v_4 = (5, 10, -12),$	$v_5 = (0, 1, 0),$	$v_6 = (-1, 20, 0).$
For X_5 ,	$v_1 = (12, -10, 10),$	$v'_1 = (11, -10, 20),$
$v_2 = (-10, 10, -6),$	$v'_2 = (-9, 10, -4),$	$v_3 = (-16, -10, 8),$
$v_4 = (5, 10, -12),$	$v'_4 = (4, 10, -10),$	$v_5 = (0, 1, 0),$
$v_6 = (-1, 20, 0).$		
For X_6 ,	$v_1 = (12, -10, 10),$	$v'_1 = (11, -10, 20),$
$v_2 = (-10, 10, -6),$	$v_3 = (-16, -10, 8),$	$v'_3 = (-14, -10, 10),$
$v_4 = (5, 10, -12),$	$v_5 = (0, 1, 0),$	$v_6 = (-1, 20, 0).$
For X_7 ,	$v_1 = (12, -10, 10),$	$v'_1 = (11, -10, 20),$
$v_2 = (-10, 10, -6),$	$v'_2 = (-9, 10, -4),$	$v_3 = (-16, -10, 8),$
$v'_3 = (-14, -10, 10)$	$v_4 = (5, 10, -12),$	$v_5 = (0, 1, 0),$
$v_6 = (-1, 20, 0).$		
For X_8 ,	$v_1 = (12, -10, 10),$	$v'_1 = (11, -10, 20),$
$v_2 = (-10, 10, -6),$	$v'_2 = (-9, 10, -4),$	$v_3 = (-16, -10, 8),$
$v'_3 = (-14, -10, 10),$	$v_4 = (5, 10, -12),$	$v'_4 = (4, 10, -10),$
$v_5 = (0, 1, 0),$	$v_6 = (-1, 20, 0).$	

Lemma 17 *Let M be a Möbius triangulation with boundary cycle $v_1 v_2 v_3 v_4$ containing one of the submaps $X_0, X_1, \dots, X_8$. Unless M has an edge $v_1 v_3$ and $v_2 v_4$, then M has a geometric realization to which we can add a flat triangular disk $v_j v_{j+1} v_{j+2}$ with no intersection except their common points for each $j = 1, 2, 3, 4$.*

Fig. 11 Moving v slightly to get $\hat{M}$

Proof Suppose that M contains one of the submaps $X_0, \dots, X_8$, say X . By Lemma 15, each X_i has an exhibition ψ , and hence M has a geometric realization $\hat{M}$, where we take the coordinates of the vertices of X contained in $\hat{M}$ by ψ the same as in Lemma 15. Suppose that $X = X_0$ and we try to add a flat triangular disk $v_1v_3v_4$. Observe that the region, denoted by D , of X formed by two triangular faces $v_1v_5v_3$ and $v_1v_3v_6$ and a triangular disk $\Delta = v_1v_3v_4$ or $\Delta' = v_2v_3v_4$ added to it does not satisfy the assumption of Proposition 16. Let $\tilde{M}$ denote the map $M \cup \Delta$ or $M \cup \Delta'$.

We first consider Δ . We note that D and Δ do not share an edge v_1v_3 in $\tilde{M}$. In this case, since D has no chord v_1v_3 in M by the assumption, we construct D in a geometric realization of $\hat{M}$ as follows: Let P be a shortest path in D corresponding to an edge v_1v_3 of X . Then P must be chordless in M . (For otherwise, we can choose a shorter one.) Since D has no diagonal v_1v_3 , P has an inner vertex, say v , in M . Hence, by Lemma 13, M has a path P_1 from v to a vertex x on the path corresponding to v_1v_5 or v_3v_5 , and a path P_2 from v to a vertex y on the path corresponding to v_1v_6 or v_3v_6 , where each of x and y is distinct from v_1 and v_3 . Then D can be divided into four regions, say D_1, D_2, D_3, D_4 . Move a position of v in $\hat{M}$ slightly toward the interior of $\psi(D)$, where $\psi(D)$ is a tetrahedron in $\hat{M}$ corresponding to D . Note that a very small movement of v in $\mathbb{R}^3$ does not yield an intersection of faces in $\hat{M}$. Hence we can take an exhibition of each of the regions D_1, D_2, D_3 , and D_4 to get $\hat{M}$, and we can get a geometric realization of $\tilde{G}$. (For example, see Fig. 11.)

Now we secondly consider $\Delta' = v_1v_2v_4$. Let D' be the region of X bounded by $v_2v_5v_4v_6$. If D' has an internal path between v_2 and v_4 , then we can do similarly as in the previous case. Otherwise, by Lemma 13, D' has a chord uv where v_2, u, v_4 and v appear on the boundary of D' in this order. In this case, let D'_1 and D'_2 be the two regions obtained from D' by dividing along uv , and consider the exhibitions of D'_1 and D'_2 . Since our coordinates of D' makes a tetrahedron in $\hat{M}$, these two exhibitions of D'_1 and D'_2 share only a segment uv , and the exhibition of $D'_1 \cup D'_2$ does not intersect a straight segment in $\mathbb{R}^3$ between v_2 and v_4 . (For example, see Fig. 12.)

Consequently, we can construct a geometric realization $\hat{M}$ avoiding a segment between v_1 and v_3 (or between v_2 and v_4) except their ends. Hence we can add each of Δ (or Δ') to X without intersections of faces, which can be checked by Programs 1 and 2.

However, we cannot add each of the triangular disks $v_1v_2v_3$ and $v_1v_2v_4$ if we use the coordinates of each vertex of X_i as in Lemma 15. In this case, we use the

Fig. 12 An exhibition of D' and two exhibitions $D'_1 \cup D'_2$

following coordinates of X_i , each of which is basically obtained from those in the proof of Lemma 15 by replacing the coordinate of each boundary node v_j (resp. v'_j) for $j = 1, 2, 3, 4$, with that of v_{j+2} (resp. v'_{j+2}) and the coordinate of each inner node v_j , for $j = 5, 6$, with that of v_{j+1} . $\square$

For X_0 , $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$,
 $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$,
 $v_6 = (0, 1, 0)$.

For X_1 , $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$,
 $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$,
 $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_2 , $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$,
 $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$,
 $v'_4 = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_3 , $v_1 = (-16, -10, 8)$, $v'_1 = (-14, -10, 10)$,
 $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$,

$$\begin{aligned}
v_5 &= (-1, 20, 0), & v_6 &= (0, 1, 0). \\
\text{For } X_4, & & v_1 &= (-16, -10, 8), & v'_1 &= (-14, -10, 10), \\
v_2 &= (5, 10, -12), & v'_2 &= (4, 10, -10), & v_3 &= (12, -10, 10), \\
v_4 &= (-10, 10, -6), & v_5 &= (-1, 20, 0), & v_6 &= (0, 1, 0). \\
\text{For } X_5, & & v_1 &= (-16, -10, 8), & v'_1 &= (-14, -10, 10), \\
v_2 &= (5, 10, -12), & v'_2 &= (4, 10, -10), & v_3 &= (12, -10, 10), \\
v_4 &= (-10, 10, -6), & v'_4 &= (-9, 10, -4), & v_5 &= (-1, 20, 0), \\
v_6 &= (0, 1, 0). \\
\text{For } X_6, & & v_1 &= (-16, -10, 8), & v'_1 &= (-14, -10, 10), \\
v_2 &= (5, 10, -12), & v_3 &= (12, -10, 10), & v'_3 &= (11, -10, 20), \\
v_4 &= (-10, 10, -6), & v_5 &= (-1, 20, 0), & v_6 &= (0, 1, 0). \\
\text{For } X_7, & & v_1 &= (-16, -10, 8), & v'_1 &= (-14, -10, 10), \\
v_2 &= (5, 10, -12), & v'_2 &= (4, 10, -10), & v_3 &= (12, -10, 10), \\
v'_3 &= (11, -10, 20) & v_4 &= (-10, 10, -6), & v_5 &= (-1, 20, 0), \\
v_6 &= (0, 1, 0). \\
\text{For } X_8, & & v_1 &= (-16, -10, 8), & v'_1 &= (-14, -10, 10), \\
v_2 &= (5, 10, -12), & v'_2 &= (4, 10, -10), & v_3 &= (12, -10, 10), \\
v'_3 &= (11, -10, 20), & v_4 &= (-10, 10, -6), & v'_4 &= (-9, 10, -4), \\
v_5 &= (-1, 20, 0), & v_6 &= (0, 1, 0).
\end{aligned}$$

Lemma 18 *Let M be a Möbius triangulation with boundary cycle $v_1v_2v_3v_4$ containing one of the submaps $X_0, \dots, X_8$. Let $\hat{M}_f$ be a geometric realization of $M \cup f$ constructed by the coordinates in Lemma 15 or Lemma 17, where $f = v_jv_{j+1}v_{j+2}$ is a triangular disk for some $j \in \{1, 2, 3, 4\}$. Then,*

- (i) *for some point p in $\mathbb{R}^3$, we can add two triangular disks pv_jv_{j+2} and $pv_{i+2}v_{i+3}$ to the body of $\hat{M}_f$, and*
- (ii) *for some points q and r in $\mathbb{R}^3$, we can add a tetrahedron $qrv_{i+2}v_{i+3}$, two triangular disks qv_iv_{i+3} and rv_iv_{i+2} to the body of $\hat{M}_f$.*

Proof If $f = v_1v_2v_3$ or $v_1v_3v_4$, then we put $p = (0, 0, -40)$, $q = (-10, 10, -6 - \frac{1}{10})$, and $r = (0, 0, -40)$. If $f = v_2v_3v_4$ or $v_1v_2v_4$, then we put $p = (0, 0, -40)$, $q = (11, -10 + \frac{1}{10}, 8)$, and $r = (0, 0, -40)$. Programs 1 and 2 verify Lemma 18. $\square$

5 Proof of the Theorem and Concluding Remarks

Proof of Theorem 3 Let G be a projective triangulation, and let f be a face of G . If G has a nesting 3-cycle of f , then by Fact 1, G has no geometric realization, and so the necessity holds.

So we consider the sufficiency. Suppose that f has no nesting 3-cycle in G . Then G has a 4-cycle $C = v_1 v_2 v_3 v_4$ satisfying (i), (ii), and (iii) of Lemma 9. Let G_M (resp., G_D) be a Möbius triangulation (resp., a near triangulation) with the boundary cycle C . Since C satisfies the condition (ii) of Lemma 9, G_D has a diagonal $v_i v_{i+2}$, and the interior of $v_i v_{i+1} v_{i+2}$ contains f . For a simple notation, let R and R' denote the 2-cell regions of G bounded by $v_i v_{i+1} v_{i+2}$ and $v_i v_{i+2} v_{i+3}$, respectively. By Lemmas 15 and 17, $G_M \cup R'$ has a geometric realization.

If $f = R$, then we are done, and hence we suppose $f \neq R$. If $E(f) \cup E(C) \neq \emptyset$, then R has an inner vertex p with $p \in V(f)$. In this case, there exists a path from p to v_{i+2} but disjoint from v_i and v_{i+1} . If $E(f) \cup E(C) = \emptyset$, then R has two inner vertices q and r with $q, r \in V(f)$. In this case, there exist two disjoint paths, say P_1 and P_2 , from q and r to v_{i+1} and v_{i+2} , not intersecting v_i . Without loss of generality, we may suppose that P_1 from q to v_{i+1} and P_2 is from r to v_{i+2} . Therefore, by Lemma 18, we can add all faces in R except for f to the body of a geometric realization of $G_M \cup R'$ by taking suitable coordinates of p or q and r in $\mathbb{R}^3$. Therefore, $G - f$ has a geometric realization. $\square$

Historically there are many papers concerning geometric realizations of orientable maps. For example, triangulations on the sphere and the torus has a geometric realization [1, 11]. Moreover, there exists a triangulation on the orientable surfaces of genus $g \geq 5$ with no geometric realization [3, 10]. However there are few papers concerning geometric realizations of nonorientable maps. We did not know the reason, but it might be by the Brehm's example, or because a nonorientable closed surface is not embeddable in $\mathbb{R}^3$.

Theorem 2 asserts that in a given projective triangulation G , if we choose a face f carefully, then $G - f$ has a geometric realization. However, this theorem never describes a structure of geometrically realizable Möbius triangulations at all. In order to answer this question, Chávez, Fijavž, Márquez, Nakamoto, and Suárez [5] showed a result that a Möbius triangulation is geometrically realizable if it is a subgraph of a 5-connected projective triangulation. In proving the result, they used a technical argument for finding a K_6 -minor with a good property in a projective triangulation G with any face f specified. However, this did not reach a complete characterization of a face f with $G - f$ geometrically realizable. The current result (Theorem 3) completely solves the problem on geometric realization of a projective triangulation with one face removed, which has been considered in the two papers [2, 5].

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