A Counterexample to Wegner's Conjecture on Good Covers

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Abstract In 1975 Wegner conjectured that the nerve of every finite good cover in $\mathbb{R}^d$ is d -collapsible. We disprove this conjecture.

A good cover is a collection of open sets in $\mathbb{R}^d$ such that the intersection of every subcollection is either empty or homeomorphic to an open d -ball. A simplicial complex is d -collapsible if it can be reduced to an empty complex by repeatedly removing a face of dimension at most $d - 1$ which is contained in a unique maximal face.

Keywords d -collapsibility · Good cover · Convex set · Simplicial complex

1 Introduction

In 1975 Wegner [9] introduced d -collapsible simplicial complexes. His definition comes from studying intersection patterns of convex sets. He proved that simplicial complexes coming from finite collections of convex sets (as their nerves) are d -collapsible. He also conjectured that his result has a topological extension when collections of convex sets are replaced by good covers. The purpose of this article is to disprove this conjecture.

We assume that the reader is familiar with simplicial complexes; otherwise we refer to introductory chapters of books like [1, 5, 7].

Good Covers A d -cell is an open subset of $\mathbb{R}^d$ homeomorphic to a ball. A *good cover* in $\mathbb{R}^d$ is a collection $\mathcal{G}$ of d -cells in $\mathbb{R}^d$ such that for every subcollection $\mathcal{G}' \subseteq \mathcal{G}$,

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the intersection of the sets in $\mathcal{G}'$ is either a d -cell or empty.¹ We remark that a collection of open convex sets in $\mathbb{R}^d$ always forms a good cover.

d-representable Complexes The *nerve* of a collection $\mathcal{F}$ of sets in $\mathbb{R}^d$ is the simplicial complex whose vertices are the sets in $\mathcal{F}$ and whose faces are collections of vertices with a nonempty intersection. A finite simplicial complex is *convexly d-representable* if it is isomorphic to a collection of convex sets in $\mathbb{R}^d$; it is *topologically d-representable* if it is isomorphic to a good cover in $\mathbb{R}^d$. This notion usually appears in the literature under the name of *d-representable* instead of convexly *d-representable*. However, we add “convexly” in order to clearly distinguish convex sets and good covers.

d-collapsible Complexes Let $\mathbf{K}$ be a simplicial complex. Assume that σ is a face of $\mathbf{K}$ of dimension at most $d - 1$ such that there is only one maximal face of $\mathbf{K}$ containing σ . Then we say that σ is *d-collapsible* and that the complex

$$\mathbf{K}' := \mathbf{K} \setminus \{\eta \in \mathbf{K} : \sigma \subseteq \eta\}$$

arises from $\mathbf{K}$ by an *elementary d-collapse*. A simplicial complex $\mathbf{K}$ is *d-collapsible* if it can be reduced to an empty complex by a sequence of elementary *d*-collapses.

The reader might be familiar with classical collapsibility of Whitehead. We warn the reader that the notion of *d-collapsibility* behaves differently in some respects. On one hand, Whitehead’s collapses do not have a dimension restriction on the collapsible face; on the other hand, they require that the collapsible face is not inclusionwise maximal. Whitehead’s collapsibility has no impact on *d-collapsibility* since the barycentric subdivision of a *d*-simplex is Whitehead-collapsible but not $(d - 1)$ -collapsible. For the reverse direction, a *d-collapsible* complex Whitehead-collapses to a complex of dimension at most $(d - 1)$; see [8, Lemma 5.2] for more details.

Wegner [9] proved that every convexly *d-representable* simplicial complex is *d-collapsible*. He also conjectured that his result has the following topological extension.

Conjecture 1.1 [9] *Every topologically d-representable simplicial complex is d-collapsible.*

We disprove this conjecture for every $d \geq 2$. We start with constructing a simplicial complex $\mathbf{L}$ which is topologically 2-representable but not 2-collapsible. In higher dimensions we obtain a counterexample by using suspensions of the complex $\mathbf{L}$.

Theorem 1.2 *For every $d \geq 2$, there is a simplicial complex which is topologically d-representable but not d-collapsible.*

¹ Different notions of a good cover appear in the literature. Sometimes the cells are assumed to be compact instead of open. The intersection of $\mathcal{G}'$ can be also assumed contractible instead of being a *d*-cell. For our purposes, there is not a big difference while constructing a counterexample. It can be easily modified to fulfill the above-mentioned conditions.

Additional Background Conjecture 1.1 is a natural question in the context of Helly-type theorems. For example, the Helly theorem [2] can be formulated in such a way that a convexly d -representable simplicial complex containing all d -faces has to be already a full simplex. Once we know that convexly d -representable simplicial complexes are d -collapsible, it is a simple consideration to prove the Helly theorem.² In this case, the Helly theorem has a topological extension for topologically d -representable simplicial complexes [3]. Many Helly-type theorems have a similar topological extension; see, e.g., the introduction to [6] for a more detailed list. However, the conclusion for d -collapsible complexes may be stronger than for topologically d -representable complexes, e.g., in [4]. Our counterexample thus shows that results for d -collapsible complexes cannot be generalized all at once for topologically d -representable complexes.

2 Planar Case

We start this section by describing the complex $\mathbf{L}$. Let $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, D, X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3$ be the (open) sets from Fig. 1. We also set $\mathcal{A} := \{A_1, A_2, A_3\}$, $\mathcal{B} := \{B_1, B_2, B_3\}$, $\mathcal{C} := \{C_1, C_2, C_3\}$, $\mathcal{D} := \{D\}$, $\mathcal{X} := \{X_1, X_2, X_3\}$, $\mathcal{Y} := \{Y_1, Y_2, Y_3\}$, and $\mathcal{Z} := \{Z_1, Z_2, Z_3\}$. Let $\mathcal{L}$ be the collection of all of these sets, i.e., $\mathcal{L} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. Finally, $\mathbf{L}$ is the nerve of $\mathcal{L}$.

We will show that $\mathbf{L}$ is topologically 2-representable but not 2-collapsible.

Before doing so, we explain how $\mathbf{L}$ is constructed. We hope that this explanation sheds some light on the properties of $\mathbf{L}$, although it is not necessary for the proof that we offer.

Let us split the collection $\mathcal{L}$ into two parts $\mathcal{L}^+ := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ and $\mathcal{L}^- := \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. The nerve of $\mathcal{L}^+$, resp. $\mathcal{L}^-$, is denoted by $\mathbf{L}^+$, resp. $\mathbf{L}^-$. Both $\mathbf{L}^+$ and $\mathbf{L}^-$ are triangulations of a disc with only three boundary edges $\{A_1, A_2\}$, $\{A_1, A_3\}$, and $\{A_2, A_3\}$, resp. $\{Z_1, Z_2\}$, $\{Z_1, Z_3\}$, and $\{Z_2, Z_3\}$; see Fig. 2. These boundary faces are the only 2-collapsible faces of $\mathbf{L}^+$, resp. $\mathbf{L}^-$.

By suitably overlapping $\mathcal{L}^+$ and $\mathcal{L}^-$ (i.e., so that we obtain $\mathcal{L}$) we also get that the above-mentioned boundary faces are not 2-collapsible anymore (in the whole $\mathbf{L}$). For instance, $Z_1 \cap Z_2$ intersects A_1 (in addition to X_1 , which is already in $\mathcal{L}^-$); however, A_1 and X_1 are disjoint. Thus $\{Z_1, Z_2\}$ is not a 2-collapsible face of $\mathcal{L}$.

It is necessary to check that merging $\mathcal{L}^+$ and $\mathcal{L}^-$ does not introduce any new problems. This is, in fact, checked in detail in Sect. 2.2. We just mention that there is no problem with the 1-faces that already appear in $\mathbf{L}^+$ or $\mathbf{L}^-$. However, new 1-faces are introduced when one vertex comes from $\mathbf{L}^+$ and the second one from $\mathbf{L}^-$. It would be perhaps possible to show that the complex is not 2-collapsible even if the

²There is also a simple geometric proof of the Helly theorem, of course.

Fig. 1 The sets $A_1, \dots, Z_3$. We rather supply a more detailed description of the sets if this picture is printed only in black and white: The sets A_* are the *ovals* on the boundary; B_* are the *small discs* close to the boundary; C_* are the *trapezoid-shaped* sets; D is the *triangle* in the center; X_* are the *circles* close to the center; Y_* are the *deltoid-shaped* sets; and Z_* are the *nonconvex* sets. (See also Fig. 2)

newly introduced faces were 2-collapsible (for more general choice of $\mathbf{L}^+$ and $\mathbf{L}^-$). However, the case analysis proof that will follow seems to be simpler for the current purpose.

2.1 Topological Representability

It is sufficient to show that $\mathcal{L}$ is a good cover. This property can be hand-checked; however, we offer an alternative approach.

First we realize that all sets of $\mathcal{L} \setminus \mathcal{Z}$ are convex. Thus $\mathcal{L} \setminus \mathcal{Z}$ is a good cover. It remains to check that adding sets of $\mathcal{Z}$ does not violate this property.

Let $Z \in \mathcal{Z}$, and let $\mathcal{L}^Z := \{L \cap Z : L \in \mathcal{L}\}$. We are done as soon as we show that $\mathcal{L}^{Z_1}$, $\mathcal{L}^{Z_2}$, and $\mathcal{L}^{Z_3}$ are good covers.

There is an obvious symmetry of $\mathcal{L}$. Namely the rotation by $2\pi/3$ around the center of D maps M_i to $M_{i+1} \pmod{3}$, where M stands for A , B , C , X , Y , or Z . Because of this symmetry, we show only that $\mathcal{L}^{Z_1}$ is a good cover. The sets of $\mathcal{L}^{Z_1}$

Fig. 2 The complexes $\mathbf{L}^+$ and $\mathbf{L}^-$

can be transformed into convex sets by a homeomorphism of $\mathbb{R}^2$. See Fig. 3. Thus they form a good cover.

2.2 Noncollapsibility by Case Analysis

Here we prove by case analysis that $\mathbf{L}$ is not 2-collapsible. The result we get is a bit stronger; that will help us for higher dimensions.

For a simplicial complex $\mathbf{K}$, we set

$$\gamma_0(\mathbf{K}) := \min\{d : \mathbf{K} \text{ has a } d\text{-collapsible face}\}.$$

The fact that $\mathbf{L}$ is not 2-collapsible is implied by the following proposition.

Fig. 3 A transformation of $\mathcal{L}^{Z_1}$. Whatever is outside of Z_1 can be ignored

Proposition 2.1 $\gamma_0(\mathbf{L}) = 3$.

In order to prove the proposition, we need a simple lemma.

Lemma 2.2 *Let $\mathbf{K}$ be a simplicial complex, and σ be a 1-face (edge) of it. Assume that u and v are vertices of $\mathbf{K}$ not belonging to σ such that $\sigma \cup \{u\} \in \mathbf{K}$, $\sigma \cup \{v\} \in \mathbf{K}$, but $\sigma \cup \{u, v\} \notin \mathbf{K}$. Then σ is not a 2-collapsible face of $\mathbf{K}$.*

Proof If τ is a unique maximal face of $\mathbf{K}$ containing σ , then $u, v \in \tau$ due to the conditions of the lemma. However, $\sigma \cup \{u, v\} \notin \mathbf{K}$. $\square$

Proof of Proposition 2.1 In the spirit of Lemma 2.2, for every 1-face $\sigma \in \mathbf{L}$, we find a pair of vertices $u, v \in \mathbf{L}$ such that $\sigma \cup \{u\}, \sigma \cup \{v\} \in \mathbf{L}$, but $\sigma \cup \{u, v\} \notin \mathbf{L}$. It is sufficient to check 1-faces since if a 0-face (vertex) w is 1-collapsible, then any 1-face containing w is 1-collapsible as well. Moreover, it is sufficient to check only some 1-faces because of the symmetry of the complex. The rest of the proof is given by the following table.

σ	u, v	σ	u, v	σ	u, v
$\{A_1, A_2\}$	B_2, Z_2	$\{A_1, B_1\}$	C_1, A_3	$\{A_1, C_1\}$	B_1, B_2
$\{A_1, Y_1\}$	B_1, Z_1	$\{A_1, Z_1\}$	C_1, A_3	$\{B_1, C_1\}$	A_1, C_3
$\{B_1, Y_1\}$	C_1, A_3	$\{C_1, C_2\}$	B_2, D	$\{C_1, D\}$	C_2, C_3
$\{C_1, X_1\}$	Y_1, Y_2	$\{C_1, Y_1\}$	B_1, Z_1	$\{C_1, Z_1\}$	Y_1, Z_2
$\{D, X_1\}$	Y_1, Y_2	$\{D, Y_1\}$	C_1, X_3	$\{X_1, X_2\}$	Y_2, X_3
$\{X_1, Y_1\}$	D, Z_1	$\{X_1, Z_1\}$	Y_1, Z_2	$\{Y_1, Z_1\}$	C_1, A_3
$\{Z_1, Z_2\}$	A_1, X_1				

□

3 Higher Dimensions

Joins of simplicial complexes will help us to generalize the counterexample to higher dimensions. Let $\mathbf{K}$ and $\mathbf{K}'$ be simplicial complexes with the vertex sets $V(\mathbf{K})$ and $V(\mathbf{K}')$. Their *join* is a simplicial complex $\mathbf{K} \star \mathbf{K}'$ whose vertex set is the disjoint union³ $V(\mathbf{K}) \sqcup V(\mathbf{K}')$ and whose set of faces is $\{\alpha \sqcup \beta : \alpha \in \mathbf{K}, \beta \in \mathbf{K}'\}$.

We need the following two lemmas.

Lemma 3.1 [6, Lemma 4.2] *For every pair of simplicial complexes $\mathbf{K}, \mathbf{K}'$, we have $\gamma_0(\mathbf{K} \star \mathbf{K}') = \gamma_0(\mathbf{K}) + \gamma_0(\mathbf{K}')$.*

Lemma 3.2 *Let $\mathbf{K}$ be a convexly (resp. topologically) d -representable complex, and $\mathbf{K}'$ be a convexly (resp. topologically) d' -representable complex. Then $\mathbf{K} \star \mathbf{K}'$ is a convexly (resp. topologically) $(d + d')$ -representable complex.*

Proof Let $\mathcal{F}$ be a collection of convex sets (resp. good cover) in $\mathbb{R}^d$ such that $\mathbf{K}$ is isomorphic to the nerve of $\mathcal{F}$. Similarly, $\mathcal{F}'$ is a suitable collection in $\mathbb{R}^{d'}$ such that $\mathbf{K}'$ is isomorphic to the nerve of $\mathcal{F}'$. Without loss of generality we can assume that the vertices of $\mathbf{K}$ (resp. $\mathbf{K}'$) are the sets of $\mathcal{F}$ (resp. $\mathcal{F}'$).

Let us set

$$\mathcal{F} \star \mathcal{F}' := \{F \times \mathbb{R}^{d'} : F \in \mathcal{F}\} \cup \{\mathbb{R}^d \times F' : F' \in \mathcal{F}'\}.$$

Then it is easy to check that $\mathbf{K} \star \mathbf{K}'$ is isomorphic to the nerve of $\mathcal{F} \star \mathcal{F}'$ (via the isomorphism $i: F \rightarrow F \times \mathbb{R}^{d'}$ for $F \in \mathcal{F}$ and $i: F' \rightarrow \mathbb{R}^d \times F'$ for $F' \in \mathcal{F}'$). Moreover, $\mathcal{F} \star \mathcal{F}'$ is a collection of convex sets (resp. good cover) in $\mathbb{R}^{d+d'}$. This is trivial for the case of convex sets; regarding good covers for $F_1, \dots, F_k \in \mathcal{F}$ and $F'_1, \dots, F'_{k'} \in \mathcal{F}'$, we have

$$i(F_1) \cap \dots \cap i(F_k) \cap i(F'_1) \cap \dots \cap i(F'_{k'}) = (F_1 \cap \dots \cap F_k) \times (F'_1 \cap \dots \cap F'_{k'}),$$

which is homeomorphic to an open $(d + d')$ -ball or empty. ($F_1 \cap \dots \cap F_0$ stands for $\mathbb{R}^d$, and similarly $F'_1 \cap \dots \cap F'_0$ stands for $\mathbb{R}^{d'}$.) □

³If A and B are sets with $A \cap B \neq \emptyset$, then their disjoint union can be defined as $A \sqcup B := A \times \{1\} \cup B \times \{2\}$.

Now we can finish the proof of our main result.

Proof of Theorem 1.2 Let $\mathbf{T}$ be the simplicial complex consisting of two isolated points. The complex $\mathbf{T}$ is topologically 1-representable, and $\gamma_0(\mathbf{T}) = 1$. Let us set

$$\mathbf{J} = \mathbf{L} \star \underbrace{\mathbf{T} \star \cdots \star \mathbf{T}}_{d-2}.$$

In topology, the complex $\mathbf{J}$ would be called $(d - 2)$ -tuple *suspension* of $\mathbf{L}$. (Recall that $\mathbf{L}$ was constructed in Sect. 2.) Then $\gamma_0(\mathbf{J}) = d + 1$ due to Proposition 2.1 and Lemma 3.1. On the other hand, $\mathbf{J}$ is topologically d -representable due to Lemma 3.2. $\square$

4 Conclusion

In the spirit of Helly-type theorems we could ask whether there is at least some weaker bound for collapsibility of topologically d -representable complexes.

Question 4.1 For which $d \geq 2$ is there a $d' \in \mathbb{N}$ (as least as possible) such that every topologically d -representable complex is d' -collapsible?

Using joins of multiple copies of $\mathbf{L}$ (instead of suspensions of $\mathbf{L}$), we obtain the following bound.

Proposition 4.2 *For every $d \geq 2$, there is a simplicial complex which is topologically $2d$ -representable but not $(3d - 1)$ -collapsible.*

Proof Consider the complex $\underbrace{\mathbf{L} \star \cdots \star \mathbf{L}}_d$. $\square$

If there is a wider gap between these notions, it will also reflect the gap between d -representable and d -Leray complexes obtained (with a similar method) by Matoušek and the author [6].

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