Straight-Line Rectangular Drawings of Clustered Graphs

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Abstract We show that every c -planar clustered graph has a straight-line c -planar drawing in which each cluster is represented by an axis-parallel rectangle, thus solving a problem posed by Eades, Feng, Lin, and Nagamochi (Algorithmica 44(1):1–32, 2006).

Keywords Clustered graphs · Planar graphs · Straight-line drawings · Rectangles

1 Introduction

A *clustered graph* is a pair $C(G, T)$, where G is a graph, called *underlying graph*, and T is a rooted tree, called *inclusion tree*, such that the leaves of T are the vertices of G . Each internal node v of T corresponds to the subset of vertices of G , called *cluster*, that are the leaves of the subtree of T rooted at v .

Clustered graphs are widely used in applications where it is needed at the same time to represent relationships between entities and to group entities with semantic affinities. For example, in the Internet network, links among routers give rise to a graph; geographically close routers are grouped into areas, which in turn are grouped into Autonomous Systems.

Visualizing clustered graphs turns out to be a difficult problem, due to the simultaneous need for a readable drawing of the underlying structure and for a good

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rendering of the recursive clustering relationship. As for the visualization of graphs, the most important aesthetic criterion for a drawing of a clustered graph to be “nice” is commonly regarded to be the *planarity*, which however needs a refinement in the context of clustered graphs, in order to deal also with the clustering structure.

A *drawing* Γ of a clustered graph $C(G, T)$ consists of a drawing of G (each vertex is a point in the plane and each edge is as Jordan curve between its end-vertices) and of a representation of each node μ of T as a simple closed region containing all and only the vertices that belong to μ . A drawing is *c-planar* if it has no edge crossings (i.e., the drawing of the underlying graph is planar), no edge-region crossings (i.e., an edge intersects the boundary of a cluster at most once), and no region-region crossings (i.e., no two cluster boundaries cross).

Given a clustered graph, testing whether it admits a *c-planar* drawing is a problem of unknown complexity, perhaps the most studied problem in the Graph Drawing community during the last ten years [1–5, 8, 15, 17–22].

Suppose that a *c-planar* clustered graph C is given together with a *c-planar* embedding, that is, together with an equivalence class of *c-planar* drawings of C , where two *c-planar* drawings are equivalent if they have the same order of the edges incident to each vertex and the same order of the edges incident to each cluster. How can the graph be drawn? Such a problem has been intensively studied in the literature and a number of papers have been presented for constructing *c-planar* drawings of *c-planar* clustered graphs within many drawing conventions.

Eades et al. show in [12] how to construct $O(n^2)$ -area *c-planar* orthogonal drawings and $O(n^2)$ -area *c-planar* poly-line drawings of *c-planar* clustered graphs with clusters drawn as axis-parallel rectangles. Di Battista et al. [6] show algorithms and bounds for constructing small-area drawings of *c-planar* clustered trees within several drawing styles. The strongest result in the area is perhaps the one that Eades et al. present in [11]. Namely, the authors show an algorithm for constructing *c-planar* straight-line drawings of *c-planar* clustered graphs in which each cluster is drawn as a convex region (see also a paper of Nagamochi and Kuroya [25]). Such an algorithm requires, in general, exponential area. However, in [16] Feng et al. have shown that such a bound is asymptotically optimal in the worst case.

In this paper we address a problem posed by Eades et al. in [11, 13, 15], namely whether every *c-planar* clustered graph has a *straight-line rectangular drawing*, i.e., a *c-planar* straight-line drawing in which each cluster is drawn as an axis-parallel rectangle. An example of a straight-line rectangular drawing of a *c-planar* clustered graph is given in Fig. 1. Eades et al. observe how pleasant and readable straight-line rectangular drawings are; however, they provide evidence that their algorithm [11] for constructing *c-planar* straight-line convex drawings of clustered graphs cannot be modified in order to get rectangular clusters without introducing edge-region crossings.

We show that every *c-planar* clustered graph has a straight-line rectangular drawing. We obtain such a result as a corollary of a stronger theorem stating that a straight-line rectangular drawing of a *c-planar* clustered graph exists for an arbitrary *convex-separated* drawing of its outer face, that is, a drawing satisfying some properties of convexity and of visibility among vertices and clusters.

Such a stronger result is proved by means of an inductive algorithm reminiscent of Fáry’s drawing algorithm for planar graphs [14]. Namely, the algorithm consists

Fig. 1 A straight-line rectangular drawing of a c -planar clustered graph

of three inductive cases. Each case considers a clustered graph C and performs an operation (removal of a cluster, split of the graph in correspondence of a separating 3-cycle, or contraction of an edge) turning C into a smaller clustered graph C' for which a straight-line rectangular drawing can be inductively constructed. Then, such a drawing can be easily augmented to a straight-line rectangular drawing of C .

When none of the three inductive cases applies, every cluster contains a vertex incident to the outer face. We call *outerclustered graph* a clustered graph satisfying this property. We prove that every outerclustered graph admits a straight-line rectangular drawing even if a convex-separated drawing of its outer face is arbitrarily fixed, thus providing a base case for the above inductive algorithm for general clustered graphs. In order to draw an outerclustered graph C , we split it into three *linearly-ordered outerclustered graphs* (an even more restricted family of clustered graphs), we separately draw such graphs, and we compose the obtained drawings to get a drawing of C .

A linearly-ordered outerclustered graph $C(G, T)$ is an outerclustered graph in which the inclusion tree T is such that removing its leaves (corresponding to the vertices of G) and their incident edges yields a path. A drawing algorithm is provided for constructing a straight-line rectangular drawing of any linearly-ordered outerclustered graph $C(G, T)$ for an arbitrary convex-separated drawing of its outer face. Such an inductive algorithm splits $C(G, T)$ into smaller linearly-ordered outerclustered graphs and combines the inductively-constructed drawings of such subgraphs to obtain a drawing of $C(G, T)$.

The rest of the paper is organized as follows. In Sect. 2 we introduce some preliminaries and definitions; in Sect. 3 we show a drawing algorithm for linearly-ordered outerclustered graphs; in Sect. 4 we show a drawing algorithm for outerclustered graphs; in Sect. 5 we show a drawing algorithm for general clustered graphs; finally, in Sect. 6 we conclude and present some open problems.

2 Preliminaries

We assume familiarity with graphs, clustered graphs and their drawings (see, e.g., [7, 24]).

A graph G is k -connected if removing any $k - 1$ vertices leaves G connected; 3-connected and 2-connected graphs are also called *triconnected* and *biconnected* graphs, respectively.

Let $C(G, T)$ be a clustered graph. An edge (u, v) of G is *incident to a cluster* μ of T if u belongs to μ and v does not belong to μ . We denote by $\sigma(u_1, u_2, \dots, u_k)$ the *smallest cluster* of T containing vertices $u_1, u_2, \dots, u_k$ of G , i.e., the node of T containing all vertices $u_1, u_2, \dots, u_k$ and such that none of its children in T , if any, contains all vertices $u_1, u_2, \dots, u_k$. A cluster is *minimal* if it contains no other clusters. A cluster μ is an *ancestor* (*descendant*) of a cluster ν if μ is an ancestor (descendant) of ν in the inclusion tree T . A clustered graph $C(G, T)$ is *c-connected* if each cluster induces a connected subgraph of G .

Throughout the paper, we allow a clustered graph $C(G, T)$ to have several clusters containing exactly the same set of vertices of G . Then, a set of clusters all containing the same set of vertices of G induces a path in T .

A *drawing* Γ of a clustered graph $C(G, T)$ consists of a drawing of G (each vertex is represented by a point in the plane and each edge is represented by a Jordan curve between its end-vertices) and of a representation of each node μ of T as a simple closed region that contains all and only the vertices that belong to μ . If two clusters μ and ν contain the same set of vertices and ν is an ancestor of μ , then the region representing ν encloses the region representing μ . In the following, when we say “cluster”, we refer both to the set of vertices contained in the cluster and to the region representing the cluster in a drawing, the meaning of the word being clear from the context. A drawing Γ has an *edge crossing* if two edges of G cross; Γ has an *edge-region crossing* if an edge crosses the boundary of a cluster more than once; Γ has a *region-region crossing* if the boundaries of two clusters cross. A drawing of a clustered graph is *c-planar* if it does not have edge crossings, edge-region crossings, and region-region crossings. A graph is *c-planar* if it has a *c-planar* drawing. In this paper we are interested in *straight-line rectangular drawings* of clustered graphs, i.e., *c-planar* drawings such that each edge is represented by a straight-line segment and each cluster is represented by an axis-parallel rectangle. From now on, even when not explicitly specified, “clustered graph” will always mean *c-planar* clustered graph, while “drawing” will always mean straight-line rectangular drawing.

A clustered graph $C(G, T)$ is *maximal* if G is a maximal planar graph. In order to prove that every clustered graph admits a straight-line rectangular drawing, it suffices to prove that every maximal clustered graph admits a straight-line rectangular drawing. In fact, every non-maximal *c-planar* clustered graph $C(G, T)$ can be augmented in linear time to a maximal *c-planar* clustered graph $C'(G', T)$ by adding dummy edges to G [23]. Notice that every maximal *c-planar* clustered graph is *c-connected*. Namely, Feng et al. proved in [17] that a clustered graph $C(G, T)$ is *c-planar* if and only if edges can be added to G so that the resulting clustered graph $C'(G', T)$ is *c-planar* and *c-connected*. As no edge can be added to a maximal *c-planar* clustered graph without losing *c-planarity*, every maximal *c-planar* clustered graph is *c-connected*. From now on, we will always assume that the embedding (that is, the order of the edges incident to each vertex) and the outer face of any considered graph G is fixed in advance (hence, G has to be meant as a *plane graph*). We denote by $o(G)$ the outer face of G . A clustered graph $C(G, T)$ is *internally-triangulated* if

Fig. 2 (a) A biconnected internally-triangulated clustered graph. (b) A biconnected internally-triangulated outerclustered graph

every internal face of G is delimited by a 3-cycle. Figure 2(a) shows a biconnected internally-triangulated clustered graph.

Let $C(G, T)$ be a clustered graph. Let f be any face of G . Denote by $C_f(G_f, T_f)$ the clustered graph whose underlying graph G_f is the cycle induced by the vertices incident to f and whose inclusion tree T_f is the subtree of T induced by the clusters containing vertices incident to f . In particular, the *outer face* of $C(G, T)$ is the clustered graph $C_{o(G)}(G_{o(G)}, T_{o(G)})$, that is simply denoted by C_o in the following. In Sects. 3, 4, and 5, we will prove that a drawing of a clustered graph can be constructed for an arbitrary drawing of its outer face that satisfies some geometric properties to be described below. Then, we say that a straight-line rectangular drawing $\Gamma(C)$ of C *completes* a straight-line rectangular drawing $\Gamma(C_o)$ of C_o if the part of $\Gamma(C)$ representing C_o coincides with $\Gamma(C_o)$.

In order to prove that every clustered graph admits a straight-line rectangular drawing, it will be useful to study the class of clustered graphs that is defined below (see Fig. 2(b)).

Definition 1 A c -planar clustered graph $C(G, T)$ is an outerclustered graph if:

- O1: every cluster contains at least one vertex incident to $o(G)$;
- O2: the boundary of every cluster μ that does not contain all the vertices incident to $o(G)$ intersects $o(G)$ exactly twice, that is, it intersects exactly two edges $e_1(\mu)$ and $e_2(\mu)$ incident to $o(G)$; and
- O3: every edge (u, v) with $\sigma(u) = \sigma(v)$ is incident to $o(G)$.

Throughout the paper, for every cluster μ of a clustered graph $C(G, T)$ such that the boundary of μ intersects $o(G)$ exactly twice, we denote by $e_1(\mu)$ and $e_2(\mu)$ the edges incident to $o(G)$ that are intersected by the boundary of μ .

A *maximal outerclustered graph* is an outerclustered graph $C(G, T)$ such that G is a maximal plane graph. Observe that the outer face of the underlying graph of a maximal clustered graph is delimited by a 3-cycle.

Let $C(G, T)$ be a biconnected internally-triangulated outerclustered graph and let $\mathcal{C}$ be any simple cycle in G such that the boundary of every cluster in T containing some but not all the vertices of $\mathcal{C}$ intersects $\mathcal{C}$ exactly twice. Denote by $C'(G', T')$ the clustered graph such that G' is the subgraph of G induced by the vertices incident to and internal to $\mathcal{C}$, and such that T' is the subtree of T induced by the clusters containing vertices of G' .

Lemma 1 $C'(G', T')$ is a biconnected internally-triangulated outerclustered graph.

Proof Since G is biconnected and internally-triangulated, G' is biconnected and internally-triangulated, as well. We prove that C' satisfies Property O1 of Definition 1. Suppose that there exists a cluster μ in T' that does not contain any vertex incident to $\partial(G')$. Then, μ contains a vertex internal to G' , otherwise it would not be a cluster in T' . Also, μ contains a vertex incident to $\partial(G)$, otherwise C would not be an outerclustered graph. Since the boundary of μ is a closed curve containing a vertex inside $\mathcal{C}$ and a vertex outside $\mathcal{C}$, then either μ contains a vertex of $\mathcal{C}$ or it intersects twice the same edge of $\mathcal{C}$, in both cases contradicting the c -planarity of C . Clustered graph C' satisfies Property O2 by hypothesis. We prove that C' satisfies Property O3. Suppose that there exists an edge (u', v') such that $\sigma(u') = \sigma(v')$ and u' is an internal vertex of G' . Then, u' is an internal vertex of G , as well, and C is not an outerclustered graph, a contradiction. $\square$

An interesting subclass of the outerclustered graphs is considered in the following (see Fig. 3).

Fig. 3 An outerclustered graph that is linearly-ordered according to the cluster sequence $\mu_1, \mu_2, \dots, \mu_{12}$. Notice that $h = 6$

Definition 2 A biconnected internally-triangulated outerclustered graph $C(G, T)$ is linearly-ordered if there exists a sequence $\mu_1, \mu_2, \dots, \mu_k$ of clusters in T and an index $1 \leq h \leq k$, such that:

- LO1: for each vertex v_j of G , $\sigma(v_j) = \mu_i$, for some $1 \leq i \leq k$;
- LO2: let v_i and v_j be any two vertices incident to $o(G)$ such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$; then $o(G)$ is delimited by two monotone paths $\mathcal{P}_1 = (v_i, v_{i+1}, \dots, v_{j-1}, v_j)$ and $\mathcal{P}_2 = (v_i, v_{i-1}, \dots, v_{j+1}, v_j)$, i.e., paths such that, if $\sigma(v_t) = \mu_a$ and $\sigma(v_{t+1}) = \mu_b$, then $a \leq b$ if $(v_t, v_{t+1}) \in \mathcal{P}_1$ and $b \leq a$ if $(v_t, v_{t+1}) \in \mathcal{P}_2$; and
- LO3: μ_{i+1} is the parent of μ_i , for each $1 \leq i < h$, and μ_{i+1} is a child of μ_i , for each $h \leq i < k$.

Let $C(G, T)$ be a biconnected internally-triangulated outerclustered graph and suppose that C is linearly-ordered according to a sequence $\mu_1, \mu_2, \dots, \mu_k$ of clusters of T . Let $v_i \in \mu_1$ and $v_j \in \mu_k$. Then, denote by V_1 (resp. by V_2) the vertex set containing all the internal vertices of $\mathcal{P}_1$ (resp. of $\mathcal{P}_2$).

In Sect. 3 we will prove that a drawing of any linearly-ordered outerclustered graph $C(G, T)$ can be obtained even if the drawing of C_o is arbitrarily fixed. However, such a drawing of C_o will be assumed to be *convex-separated*, i.e., the drawing of C_o satisfies certain geometric properties, as the one that the polygon representing $o(G)$ is convex and as the one that, for every cluster $\mu \in T$, there exists a convex region $R(\mu, v)$ “separating” the vertices belonging to a certain cluster v from the vertices not belonging to such a cluster (in the notation $R(\mu, v)$, μ is the parent of v). Convex-separated drawings are formally defined below (see Fig. 4).

Definition 3 A straight-line rectangular drawing $\Gamma(C_o)$ of C_o is a convex-separated drawing if:

- CS1: the polygon P representing $o(G)$ is convex;

Fig. 4 A convex-separated drawing of the outer face of a linearly-ordered outerclustered graph. In this and other pictures of the paper, the polygonal lines composed of horizontal and vertical segments represent parts of the boundaries of the rectangular clusters

- CS2: there exist two vertices v_i and v_j , with $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$, such that the angle of P incident to v_i and the angle of P incident to v_j are strictly less than 180° ; and
- CS3: for every pair of clusters μ and ν such that μ is the parent of ν in T and such that μ is not an ancestor of the smallest cluster containing all the vertices of $o(G)$, there exists a convex region $R(\mu, \nu)$ such that: (i) $R(\mu, \nu)$ is entirely contained inside $\mu \cap (P \cup \text{int}(P))$, where $\text{int}(P)$ denotes the interior of polygon P ; (ii) for any cluster $\mu' \neq \mu$ and any child ν' of μ' , $R(\mu, \nu)$ intersects neither $R(\mu', \nu')$ nor the boundary of μ' ; (iii) $R(\mu, \nu) \cap P$ consists of two polygonal lines $l_1(\mu, \nu)$ and $l_2(\mu, \nu)$ such that $l_1(\mu, \nu)$ belongs to the polygonal line representing $\mathcal{P}_1$ in $\Gamma(C_o)$ and $l_2(\mu, \nu)$ belongs to the polygonal line representing $\mathcal{P}_2$ in $\Gamma(C_o)$; further, at least one endpoint of $l_1(\mu, \nu)$ (resp. of $l_2(\mu, \nu)$) lies on $e_1(\nu)$ (resp. on $e_2(\nu)$).

Let $C(G, T)$ be an outerclustered graph with outer face $o(G)$ delimited by a cycle $C = (v_i, v_{i+1}, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_{i-1}, v_i)$. Suppose that C is linearly-ordered according to a sequence $\Sigma = \mu_1, \mu_2, \dots, \mu_k$ of clusters in T . Let (v_x, v_y) be a chord of C . Consider the clustered graphs $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ such that G^1 (resp. G^2) is the subgraph of G induced by the vertices incident to and internal to cycle $C^1 = (v_x, v_{x+1}, \dots, v_{y-1}, v_y, v_x)$ (resp. to cycle $C^2 = (v_y, v_{y+1}, \dots, v_{x-1}, v_x, v_y)$), and such that T^1 (resp. T^2) is the subtree of T induced by the clusters containing vertices of G^1 (resp. of G^2).

Lemma 2 $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ are linearly-ordered outerclustered graphs.

Proof We prove the statement for C^1 , the proof for C^2 being analogous. Refer to Fig. 5.

We claim that C^1 is a biconnected internally-triangulated outerclustered graph. In order to prove the claim, by Lemma 1, it suffices to show that no cluster in T^1 exists that contains some but not all the vertices of C^1 and that does not intersect C^1

Fig. 5 The linearly-ordered outerclustered graphs obtained by splitting the linearly-ordered outerclustered graph of Fig. 3 by chord (v_x, v_y) . (a) $C^1(G^1, T^1)$. (b) $C^2(G^2, T^2)$

exactly twice. Namely, the boundary of every cluster μ is a simple closed curve, and hence it intersects C^1 an even number of times. Suppose that the boundary of μ does not intersect C^1 . Then, since μ contains vertices of G^1 , it contains all such vertices. Suppose that μ intersects C^1 at least four times. At most two of such intersections can be on the edges of $C^1 \setminus (v_x, v_y)$, since C is an outerclustered graph. Then, the boundary of μ intersects (v_x, v_y) at least twice, contradicting the c -planarity of C .

Consider the subsequence Σ_1 of Σ induced by the clusters in T^1 . Obtain a sequence Σ'_1 by removing from Σ_1 all the clusters that contain all the vertices of G^1 and that are different from $\sigma(v_x, v_{x+1}, \dots, v_y)$, if any such a cluster exists. We claim that C^1 is linearly-ordered according to Σ'_1 .

We prove that C^1 satisfies Property LO1 of Definition 2. By definition, Σ'_1 contains all the clusters of T^1 that contain vertices of G^1 , except for each cluster μ that contains all the vertices of G^1 and that is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$. However, for any vertex v of G^1 , $\sigma(v) \neq \mu$, because $\sigma(v_x, v_{x+1}, \dots, v_y)$ is a descendant of μ and contains v .

We prove that C^1 satisfies Property LO2. Since C is linearly-ordered, $o(G)$ is delimited by two monotone paths $\mathcal{P}_1 = (v_i, v_{i+1}, \dots, v_j)$ and $\mathcal{P}_2 = (v_i, v_{i-1}, \dots, v_j)$. Suppose that (see Fig. 6(a)) both v_x and v_y belong to $\mathcal{P}_1$ and v_x precedes v_y in $\mathcal{P}_1$ (the other cases in which both v_x and v_y belong to $\mathcal{P}_1$ or both belong to $\mathcal{P}_2$ being analogous); then, the subpath of $\mathcal{P}_1$ between v_x and v_y and edge (v_x, v_y) are monotone paths delimiting $o(G^1)$; further, $\mathcal{P}_2$ and the path obtained from $\mathcal{P}_1$ by replacing the subpath between v_x and v_y with edge (v_x, v_y) are monotone paths delimiting $o(G^2)$. Suppose that v_x belongs to $\mathcal{P}_1$ and v_y belongs to $\mathcal{P}_2$. If $\sigma(v_x)$ precedes $\sigma(v_y)$ in Σ (see Fig. 6(b)), then the two monotone paths delimiting $o(G^1)$ are (i) the subpath of $\mathcal{P}_1$ between v_x and v_j and (ii) edge (v_x, v_y) plus the subpath of $\mathcal{P}_2$ between v_y and v_j ; the two monotone paths delimiting $o(G^2)$ are (i) the subpath of $\mathcal{P}_1$ between v_i and v_x plus edge (v_x, v_y) and (ii) the subpath of $\mathcal{P}_2$ between v_i and v_y . If $\sigma(v_y)$ precedes $\sigma(v_x)$ in Σ , then the two monotone paths delimiting $o(G^1)$ are (i) edge (v_y, v_x) plus the subpath of $\mathcal{P}_1$ between v_x and v_j and (ii) the subpath of $\mathcal{P}_2$ between v_y and v_j ;

Fig. 6 C^1 and C^2 satisfy LO2. (a) v_x and v_y belong to $\mathcal{P}_1$, and v_x precedes v_y in $\mathcal{P}_1$. (b) v_x belongs to $\mathcal{P}_1$, v_y belongs to $\mathcal{P}_2$, and $\sigma(v_x)$ precedes $\sigma(v_y)$ in Σ

the two monotone paths delimiting $o(G^2)$ are (i) the subpath of $\mathcal{P}_1$ between v_i and v_x and (ii) the subpath of $\mathcal{P}_2$ between v_i and v_y plus edge (v_y, v_x) .

We prove that C^1 satisfies Property LO3. Since C is linearly-ordered according to Σ and Σ'_1 is a subsequence of Σ , it suffices to show that any two consecutive clusters μ_x and μ_y of Σ'_1 are adjacent in T^1 . Suppose, for a contradiction, that μ_y and μ_x are not adjacent in T^1 . First, consider the case in which μ_x and μ_y are comparable, that is, μ_y is either an ancestor or a descendant of μ_x . If μ_y is an ancestor of μ_x (the case in which μ_x is an ancestor of μ_y being analogous), consider the parent μ_{x+1} of μ_x . Such a cluster contains all the vertices contained in μ_x , hence either μ_{x+1} belongs to Σ'_1 , contradicting the fact that μ_x and μ_y are consecutive in Σ'_1 , or μ_{x+1} contains all the vertices of G^1 and is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$. However, this would imply that also μ_y contains all the vertices of G^1 and is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$, a contradiction to the fact that μ_y is in Σ'_1 . Second, consider the case in which μ_x and μ_y are incomparable, that is, μ_y is neither an ancestor nor a descendant of μ_x . Again, consider the parent μ_{x+1} of μ_x . Such a cluster contains all the vertices contained in μ_x , still not being in Σ'_1 . Hence, μ_{x+1} contains all the vertices of G^1 and is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$; this implies that $\mu_x = \sigma(v_x, v_{x+1}, \dots, v_y)$, hence μ_y is a descendant of μ_x , a contradiction. $\square$

Let $C(G, T)$, $C^1(G^1, T^1)$, and $C^2(G^2, T^2)$, Σ , v_x , and v_y be defined as above. Let Γ be any convex-separated drawing of C_o and let P be the polygon representing $o(G)$ in Γ ; let v_i and v_j be any two vertices with $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$ such that the angle of P incident to v_i and the angle of P incident to v_j are strictly less than 180° (such vertices exist by Property CS2 of Definition 3). Suppose that v_x and v_y are not collinear with any vertex of $o(G)$. Let Γ_1 and Γ_2 be the drawings of C_o^1 and C_o^2 obtained by drawing (v_x, v_y) in Γ as a straight-line segment. Denote by P^1 and P^2 the polygons representing $o(G^1)$ and $o(G^2)$ in Γ_1 and in Γ_2 , respectively.

Lemma 3 Γ_1 and Γ_2 are convex-separated drawings.

Proof We prove the statement for Γ_1 , the proof for Γ_2 being analogous. Refer to Fig. 7. The drawing is straight-line and rectangular by construction. Since Γ is a convex-separated drawing, by Property CS1 of Definition 3, the polygon P representing $o(G)$ in Γ is convex. Further, by hypothesis, v_x and v_y are not collinear with any vertex of G . Hence, P^1 is convex, thus satisfying Property CS1.

Drawing Γ_1 has no region-region crossings, since each cluster is represented in Γ_1 by the same rectangle as in Γ . We prove that Γ_1 has no edge-region crossings. Refer to Fig. 8. Suppose that an edge-region crossing exists between an edge e and a cluster ν . Then, e is edge (v_x, v_y) , otherwise Γ would not be c -planar. Cluster ν does not contain both of v_x and v_y otherwise, by the convexity of ν , e would be internal to ν ; further, ν does not contain exactly one of v_x and v_y otherwise, by the convexity of ν , e would cross ν exactly once. It follows that ν contains neither v_x nor v_y . Consider the parent μ of ν in T_1 . Such a parent exists, otherwise ν would be the root of T , contradicting the fact that ν contains neither v_x nor v_y . Since Γ satisfies Property CS3, there exists a convex region $R(\mu, \nu)$ with the properties described in

Fig. 7 (a) A convex-separated drawing Γ of the outer face C_o of a linearly-ordered outerclustered graph C . (b) and (c) Convex-separated drawings Γ_1 and Γ_2 of the outer faces C_o^1 and C_o^2 of the linearly-ordered outerclustered graphs C^1 and C^2

Fig. 8 Drawing Γ_1 has no crossing between an edge e and a cluster containing none of the end-vertices of e . The *thick line* represents segment $p(l_1)p(l_2)$

Definition 3 which “separates” v from the rest of the drawing, thus avoiding an edge-region crossing between e and v . More precisely, since Γ satisfies Property O2 of Definition 1, v has exactly two incident edges $e_1(v)$ and $e_2(v)$ belonging to $o(G)$. Denote by $u(e_1(v))$ and $u(e_2(v))$ the end-vertices of $e_1(v)$ and $e_2(v)$ belonging to v . Denote by $p(l_1)$ the endpoint of $l_1(\mu, v)$ that lies on $e_1(v)$ (if both endpoints of

$l_1(\mu, v)$ lie on $e_1(v)$, then $p(l_1)$ is the one that is closer to $u(e_1(v))$). Note that an endpoint of $l_1(\mu, v)$ lying on $e_1(v)$ exists as Γ satisfies Property CS3. Analogously define $p(l_2)$. Then, segment $\overline{p(l_1)p(l_2)}$ splits P into two disjoint convex polygons P' and P'' , where P' contains all and only the vertices in v and P'' contains all and only the vertices not in v , as Γ satisfies Property CS3. By the convexity of P' and P'' , e is internal to P'' , while the part of v inside P is internal to P' . Hence, e does not cross v .

We prove that Γ_1 satisfies Property CS2. First, observe that the angles incident to v_i and v_j in P^1 are strictly less than 180° , since they are strictly less than 180° in Γ ; further, the angles incident to v_x and v_y in P^1 are strictly less than 180° , since they are strictly less than the angles incident to v_x and v_y in P , that are at most 180° , by the convexity of P . Suppose that $\sigma(v_x)$ precedes $\sigma(v_y)$ in Σ , the opposite case being analogous. It suffices to observe that: (i) if C^1 contains both v_i and v_j , then $\Sigma'_1 = \Sigma$; hence, v_i and v_j are vertices satisfying the desired properties; (ii) if C^1 contains neither v_i nor v_j , then Σ'_1 is the subsequence of Σ that starts at $\sigma(v_x)$ and ends at $\sigma(v_y)$; hence, v_x and v_y are vertices satisfying the desired properties; (iii) if C^1 contains v_j and does not contain v_i (as in Fig. 7), then Σ'_1 is the subsequence of Σ that starts at $\sigma(v_x)$ and ends at $\sigma(v_j) = \mu_k$; hence, v_x and v_j are vertices satisfying the desired properties.

We prove that Γ_1 satisfies Property CS3. The existence of regions $R(\mu, v)$ inside P^1 , for every cluster v , is easily deduced from the existence of regions $R(\mu, v)$ inside P , which is guaranteed as Γ satisfies Property CS3. Namely, if $R(\mu, v)$ is not intersected by e , then a region $R(\mu, v)$ inside P^1 can be constructed coincident with the same region inside P . Further, if $R(\mu, v)$ is cut by e , then two regions are created, one inside P^1 and the other one inside P^2 . The properties that have to be satisfied by $R(\mu, v)$ inside P^1 easily descend from the analogous properties satisfied by $R(\mu, v)$ inside P . $\square$

When dealing with outerclustered graphs and general clustered graphs, it is sufficient to consider clustered graphs whose underlying graphs have triangular outer faces. We exploit such a fact to refine the definition of convex-separated drawing of the outer face of a clustered graph into a similar definition that only deals with clustered graphs whose underlying graphs are 3-cycles (such a definition will be later used to ensure that a drawing of the outer face of a considered clustered graph satisfies certain properties). See Fig. 9.

Definition 4 Let $C(G, T)$ be a clustered graph such that G is a 3-cycle (u, v, z) . A straight-line rectangular drawing $\Gamma(C)$ of C is a triangular-convex-separated drawing if:

- TCS1: for every pair of clusters μ and v such that μ is the parent of v in T and such that μ is not an ancestor of $\sigma(u, v, z)$, there exists a convex region $R(\mu, v)$ such that: (i) $R(\mu, v)$ is entirely contained inside $\mu \cap (P \cup \text{int}(P))$, where P is the triangle representing G in $\Gamma(C)$; (ii) for any cluster $\mu' \neq \mu$ and any child v' of μ' , $R(\mu, v)$ intersects neither $R(\mu', v')$ nor the boundary of μ' ; (iii) $R(\mu, v) \cap P$ consists of two polygonal lines $l_1(\mu, v)$ and $l_2(\mu, v)$ such that at least one endpoint of $l_1(\mu, v)$ (resp. of $l_2(\mu, v)$) belongs to $e_1(v)$ (resp. to $e_2(v)$).

Fig. 9 A triangular-convex-separated drawing of a clustered graph whose underlying graph is a 3-cycle

The interplay between a triangular-convex-separated drawing of a clustered graph whose underlying graph is a 3-cycle and a convex-separated drawing of a linearly-ordered maximal outerclustered graph is clarified in the following lemma.

Lemma 4 *Let $C(G, T)$ be a linearly-ordered maximal outerclustered graph. Then, a triangular-convex-separated drawing of C_o is a convex-separated drawing of C_o .*

Proof Consider any triangular-convex-separated drawing $\Gamma(C_o)$ of C_o . We prove that $\Gamma(C_o)$ is a convex-separated drawing. Properties CS1 and CS2 of Definition 3 easily descend from the fact that, since $o(G)$ is a 3-cycle, it is represented as a convex polygon P whose internal angles are smaller than 180° . Property CS3 coincides with Property TCS1. $\square$

Finally, we define a class of drawings in which the properties of convexity and visibility among vertices and clusters are imposed on all the internal faces rather than on the outer face.

Definition 5 Let $C(G, T)$ be an internally-triangulated clustered graph. A drawing $\Gamma(C)$ of C is an internally-convex-separated drawing if, for every internal face f of G , the part $\Gamma(C_f)$ of $\Gamma(C)$ representing C_f is a triangular-convex-separated drawing.

3 Drawing Linearly-Ordered Outerclustered Graphs

In this section we show how to construct an internally-convex-separated drawing of any linearly-ordered outerclustered graph C for an arbitrary triangular-convex-separated drawing of the outer face C_o of C . This is done by means of an inductive algorithm that uses the following lemma as the main tool (see Fig. 10):

Lemma 5 *Let $C(G, T)$ be an internally-triangulated triconnected outerclustered graph. Suppose that C is linearly-ordered according to a sequence $\mu_1, \mu_2, \dots, \mu_k$ of clusters of T . Let v_i and v_j be any two vertices such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$. Let V_1 (resp. V_2) be the set of vertices between v_i and v_j (resp. between v_j and v_i) in*

Fig. 10 A path satisfying Properties P1–P5

the clockwise order of the vertices around $o(G)$. Then, if $V_1 \neq \emptyset$, there exists a path $\mathcal{P}_u = (u_1, u_2, \dots, u_r)$ such that:

- P1: u_1 and u_r belong to $V_2 \cup \{v_i, v_j\}$;
- P2: u_i is an internal vertex of G , for each $2 \leq i \leq r-1$;
- P3: if $\sigma(u_i) = \mu_{j_1}$ and $\sigma(u_{i+1}) = \mu_{j_2}$, then $j_1 < j_2$, for each $1 \leq i \leq r-1$;
- P4: there exists exactly one vertex u_x , where $2 \leq x \leq r-1$, that is adjacent to at least one vertex v_x in V_1 ;
- P5: there exists no chord among the vertices of path $(u_1, u_2, \dots, u_x)$ and no chord among the vertices of path $(u_x, u_{x+1}, \dots, u_r)$.

Proof We show an algorithm to construct a path $\mathcal{P}_u$ as required by the lemma. The algorithm derives path $\mathcal{P}_u$ in several steps. At step $s+1$ a path $\mathcal{P}^{s+1}$ is found by modifying the path $\mathcal{P}^s$ obtained at the previous step. At the last step m of the algorithm, a path $\mathcal{P}_u$ can be obtained as a subpath of $\mathcal{P}^m$ satisfying the properties required by the lemma. A path satisfying Properties P1–P5 is shown in Fig. 10.

In more detail, at each step $s \leq m$, a path $\mathcal{P}^s = (u_1^s, u_2^s, \dots, u_{r(s)}^s)$ is found satisfying the properties described below (see Fig. 11).

Denote by $\mathcal{P}_1 = (v_i, v_{i+1}, \dots, v_{j-1}, v_j)$ and $\mathcal{P}_2 = (v_i, v_{i-1}, \dots, v_{j+1}, v_j)$ the monotone paths on the vertices of $V_1 \cup \{v_i, v_j\}$ and of $V_2 \cup \{v_i, v_j\}$, respectively, delimiting $o(G)$.

- PP1: $u_1^s = v_i$ and $u_{r(s)}^s = v_j$;
- PP2: each vertex u_t^s is either an internal vertex of G or a vertex of $\mathcal{P}_2$, for each $2 \leq t \leq r(s)-1$;
- PP3: if $\sigma(u_t) = \mu_{j_1}$ and $\sigma(u_{t+1}) = \mu_{j_2}$, then $j_1 \leq j_2$, for each $1 \leq t \leq r(s)-1$; and
- PP4: there exists no chord inside cycle $\mathcal{P}^s \cup \mathcal{P}_1$.

At the first step, set $\mathcal{P}^1 = \mathcal{P}_2$. Path $\mathcal{P}^1$ satisfies Properties PP1 and PP2 by definition, Property PP3 since C satisfies Property LO2 of Definition 2, and Property PP4 because G is triconnected.

Fig. 11 A path $\mathcal{P}^s$ satisfying Properties PP1–PP4

Fig. 12 Obtaining $\mathcal{P}^{s+1}$ from $\mathcal{P}^s$ if $\sigma(u_a^s)$, $\sigma(z_1^*)$, and $\sigma(u_b^s)$ appear in this order in Σ . The *dashed segments* show the subpath of $\mathcal{P}^s$ that does not belong to $\mathcal{P}^{s+1}$

Suppose that a path $\mathcal{P}^s$ has been found at step s of the algorithm. We show how to determine $\mathcal{P}^{s+1}$ at step $s + 1$ of the algorithm.

Consider any edge (u_t^s, u_{t+1}^s) of $\mathcal{P}^s$. Such an edge is incident to a face internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$. Let z_1^* be the third vertex of such a face. As $\mathcal{P}^s$ satisfies Property PP4, z_1^* is neither a vertex of $\mathcal{P}_1$ nor a vertex of $\mathcal{P}^s$. Hence, z_1^* is internal to $\mathcal{P}^s \cup \mathcal{P}_1$.

Denote by u_a^s and by u_b^s the first and the last vertex of $\mathcal{P}^s$ adjacent to z_1^* . We distinguish two cases.

In the first case, $\sigma(u_a^s)$, $\sigma(z_1^*)$, and $\sigma(u_b^s)$ appear in this order in Σ . Then, path $\mathcal{P}^{s+1}$ is obtained by replacing the subpath of $\mathcal{P}^s$ between u_a^s and u_b^s with path (u_a^s, z_1^*, u_b^s) . An example of this case is shown in Fig. 12.

Suppose that z_1^* is adjacent to at least one vertex in V_1 . Since the first and the last vertex of $\mathcal{P}^{s+1}$ (that are v_i and v_j , respectively) belong to $\mathcal{P}_2$, the vertices shared by $\mathcal{P}^{s+1}$ and $\mathcal{P}_2$ partition $\mathcal{P}^{s+1}$ into subpaths, where the internal vertices of each

Fig. 13 Three repetitions of the argument for the case in which $\sigma(z_1^*)$ (resp. $\sigma(z_2^*)$ and $\sigma(z_3^*)$) precedes the smallest clusters containing the first and last neighbor of z_1^* (resp. of z_2^* or of z_3^*) in $\mathcal{P}^s$. Thick segments represent $\mathcal{P}^s$

subpath are internal to $\mathcal{P}_1 \cup \mathcal{P}_2$. Let $\mathcal{P}_u$ be the one of such subpaths containing z_1^* . Path $\mathcal{P}_u = (u_1, u_2, \dots, u_x, u_{x+1}, \dots, u_r)$ defined as above is easily shown to satisfy Properties P1–P4 (where $u_x = z_1^*$). In particular, notice that $\mathcal{P}_u$ satisfies Property P3 since $\mathcal{P}^{s+1}$ satisfies Property PP3 and no two adjacent vertices of $\mathcal{P}_u$ have the same smallest containing cluster, as C satisfies Property O3 of Definition 1. Further, if there exists a chord (u_a, u_b) among the vertices of path $(u_1, u_2, \dots, u_x)$ (resp. among the vertices of path $(u_x, u_{x+1}, \dots, u_r)$), then replace the subpath of $(u_1, u_2, \dots, u_x)$ (resp. of $(u_x, u_{x+1}, \dots, u_r)$) between u_a and u_b with such a chord. The repetition of such an argument eventually leads to a path $\mathcal{P}_u$ also satisfying Property P5. Hence, the algorithm stops because a path with the properties required by the lemma has been found.

Suppose that z_1^* is not adjacent to any vertex in V_1 . Then, $\mathcal{P}^{s+1}$ is easily shown to satisfy Properties PP1–PP4. Hence, the algorithm continues with step $s + 2$.

In the second case, either $\sigma(z_1^*)$, $\sigma(u_a^s)$, and $\sigma(u_b^s)$ appear in this order in Σ , or they appear in the order $\sigma(u_a^s)$, $\sigma(u_b^s)$, $\sigma(z_1^*)$. Suppose that $\sigma(z_1^*)$, $\sigma(u_a^s)$, and $\sigma(u_b^s)$ appear in this order in Σ , the other case being analogous. See Fig. 13.

Consider edge (u_{a-1}^s, u_a^s) of $\mathcal{P}^s$. Such an edge is incident to a face internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$. Let z_2^* be the third vertex of such a face. By Property PP4, z_2^* is neither a vertex of $\mathcal{P}_1$ nor a vertex of $\mathcal{P}^s$. Hence, z_2^* is internal to $\mathcal{P}^s \cup \mathcal{P}_1$. Further, $\sigma(z_2^*)$ is not the same cluster of $\sigma(u_a^s)$ (since C satisfies Property O3 of Definition 1), and $\sigma(z_2^*)$ does not follow $\sigma(u_a^s)$ in Σ , otherwise edge (u_a^s, z_2^*) would cross twice the boundary of $\sigma(u_a^s)$ or the boundary of the cluster coming before $\sigma(u_a^s)$ in Σ (depending on whether $\sigma(u_a^s)$ is a child or is the parent of the cluster coming before $\sigma(u_a^s)$ in Σ). Hence, $\sigma(z_2^*)$ precedes $\sigma(u_a^s)$ in Σ .

Then, the whole argument can be repeated, namely denote by u_c^s and by u_d^s the first and the last vertex adjacent to z_2^* in $\mathcal{P}^s$. Since C satisfies Property O3, then $\sigma(u_c^s) \neq \sigma(z_2^*)$; since $\sigma(z_2^*)$ precedes $\sigma(u_a^s)$ in Σ and since $\sigma(u_a^s)$ does not follow $\sigma(u_d^s)$ in Σ , then $\sigma(z_2^*)$ precedes $\sigma(u_d^s)$ in Σ . Then, either $\sigma(u_c^s)$ precedes $\sigma(z_2^*)$ in Σ or $\sigma(z_2^*)$ precedes $\sigma(u_c^s)$ in Σ . If $\sigma(u_c^s)$ precedes $\sigma(z_2^*)$, then a path $\mathcal{P}^{s+1}$ is found

by replacing the subpath of $\mathcal{P}^s$ between u_c^s and u_d^s with path (u_c^s, z_2^*, u_d^s) ; then, either a path $\mathcal{P}_u$ satisfying Properties P1–P5 can be derived from $\mathcal{P}^{s+1}$ or $\mathcal{P}^{s+1}$ satisfies Properties PP1–PP4, depending on whether z_2^* is adjacent to at least one vertex in V_1 or not. If $\sigma(z_2^*)$ precedes $\sigma(u_c^s)$, then edge (u_{c-1}^s, u_c^s) and the vertex z_3^* incident to the face $(u_{c-1}^s, u_c^s, z_3^*)$ internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$ are considered and the argument is repeated again. The repetition of such an argument eventually leads to find a vertex z_f^* that is incident to a face $(u_{y-1}^s, u_y^s, z_f^*)$ and such that, denoting by u_p^s and u_q^s the first and the last neighbor of z_f^* in $\mathcal{P}^s$, $\sigma(u_p^s)$, $\sigma(z_f^*)$, and $\sigma(u_q^s)$ appear in this order in Σ . Namely, at every repetition of such an argument, the considered edge (that is equal to (u_i^s, u_{i+1}^s) at the first repetition, to (u_{a-1}^s, u_a^s) at the second repetition, to (u_{c-1}^s, u_c^s) at the third repetition, and to (u_{y-1}^s, u_y^s) at the last repetition) gets closer to v_i in $\mathcal{P}^s$; then, after a certain number of repetitions of the algorithm, vertex u_{y-1}^s eventually belongs to cluster μ_1 and no cluster preceding μ_1 exists in Σ .

It remains to observe that, after a certain number m of steps of the algorithm, a path $\mathcal{P}_u$ satisfying Properties P1–P5 can be derived from $\mathcal{P}^m$. Namely, the number of vertices internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$ decreases at every step of the algorithm, hence a vertex z^* adjacent to a vertex in V_1 is eventually added to a path $\mathcal{P}^{m-1}$ to form a path $\mathcal{P}^m$, from which a path $\mathcal{P}_u$ satisfying Properties P1–P5 can then be derived. Notice that z^* is the only vertex of $\mathcal{P}^m$ adjacent to a vertex in V_1 , since no vertex of $\mathcal{P}^{m-1}$ is adjacent to a vertex in V_1 , as $\mathcal{P}^{m-1}$ satisfies Property PP4. $\square$

A pseudo-code description of the algorithm for finding a path $\mathcal{P}_u$ satisfying the properties required by Lemma 5 (supposing that $V_1 \neq \emptyset$) is presented in Algorithm 1.

A lemma similar to Lemma 5 is presented in the following. The proof of such a lemma can be obtained analogously to the one of Lemma 5, where V_1 replaces V_2 and vice versa.

Lemma 6 *Let $C(G, T)$ be an internally-triangulated triconnected outerclustered graph. Suppose that C is linearly-ordered according to a sequence $\mu_1, \mu_2, \dots, \mu_k$ of clusters of T . Let v_i and v_j be any two vertices such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$. Let V_1 (resp. V_2) be the set of vertices between v_i and v_j (resp. between v_j and v_i) in the clockwise order of the vertices around $o(G)$. Then, if $V_2 \neq \emptyset$, there exists a path $\mathcal{P}_u = (u_1, u_2, \dots, u_r)$ such that:*

- P1: u_1 and u_r belong to $V_1 \cup \{v_i, v_j\}$;
- P2: u_i is an internal vertex of G , for each $2 \leq i \leq r-1$;
- P3: if $\sigma(u_i) = \mu_{j_1}$ and $\sigma(u_{i+1}) = \mu_{j_2}$, then $j_1 < j_2$, for each $1 \leq i \leq r-1$;
- P4: there exists exactly one vertex u_x , where $2 \leq x \leq r-1$, that is adjacent to at least one vertex v_x in V_2 ;
- P5: there exists no chord among the vertices of path $(u_1, u_2, \dots, u_x)$ and no chord among the vertices of path $(u_x, u_{x+1}, \dots, u_r)$.

When dealing with a linearly-ordered outerclustered graph C , we will always denote by v_i and v_j two vertices such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$, where $\mu_1, \mu_2, \dots, \mu_k$ is the sequence of clusters C is linearly-ordered according to.

We now present the main theorem of this section.

Algorithm 1 PATHS IN LINEARLY-ORDERED OUTERCLUSTERED GRAPHS**Input:** $C(G, T)$, Σ , $V_1 \neq \emptyset$, $\mathcal{P}_1$, and $\mathcal{P}_2$, as in Lemma 5.**Output:** A path $\mathcal{P}_u = (u_1, u_2, \dots, u_r)$ satisfying Properties P1–P5 of Lemma 5.

```

1:  $s \leftarrow 1$ ;  $\mathcal{P}^s \leftarrow \mathcal{P}_2$ ;  $path\_found \leftarrow \text{FALSE}$ ;  $e \leftarrow$  any edge  $(u_t^s, u_{t+1}^s)$  of  $\mathcal{P}^s$ ;
2: while  $path\_found = \text{FALSE}$  do
3:    $i \leftarrow 1$ ;
4:    $z_i^* \leftarrow$  the vertex creating a face with  $e$  inside  $\mathcal{P}^s \cup \mathcal{P}_1$ ;
5:    $u_L^s \leftarrow$  the first vertex of  $\mathcal{P}^s$  adjacent to  $z_i^*$ ;
6:    $u_R^s \leftarrow$  the last vertex of  $\mathcal{P}^s$  adjacent to  $z_i^*$ ;
7:   if  $\sigma(u_L^s)$ ,  $\sigma(z_i^*)$ , and  $\sigma(u_R^s)$  do not appear in this order in  $\Sigma$  then
8:     while  $\sigma(z_i^*)$ ,  $\sigma(u_L^s)$ , and  $\sigma(u_R^s)$  appear in this order in  $\Sigma$  do
9:        $e \leftarrow$  edge  $(u_{L-1}^s, u_L^s)$  of  $\mathcal{P}^s$ ;  $i \leftarrow i + 1$ ;
10:       $z_i^* \leftarrow$  the vertex creating a face with  $e$  inside  $\mathcal{P}^s \cup \mathcal{P}_1$ ;
11:       $u_L^s \leftarrow$  the first vertex of  $\mathcal{P}^s$  adjacent to  $z_i^*$ ;
12:       $u_R^s \leftarrow$  the last vertex of  $\mathcal{P}^s$  adjacent to  $z_i^*$ ;
13:    end while
14:    while  $\sigma(u_L^s)$ ,  $\sigma(u_R^s)$ , and  $\sigma(z_i^*)$  appear in this order in  $\Sigma$  do
15:       $e \leftarrow$  edge  $(u_R^s, u_{R+1}^s)$  of  $\mathcal{P}^s$ ;  $i \leftarrow i + 1$ ;
16:       $z_i^* \leftarrow$  the vertex creating a face with  $e$  inside  $\mathcal{P}^s \cup \mathcal{P}_1$ ;
17:       $u_L^s \leftarrow$  the first vertex of  $\mathcal{P}^s$  adjacent to  $z_i^*$ ;
18:       $u_R^s \leftarrow$  the last vertex of  $\mathcal{P}^s$  adjacent to  $z_i^*$ ;
19:    end while
20:  end if
21:   $z^* \leftarrow z_i^*$ ;
22:   $\mathcal{P}^{s+1} \leftarrow$  the path obtained from  $\mathcal{P}^s$  by replacing the subpath between  $u_L^s$  and  $u_R^s$  with path  $(u_L^s, z^*, u_R^s)$ ;
23:  if  $z^*$  is adjacent to at least one vertex in  $V_1$  then
24:     $m \leftarrow s + 1$ ;
25:    partition  $\mathcal{P}^m$  into subpaths, based on the vertices of  $\mathcal{P}^m \cap \mathcal{P}_2$ ;
26:     $\mathcal{P}_u \leftarrow$  the subpath of  $\mathcal{P}^m$  containing  $z^*$ ;
27:    while  $(u_1, u_2, \dots, u_x = z^*)$  has a chord  $(u_a, u_b)$  do
28:      replace the subpath of  $\mathcal{P}_u$  between  $u_a$  and  $u_b$  with  $(u_a, u_b)$ ;
29:    end while
30:    while  $(u_x = z^*, u_{x+1}, \dots, u_r)$  has a chord  $(u_a, u_b)$  do
31:      replace the subpath of  $\mathcal{P}_u$  between  $u_a$  and  $u_b$  with  $(u_a, u_b)$ ;
32:    end while
33:     $path\_found \leftarrow \text{TRUE}$ ;
34:  else
35:     $s \leftarrow s + 1$ ;  $e \leftarrow$  any edge  $(u_t^s, u_{t+1}^s)$  of  $\mathcal{P}^s$ ;
36:  end if
37: end while
38: return  $\mathcal{P}_u$ ;

```

Theorem 1 Let $C(G, T)$ be a linearly-ordered internally-triangulated triconnected outerclustered graph. Then, for every convex-separated drawing $\Gamma(C_o)$ of C_o , there exists an internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.

Proof We prove the statement by induction on the number of internal vertices of G . First observe that, since G is triconnected, $o(G)$ has no chords. Hence, if G has no internal vertices, then C_o and C are the same graph and the statement trivially follows. Otherwise, G has internal vertices. Denote by $\mathcal{C}$ the cycle delimiting $o(G)$. We show how to split C into smaller linearly-ordered internally-triangulated triconnected outerclustered graphs.

Let $\Gamma(C_o)$ be an arbitrary convex-separated drawing of C_o . Denote by P the polygon representing $o(G)$ in $\Gamma(C_o)$. Since $\Gamma(C_o)$ satisfies Property CS1 of Definition 3, P is convex. Suppose that C is linearly-ordered according to a sequence $\Sigma = \mu_1, \mu_2, \dots, \mu_h, \dots, \mu_k$ of clusters of T . Let v_i and v_j be two vertices of $\mathcal{C}$ with $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$ such that the angle of P incident to v_i and the angle of P incident to v_j are strictly less than 180° . Such vertices exist since $\Gamma(C_o)$ satisfies Property CS2 of Definition 3.

We distinguish two cases:

Case 1 applies when the vertices of $V_1 \cup \{v_i, v_j\}$ are not all collinear. In such a case, $V_1 \neq \emptyset$, otherwise the only two vertices of $V_1 \cup \{v_i, v_j\}$ would be collinear. Hence, a path $(u_1, u_2, \dots, u_x, \dots, u_r)$ can be found as in Lemma 5. Let v_x be any vertex of V_1 adjacent to u_x .

Lemma 7 Vertex v_x does not lie on the line $l(u_1, u_r)$ through u_1 and u_r .

Proof Denote by $\mathcal{P}(u_1, v_x, u_r)$ the subpath of $\mathcal{C}$ between vertices u_1 and u_r containing v_x . Such a path is hence part of the boundary of $o(G)$. If there exists an internal vertex of $\mathcal{P}(u_1, v_x, u_r)$ lying on the segment $\overline{u_1 u_r}$, then all the vertices in $\mathcal{P}(u_1, v_x, u_r)$ lie on $\overline{u_1 u_r}$, otherwise polygon P would not be convex (see Fig. 14(a)). However, $\mathcal{P}(u_1, v_x, u_r)$ contains all the vertices of $V_1 \cup \{v_i, v_j\}$, that are not all collinear, by hypothesis. Now suppose that v_x lies on the half-line that is part of $l(u_1, u_r)$, that starts at u_1 , and that does not contain u_r (the case in which v_x lies on the half-line that is part of $l(u_1, u_r)$, that starts at u_r , and that does not contain u_1 being analogous). By the convexity of P , all the vertices in the path $\mathcal{P}(v_x, u_1, u_r)$ between v_x and u_r containing u_1 lie on $l(u_1, u_r)$ (see Fig. 14(b)). However, since

Fig. 14 Illustration for the proof of Lemma 7. (a) Vertex v_x lies on $\overline{u_1 u_r}$. (b) Vertex v_x lies on the half-line that is part of $l(u_1, u_r)$, that starts at u_1 , and that does not contain u_r

$v_x \in V_1$ and since $u_1, u_r \in V_2 \cup \{v_i, v_j\}$, it follows that v_i is one of the internal vertices of $\mathcal{P}(v_x, u_1, u_r)$, thus its incident angle in P is exactly 180° , contradicting the hypothesis. $\square$

Consider the triangle $T(v_x, u_1, u_r)$ with vertices v_x, u_1 , and u_r . By Lemma 7, such a triangle is non-degenerate and, by the convexity of P , it is entirely contained inside P (observe that parts of triangle $T(v_x, u_1, u_r)$ and of polygon P could coincide). Denote by $\sigma'(u_i)$ any cluster in Σ which is a child of $\sigma(u_i)$, for each $2 \leq i \leq r-1$.

Lemma 8 *There exists a small disk D entirely contained inside $\text{int}(T(v_x, u_1, u_r)) \cap R(\sigma(u_x), \sigma'(u_x))$.*

Proof Let $\mu_{j_1}, \mu_{j_2}, \mu_{j_3}$, and μ_{j_4} be clusters $\sigma(u_1), \sigma(u_x), \sigma(u_r)$, and $\sigma(v_x)$ in Σ , respectively. Denote by μ'_j any cluster in Σ which is a child of μ_j . Since $2 \leq x \leq r-1$ and since $\sigma(u_i) = \mu_{l_1}$ and $\sigma(u_{i+1}) = \mu_{l_2}$ implies $l_1 < l_2$, we have $j_1 < j_2 < j_3$. Since C satisfies Property O3 of Definition 1, $\sigma(u_x) \neq \sigma(v_x)$ and hence $j_2 \neq j_4$. Suppose that $j_2 < j_4$, the case $j_2 > j_4$ being analogous.

We claim that $s_1 = R(\mu_{j_2}, \mu'_{j_2}) \cap \overline{u_1 u_r}$ and $s_2 = R(\mu_{j_2}, \mu'_{j_2}) \cap \overline{u_1 v_x}$ are straight-line segments. The claim implies the lemma, namely if s_1 and s_2 are straight-line segments, then the interior of the quadrilateral having s_1 and s_2 as opposite sides entirely belongs to $\text{int}(T(v_x, u_1, u_r)) \cap R(\mu_{j_2}, \mu'_{j_2})$ and any disk entirely contained inside such a quadrilateral satisfies the requirements of the lemma. See Fig. 15.

By Property CS3 of Definition 3, $R(\mu_{j_2}, \mu'_{j_2}) \cap P$ consists of two line segments $l_1(\mu_{j_2}, \mu'_{j_2})$ and $l_2(\mu_{j_2}, \mu'_{j_2})$. Since $j_2 \neq j_1, j_3, j_4$, none of vertices u_1, u_r , and v_x

Fig. 15 Illustration for the proof of Lemma 8

belongs to $R(\mu_{j_2}, \mu'_{j_2})$. Consider the two polygonal lines l_a and l_b obtained by removing $l_1(\mu_{j_2}, \mu'_{j_2})$ and $l_2(\mu_{j_2}, \mu'_{j_2})$ from P . Since P is convex, it suffices to show that u_1 is in l_a and u_r is in l_b , or vice versa, in order to prove that s_1 is a straight-line segment; analogously, it suffices to show that u_1 is in l_a and v_x is in l_b , or vice versa, in order to prove that s_2 is a straight-line segment. However, this directly follows from the fact that one of the sides of region $R(\mu_{j_2}, \mu'_{j_2})$ separates the vertices whose smallest containing cluster is μ_j , with $j < j_2$, from the vertices whose smallest containing cluster is μ_j , with $j > j_2$. $\square$

Place u_x at any point inside D . Draw edge (u_x, v_x) as a straight-line segment. Consider the straight-line segment $\overline{u_1 u_x}$ and consider any cluster μ_j such that $j_1 < j < j_2$. Then, $R(\mu_j, \mu'_j) \cap \overline{u_1 u_x}$ is a straight-line segment, namely one of the sides of region $R(\mu_j, \mu'_j)$ separates the vertices whose smallest containing cluster is μ_{j^*} , with $j^* < j$, from the vertices whose smallest containing cluster is μ_{j^*} , with $j^* > j$. Analogously, consider the straight-line segment $\overline{u_x u_r}$ and consider any cluster μ_j such that $j_2 < j < j_3$. Then, $R(\mu_j, \mu'_j) \cap \overline{u_x u_r}$ is a straight-line segment, namely one of the sides of region $R(\mu_j, \mu'_j)$ separates the vertices whose smallest containing cluster is μ_{j^*} , with $j^* < j$, from the vertices whose smallest containing cluster is μ_{j^*} , with $j^* > j$. Draw each vertex u_i , with $2 \leq i \leq x-1$, at any internal point of $\overline{u_1 u_x} \cap R(\sigma(u_i), \sigma'(u_i))$. Analogously, draw each vertex u_i , with $x+1 \leq i \leq r-1$, at any point of $\overline{u_x u_r} \cap R(\sigma(u_i), \sigma'(u_i))$. Denote by Γ the constructed drawing. The construction of Γ is depicted in Fig. 16.

Let C_1 be the cycle composed of path $(u_1, u_2, \dots, u_x)$, of edge (u_x, v_x) , and of the path between u_1 and v_x in $\mathcal{C}$ not containing u_r . Let C_2 be the cycle composed of path

Fig. 16 Drawing path $(u_1, u_2, \dots, u_x, \dots, u_r)$ and edge (u_x, v_x)

$(u_x, u_{x+1}, \dots, u_r)$, of edge (u_x, v_x) , and of the path between v_x and u_r in $\mathcal{C}$ not containing u_1 . Let $\mathcal{C}_3$ be the cycle composed of path $(u_1, u_2, \dots, u_x, \dots, u_r)$ and of the path between u_1 and u_r in $\mathcal{C}$ not containing v_x . Let $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ be the subtrees of T induced by the clusters containing vertices of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively. Denote by G_1 , G_2 , and G_3 the subgraphs of G induced by the vertices inside or on the border of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively. Finally, let C_1 , C_2 , and C_3 be the clustered graphs whose underlying graphs are G_1 , G_2 , and G_3 , respectively, and whose inclusion trees $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ are the subtrees of T induced by the clusters containing vertices of G_1 , G_2 , and G_3 , respectively.

Lemma 9 C_1 , C_2 , and C_3 are linearly-ordered outerclustered graphs.

Proof Denote by $\mathcal{C}_{1,2}$ the cycle composed of path $(u_1, u_2, \dots, u_x, \dots, u_r)$ and of the path between u_1 and u_r in $\mathcal{C}$ containing v_x . Let $C_{1,2}$ be the clustered graph whose underlying graph $G_{1,2}$ is the subgraph of G whose vertices are incident to or internal to $\mathcal{C}_{1,2}$, and whose inclusion tree $\mathcal{T}_{1,2}$ is the subtree of T induced by the clusters containing vertices of $G_{1,2}$. It suffices to prove that $C_{1,2}$ and C_3 are linearly-ordered outerclustered graphs. Namely, by Lemma 2, if $C_{1,2}$ is a linearly-ordered outerclustered graph, then C_1 and C_2 are linearly-ordered outerclustered graphs, as well.

Since C is linearly-ordered, $o(G)$ is delimited by two monotone paths $\mathcal{P}_1$ and $\mathcal{P}_2$, where $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) has the vertices in V_1 (resp. in V_2) as internal vertices.

Consider the sequence $\Sigma = \mu_1, \mu_2, \dots, \mu_k$ according to which C is linearly-ordered. Recall that clusters μ_i and μ_{i+1} are one the parent of the other and that they could contain the same set of vertices. Denote by Σ_3 the subsequence of Σ that starts at $\sigma(u_1)$ and ends at $\sigma(u_r)$. We prove that $C_{1,2}$ is an outerclustered graph linearly-ordered according to Σ and that C_3 is an outerclustered graph linearly-ordered according to Σ_3 .

Each cluster μ_i that is not crossed by path $(u_1, u_2, \dots, u_x, \dots, u_r)$ has two intersections with $\mathcal{C}_{1,2}$ and none with $\mathcal{C}_3$. Consider any cluster μ_i that is crossed by path $(u_1, u_2, \dots, u_x, \dots, u_r)$. Since C satisfies Property O2, the boundary $B(\mu_i)$ of each cluster μ_i intersects $\mathcal{C}$ exactly twice. One of such intersections belongs to $\mathcal{C}_{1,2}$, the other one to $\mathcal{C}_3$. Hence, each of $\mathcal{C}_{1,2}$ and $\mathcal{C}_3$ crosses $B(\mu_i)$ exactly twice. Then, Lemma 1 ensures that $C_{1,2}$ and C_3 are biconnected internally-triangulated outerclustered graphs.

We prove that $C_{1,2}$ and C_3 satisfy Property LO2 of Definition 2. Path $\mathcal{P}_1$ and the path obtained by replacing the subpath of $\mathcal{P}_2$ between u_1 and u_r with path $(u_1, u_2, \dots, u_x, \dots, u_r)$ are monotone paths delimiting $o(G_{1,2})$. Further, path $(u_1, u_2, \dots, u_x, \dots, u_r)$ and the subpath of $\mathcal{P}_2$ between u_1 and u_r are monotone paths delimiting $o(G_3)$.

We prove that $C_{1,2}$ and C_3 satisfy Property LO1. Since C is linearly-ordered, the smallest cluster containing each vertex in G (and each vertex in $G_{1,2}$) is a cluster μ_i , that by hypothesis belongs to Σ . Concerning C_3 , observe that $o(G_3)$ is delimited by two monotone paths. Hence, if the smallest cluster containing a vertex in G_3 does not belong to Σ_3 , then such a vertex is outside $o(G_3)$, a contradiction.

We prove that $C_{1,2}$ and C_3 satisfy Property LO3. Sequence Σ for $C_{1,2}$ coincides with sequence Σ for C , hence the property follows. Concerning C_3 , observe that the

existence of an index h such that (i) μ_{i+1} is the parent of μ_i , for each $i < h$ such that μ_i belongs to Σ_3 , and (ii) μ_{i+1} is a child of μ_i , for each $i \geq h$ such that μ_{i+1} belongs to Σ_3 , easily descends from the existence of index h for the sequence Σ of C and from the fact that Σ_3 is a subsequence of Σ . $\square$

Now we turn our attention to the geometry supporting the above topological results.

Lemma 10 *The drawings of (C_1, T_1) , (C_2, T_2) , and (C_3, T_3) in Γ are convex-separated drawings.*

Proof Drawing Γ is straight-line and rectangular by construction. By construction, u_x lies in $\text{int}(T(v_x, u_1, u_r))$ that in turn is entirely contained in $\text{int}(P)$. By the convexity of P , straight-line segments can be drawn from u_x to any vertex of P (and hence to u_1 , v_x , and u_r) not causing crossings; hence, the drawings of C_1 , C_2 , and C_3 have no edge crossings. Since $\Gamma(C_o)$ has no region-region crossings, Γ has no region-region crossings (namely, each cluster has the same drawing in Γ and in $\Gamma(C_o)$).

We prove that Γ has no edge-region crossing. Suppose that there is an edge-region crossing between an edge e and a cluster v . Then, e is either an edge of path $(u_1, u_2, \dots, u_x)$, or an edge of path $(u_x, u_{x+1}, \dots, u_r)$, or edge (u_x, v_x) . Namely, all other edge-cluster pairs have the same drawings in Γ and in $\Gamma(C_o)$, hence they do not cross more than once by hypothesis. Suppose that $e = (u_i, u_{i+1})$ is an edge of $(u_1, u_2, \dots, u_x)$, the other cases being analogous. If both u_i and u_{i+1} belong to v then, by the convexity of v , e is internal to v . If exactly one of u_i and u_{i+1} belongs to v then, by the convexity of v , e crosses v exactly once. It follows that v contains neither u_i nor u_{i+1} . Consider the parent μ of v in T . Such a parent exists since otherwise v would be the root of T , contradicting the fact that v contains neither u_i nor u_{i+1} . By definition of convex-separated drawing, there exists a convex region $R(\mu, v)$ with the properties described in Definition 3 which separates v from the rest of the drawing, thus avoiding an edge-region crossing between e and v . More precisely, by definition of outerclustered graph, v has exactly two incident edges $e_1(v)$ and $e_2(v)$ belonging to $o(G)$. Denote by $u(e_1(v))$ and $u(e_2(v))$ the end-vertices of $e_1(v)$ and $e_2(v)$ belonging to v . Denote by $p(l_1)$ the endpoint of $l_1(\mu, v)$ that lies on $e_1(v)$ (if both endpoints of $l_1(\mu, v)$ lie on $e_1(v)$, then $p(l_1)$ is the one that is closer to $u(e_1(v))$). Note that an endpoint of $l_1(\mu, v)$ lying on $e_1(v)$ exists as Γ satisfies Property CS3. Analogously define $p(l_2)$. Then, segment $\overline{p(l_1)p(l_2)}$ splits P into two disjoint convex polygons P' and P'' , such that, out of $o(G)$ and of path $(u_1, u_2, \dots, u_x)$, P' contains all and only the vertices in v and P'' contains all and only the vertices not in v , as Γ satisfies Property CS3. By the convexity of P' and P'' , e is internal to P'' , while the part of v inside P is internal to P' . Hence, e does not cross v .

We prove that Γ satisfies Property CS1 of Definition 3. Denote by P_i the polygon representing C_i in Γ , for each $i \in \{1, 2, 3\}$. Every angle that is incident to a vertex in C different from u_1 , u_r , and v_x in P_1 , P_2 , or P_3 is no more than 180° , since it is the same angle as in P . Every angle that is incident to a vertex in the path $(u_1, u_2, \dots, u_r)$ different from u_1 , u_x , and u_r in P_1 , P_2 , or P_3 is exactly 180° , by construction. Every angle incident to u_1 , v_x , and u_r in P_1 , P_2 , or P_3 is strictly less than 180° , since it

is strictly less than an angle that is at most 180° (namely the angle incident to the same vertex in P); finally, the three angles $\widehat{u_1 u_x v_x}$, $\widehat{u_1 u_x u_r}$, and $\widehat{u_r u_x v_x}$ incident to u_x are all strictly less than 180° , since they are angles in the non-degenerate triangles $T(u_1 u_x v_x)$, $T(u_1 u_x u_r)$, and $T(u_r u_x v_x)$, respectively.

We prove that Γ satisfies Property CS2. Concerning the drawing of C_1 (the arguments for the drawing of C_2 being analogous), observe that C_1 is linearly-ordered according to the subsequence Σ_1 of Σ that starts at $\sigma(v_i) = \mu_1$ and ends at the one of $\sigma(u_x)$ and $\sigma(v_x)$ that comes after in Σ . The angle incident to v_i in P_1 is strictly less than 180° , by hypothesis; further, as proved above, the angles incident to u_x and v_x in P_1 are strictly less than 180° . Concerning the drawing of C_3 , observe that $\sigma(u_1)$ and $\sigma(u_r)$ are the first and the last cluster in Σ_3 . Further, the angles incident to u_1 and to u_r in P_3 are strictly smaller than 180° , as proved above.

We prove that Γ satisfies Property CS3. For each $i = 1, 2, 3$, let μ_j be any cluster (except for the first and the last one) belonging to the sequence according to which C_i is linearly-ordered and let μ'_j be any child of μ_j . The existence of a region $R(\mu_j, \mu'_j)$ inside P_i is easily deduced from the existence of region $R(\mu_j, \mu'_j)$ inside P . Namely, either $R(\mu_j, \mu'_j)$ inside P is intersected neither by path $(u_1, u_2, \dots, u_r)$ nor by edge (u_x, v_x) , implying that a region $R(\mu_j, \mu'_j)$ inside P_i can be constructed coincident with the same region inside P , or $R(\mu_j, \mu'_j)$ inside P is intersected by one or both of path $(u_1, u_2, \dots, u_r)$ and edge (u_x, v_x) , that thus split $R(\mu_j, \mu'_j)$ into two or three regions inside the polygons P_i . The properties that have to be satisfied by $R(\mu_j, \mu'_j)$ inside P_i easily descend from the analogous properties satisfied by $R(\mu_j, \mu'_j)$ inside P . $\square$

Graphs C_1 , C_2 , and C_3 are, in general, not triconnected; namely, there could exist chords between any vertex in $(u_1, u_2, \dots, u_{x-1})$ and any vertex in $(u_{x+1}, u_{x+2}, \dots, u_r)$, or chords between any vertex in $(u_1, u_2, \dots, u_r)$ and any vertex in the path of C connecting u_1 and u_r and not containing v_x , or chords between u_x and any vertex in the path of C connecting u_1 and u_r and containing v_x . By Lemma 2, each of these chords splits a linearly-ordered outerclustered graph into two smaller linearly-ordered outerclustered graphs. Further, by construction the end-vertices of each of such chords are not collinear with any other vertex of the cycle. Hence, by Lemma 3, drawing the chords as straight-line segments splits the drawings of C_1 , C_2 , and C_3 , that are convex-separated by Lemma 10, into convex-separated drawings. When all the chords have been added, the underlying graphs of the resulting internally-triangulated linearly-ordered outerclustered graphs are all triconnected. Hence, the induction applies and an internally-convex-separated drawing of each of such linearly-ordered clustered graphs can be constructed inside the corresponding outer face, thus obtaining an internally-convex-separated drawing of C .

Case 2 applies when the vertices of $V_2 \cup \{v_i, v_j\}$ are not all collinear. In such a case, $V_2 \neq \emptyset$ and a path $(u_1, u_2, \dots, u_x, \dots, u_r)$ can be found as in Lemma 6. Analogously to Case 1, C is decomposed into smaller triconnected internally-triangulated linearly-ordered outerclustered graphs; path $(u_1, u_2, \dots, u_x, \dots, u_r)$ and all the edges connecting vertices of such a path with vertices of C can be drawn so that the outer faces of the corresponding outerclustered graphs are represented by convex-separated

Algorithm 2 DRAWING LINEARLY-ORDERED OUTERCLUSTERED GRAPHS

Input: $C(G, T)$, $\Gamma(C_o)$, Σ , $V_1 \neq \emptyset$, $\mathcal{P}_1$, and $\mathcal{P}_2$.

Output: An internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.

```

1:  $\mathcal{P}_u = (u_1, \dots, u_x, \dots, u_r) \leftarrow$  the path obtained from Algorithm 1 with input
    $C(G, T)$ ,  $\Sigma$ ,  $V_1 \neq \emptyset$ ,  $\mathcal{P}_1$ , and  $\mathcal{P}_2$ ;
2: for  $2 \leq i \leq r - 1$  do
3:    $\sigma'(u_i) \leftarrow$  any cluster in  $\Sigma$  child of  $\sigma(u_i)$ ;
4: end for
5:  $v_x \leftarrow$  any vertex in  $V_1$  adjacent to  $u_x$ ;
6:  $D \leftarrow$  disk in the interior of  $T(v_x, u_1, u_r)$ ; place  $u_x$  at any point inside  $D$ ;
7: for  $2 \leq i \leq x - 1$  do
8:   place  $u_i$  at any internal point of  $\overline{u_1 u_x} \cap R(\sigma(u_i), \sigma'(u_i))$ ;
9: end for
10: for  $x + 1 \leq i \leq r - 1$  do
11:   place  $u_i$  at any internal point of  $\overline{u_x u_r} \cap R(\sigma(u_i), \sigma'(u_i))$ ;
12: end for
13: draw each edge between two vertices among the ones in  $\mathcal{C}$  and in  $\mathcal{P}_u$  as a straight-
    line segment;
14: apply Algorithm 2 on each resulting linearly-ordered outerclustered graph com-
    pleting a drawing of the corresponding outer face;
15:  $\Gamma(C) \leftarrow$  the resulting internally-convex-separated drawing of  $C$ ;
16: return  $\Gamma(C)$ ;

```

drawings. Hence, the induction applies and an internally-convex-separated drawing of each of such linearly-ordered clustered graphs can be constructed inside the corresponding outer face, thus obtaining an internally-convex-separated drawing of C .

A pseudo-code description of the algorithm for drawing a linearly-ordered outerclustered graph C (supposing that $V_1 \neq \emptyset$) is presented in Algorithm 2.

It remains to prove that one out of Case 1 and Case 2 applies. If Case 1 does not apply, then the vertices of $V_1 \cup \{v_i, v_j\}$ are collinear, and if Case 2 does not apply, then the vertices of $V_2 \cup \{v_i, v_j\}$ are collinear. However, this implies that all the vertices of $\mathcal{C}$ are collinear, contradicting the fact that $\Gamma(C_o)$ is a convex-separated drawing. This concludes the proof of Theorem 1. $\square$

4 Drawing Maximal Outerclustered Graphs

In this section we show how to draw maximal outerclustered graphs, exploiting the results obtained on linearly-ordered outerclustered graphs. Namely, in order to show that any maximal outerclustered graph has an internally-convex-separated drawing completing an arbitrary triangular-convex-separated drawing of its outer face, we show how to reduce the problem of drawing a maximal outerclustered graph to the one of drawing some linearly-ordered outerclustered graphs.

First, we need the following preliminary results.

Fig. 17 Illustration for the proof of Lemma 11

Lemma 11 Let $C(G, T)$ be a maximal outerclustered graph and let u , v , and z be the vertices incident to $o(G)$. Let $u_i \neq u$ be any internal vertex of G such that $\sigma(u_i)$ contains u and contains neither v nor z . Then, there exists an edge (u_i, u_{i+1}) in G such that $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$.

Proof Refer to Fig. 17. Let μ_{i+1} be the only cluster child of $\sigma(u_i)$ and containing u . Notice that such a child exists. Namely, if such a child does not exist, then $\sigma(u_i) = \sigma(u)$; however, this would imply that $u_i = u$, since $\sigma(u)$ induces a connected graph, by the c -planarity and the maximality of C , and since an outerclustered graph does not contain two adjacent vertices having the same smallest containing cluster if one of them is internal, by Property O3 of Definition 1. Suppose, for a contradiction, that u_i is not adjacent to any vertex in μ_{i+1} . By Property O3, u_i is not adjacent to any vertex v_j such that $\sigma(v_j) = \sigma(u_i)$. It follows that each edge e_l incident to u_i intersects the boundary $B(\sigma(u_i))$ of $\sigma(u_i)$ in a point p_l belonging to the polygonal line $l(\sigma(u_i))$ that is the part of $B(\sigma(u_i))$ lying in the interior of $o(G)$. Order the points p_l as they are encountered walking on $l(\sigma(u_i))$ from one endpoint to the other one. Denote by $p_1, p_2, \dots, p_m$ such points. By the maximality of G and by the fact that G contains internal vertices, every vertex of G has degree at least three. Hence, $m \geq 3$ and there exists an internal face delimited by a 3-cycle containing edges e_1 and e_m . However, the third edge e^* of such a cycle crosses twice $l(\sigma(u_i))$ and hence $B(\sigma(u_i))$, contradicting the c -planarity of the considered embedding. $\square$

Corollary 1 Let $C(G, T)$ be any maximal outerclustered graph and let u , v , and z be the vertices incident to $o(G)$. Suppose that $\sigma(u) \neq \sigma(v), \sigma(z)$. Let u_1 be any internal vertex of G such that $\sigma(u_1)$ contains u and contains neither v nor z . Then, there exists a chordless path $(u_1, u_2, \dots, u_k)$ in G such that $u_k = u$ and such that $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$, for each $i = 1, 2, \dots, k - 1$.

Proof Suppose that a path $(u_1, u_2, \dots, u_i)$ has already been determined such that u_{j+1} is a descendant of u_j , for each $j = 1, 2, \dots, i - 1$. By Lemma 11, there exists

Algorithm 3 CHORDLESS PATHS IN MAXIMAL OUTERCLUSTERED GRAPHS

Input: $C(G, T)$, u , and u_1 , as in Corollary 1.

Output: A path $(u_1, u_2, \dots, u_k)$ such that $u_k = u$ and such that $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$.

```

1:  $u_j \leftarrow u_1$ ;
2: while  $u_j \neq u$  do
3:    $u_i \leftarrow u_j$ ;
4:    $(u_i, u_{i+1}) \leftarrow$  an edge in  $G$  such that  $\sigma(u_{i+1})$  is a descendant of  $\sigma(u_i)$ ;
5:   add  $(u_i, u_{i+1})$  to the current path;
6:    $u_j \leftarrow u_{i+1}$ ;
7: end while
8:  $(u_1, u_2, \dots, u_k) \leftarrow$  the path obtained by Algorithm 4 on  $G$  and the current path;
9: return  $(u_1, u_2, \dots, u_k)$ ;

```

Algorithm 4 REPLACE CHORDS IN PATHS

Input: A graph G and a path $(u_1, u_2, \dots, u_k)$ in G .

Output: A chordless path in G whose vertices are a subset of $\{u_1, u_2, \dots, u_k\}$ in the same order as in $(u_1, u_2, \dots, u_k)$.

```

1: while the current path  $(u_1, u_2, \dots, u_k)$  has a chord  $(u_i, u_j)$  in  $G$  do
2:   replace the subpath of  $(u_1, u_2, \dots, u_k)$  between  $u_i$  and  $u_j$  with edge  $(u_i, u_j)$ ;
3: end while
4: return the obtained path;

```

an edge (u_i, u_{i+1}) in G such that $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$. Repeating such an argument eventually leads to choose a vertex u_k in $\sigma(u)$. Since $\sigma(u) \neq \sigma(v)$, $\sigma(z)$, then, by Property O3 of Definition 1, u is the only vertex in $\sigma(u)$. It follows that $u_k = u$. The obtained path $\mathcal{P}_u$ may have chords. Suppose that the currently considered path $\mathcal{P}_u^l = (u_1^l, u_2^l, \dots, u_k^l)$ has a chord (u_i^l, u_j^l) , with $j > i + 1$ and with $\mathcal{P}_u^l = \mathcal{P}_u$. Obtain a new path $\mathcal{P}_u^{l+1}$ by replacing the subpath $(u_i^l, u_{i+1}^l, \dots, u_j^l)$ of $\mathcal{P}_u^l$ with the chord (u_i^l, u_j^l) . Clearly, $\sigma(u_j^l)$ is a descendant of $\sigma(u_i^l)$. Further, $\mathcal{P}_u^{l+1}$ has at least one chord less than $\mathcal{P}_u^l$ which implies that, after a certain number of steps, the current path is chordless. $\square$

A pseudo-code description of the algorithm for finding a path satisfying the requirements of Corollary 1 in a maximal outerclustered graph is presented in Algorithms 3–4.

Let $C(G, T)$ be a maximal outerclustered graph and let u , v , and z be the vertices incident to $o(G)$. Suppose that: (i) $\sigma(u) \neq \sigma(v)$, $\sigma(u) \neq \sigma(z)$, and $\sigma(v) \neq \sigma(z)$; (ii) if there exists a cluster containing exactly two vertices incident to $o(G)$, then such vertices are u and v ; (iii) if the smallest containing cluster of one of u and v contains the other one, then $\sigma(v)$ contains u ; and (iv) G has internal vertices. The following lemma states that in a maximal outerclustered graph some edge-disjoint

Fig. 18 (a)–(c) Three paths satisfying Condition 1 of Lemma 12. In (a) $\sigma(z_1)$ is a cluster containing z and not containing u and v ; in (b) $\sigma(z_1) = \sigma(u, v, z)$; in (c) $\sigma(z_1)$ is a cluster not containing z and containing u and v . (d) Two paths satisfying Condition 2 of Lemma 12

paths can be found satisfying certain properties of “monotonicity” with respect to the cluster hierarchy. After the lemma, it will be shown how such paths, together with their chords, split C into linearly-ordered outerclustered graphs.

Lemma 12 *One of the following holds:*

1. *There exist three paths $\mathcal{P}_u = (u_1, u_2, \dots, u_U)$, $\mathcal{P}_v = (v_1, v_2, \dots, v_V)$, and $\mathcal{P}_z = (z_1, z_2, \dots, z_Z)$ such that (see Figs. 18(a)–18(c)):*
 - (a) $u_U = u$, $v_V = v$, $z_Z = z$, and $u_1 = v_1 = z_1$;
 - (b) *the vertices of $\mathcal{P}_u \setminus \{u_1\}$, $\mathcal{P}_v \setminus \{v_1\}$, and $\mathcal{P}_z \setminus \{z_1\}$ are distinct;*
 - (c) *each of paths $\mathcal{P}_u \setminus \{u_1\}$, $\mathcal{P}_v \setminus \{v_1\}$, and $\mathcal{P}_z$ has no chords;*
 - (d) $\sigma(u_i)$ contains neither v nor z , for each $2 \leq i \leq U$; $\sigma(v_i)$ contains neither u nor z , for each $2 \leq i \leq V$; $\sigma(z_i)$ contains neither u nor v , for each $Z^* \leq i \leq Z$, where Z^* is an index such that $1 \leq Z^* \leq Z$;
 - (e) $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$, for each $2 \leq i \leq U - 1$; $\sigma(v_{i+1})$ is a descendant of $\sigma(v_i)$, for each $2 \leq i \leq V - 1$;
 - (f) $\sigma(z_1)$ is either a cluster containing z and not containing u and v (then $Z^* = 1$ and $\sigma(z_{i+1})$ is a descendant of $\sigma(z_i)$, for each $1 \leq i \leq Z - 1$), or $\sigma(u, v, z)$ (then $Z^* = 2$ and $\sigma(z_{i+1})$ is a descendant of $\sigma(z_i)$, for each $1 \leq i \leq Z - 1$), or a cluster not containing z and containing u and v . In the latter case $Z^* \geq 2$, $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$, for each $1 \leq i \leq Z^* - 2$, $\sigma(z_{i+1})$ is a descendant of $\sigma(z_i)$, for each $Z^* \leq i \leq Z - 1$, and either $\sigma(z_{Z^*})$ is a descendant

- of $\sigma(z_{Z^*-1})$ (if $\sigma(z_{Z^*-1}) = \sigma(u, v, z)$), or $\sigma(z_{Z^*})$ is not comparable with $\sigma(z_{Z^*-1})$ (if $\sigma(z_{Z^*-1})$ contains u and v and does not contain z);
- (g) contains an internal face having incident vertices u_1, u_2 , and v_2 .
2. There exist two paths $\mathcal{P}_u = (u_1, u_2, \dots, u_U)$ and $\mathcal{P}_v = (v_1, v_2, \dots, v_V)$ such that (see Fig. 18(d)):
- $u_U = u, v_V = v$, and $u_1 = v_1 = z$;
 - the vertices of $\mathcal{P}_u \setminus \{u_1\}$ and $\mathcal{P}_v \setminus \{v_1\}$ are distinct;
 - each of paths $\mathcal{P}_u \setminus \{u_1\}$ and $\mathcal{P}_v \setminus \{v_1\}$ has no chords;
 - $\sigma(u_i)$ contains neither v nor z , for each $2 \leq i \leq U$; $\sigma(v_i)$ contains neither u nor z , for each $2 \leq i \leq V$;
 - $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$, for each $2 \leq i \leq U - 1$; $\sigma(v_{i+1})$ is a descendant of $\sigma(v_i)$, for each $2 \leq i \leq V - 1$;
 - G contains an internal face having incident vertices u_2, v_2 , and z .

Proof Consider the biggest cluster μ_u containing u and not containing v . Notice that such a cluster exists. Namely, $\sigma(u)$ does not contain v by hypothesis. Consider the biggest cluster μ_v containing v and not containing u , if any such a cluster exists. If μ_v exists, let E' be the set of edges whose end-vertices are one in μ_u and one in μ_v . If μ_v does not exist, let E' be the set of edges incident to v whose other end-vertex is in μ_u . In both cases, there exists at least one of such edges, in fact (u, v) . Order the edges in E' as they are encountered when walking on the part $l(\mu_u)$ of the boundary $B(\mu_u)$ of μ_u that is internal to G , starting from the intersection of (u, v) with $B(\mu_u)$. See Fig. 19.

Consider the last edge (u', v') in E' and consider the internal face (u', v', z') such that (u', z') is the edge following (u', v') in the order in which the edges incident to μ_u are encountered when walking on the part of $B(\mu_u)$ that is internal to G , starting from the intersection of (u, v) with $B(\mu_u)$. Let $\mathcal{P}_u = (u_1, u_2, \dots, u_U)$ be a path such that $u_1 = z', u_2 = u'$, and $(u_2, u_3, \dots, u_U)$ is a chordless path such that $u_U = u$ and $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$, for $2 \leq i \leq U - 1$. Such a path exists by Corollary 1 (notice that $\sigma(u_2)$ contains u and contains neither v nor z). Further, let $\mathcal{P}_v = (v_1, v_2, \dots, v_V)$ be a path such that $v_1 = z', v_2 = v'$, and $(v_2, v_3, \dots, v_V)$ is a chordless path such that $v_V = v$ and $\sigma(v_{i+1})$ is a descendant of $\sigma(v_i)$, for $2 \leq i \leq$

Fig. 19 The edges in E' . The numbers on such edges indicate the order in which such edges are encountered when walking on the part of $B(\mu_u)$ internal to G , starting from the intersection of (u, v) with $B(\mu_u)$. (a) μ_v exists. (b) μ_v does not exist

Fig. 20 If $z' = z$, then paths P_u and P_v satisfy Condition 2 of Lemma 12

Fig. 21 If $z' \neq z$ and $\sigma(z')$ contains z and contains neither u nor v , then paths P_u , P_v , and P_z satisfy Condition 1 of Lemma 12

$V - 1$. Notice that if μ_v exists, then such a path exists by Corollary 1 (notice that $\sigma(v_2)$ contains v and contains neither u nor z). Otherwise, $v_2 = v_V = v$.

Suppose that $z' = z$ (see Fig. 20). Then, G contains an internal face having incident vertices u_2 , v_2 , and z . By construction, paths P_u and P_v satisfy Condition 2 of the lemma.

Suppose that $z \neq z'$. Vertex z' belongs neither to μ_u nor to μ_v , otherwise edge (z', v') or edge (z', u') would follow (u', v') in E' , a contradiction. Hence, $\sigma(z')$ is either a cluster containing z and not containing u and v , or is $\sigma(u, v, z)$, or is a cluster containing u and v and not containing z .

- Suppose that $\sigma(z')$ contains z and contains neither u nor v (see Fig. 21). Let $P_z = (z_1, z_2, \dots, z_Z)$ be a chordless path such that $z_1 = z'$, $z_Z = z$, and $\sigma(z_{i+1})$ is a descendant of $\sigma(z_i)$, for $2 \leq i \leq Z - 1$. Such a path exists by Corollary 1 (notice

Fig. 22 If $z' \neq z$, if $\sigma(z') = \sigma(u, v, z)$, and if a cluster containing both u and v and not containing z exists, then μ_z exists

that $\sigma(z_1)$ contains z and contains neither u nor v). By construction, paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfy Condition 1 of the lemma.

- Suppose that $\sigma(z') = \sigma(u, v, z)$. Let μ_z be the biggest cluster containing z and containing neither u nor v .

We claim that μ_z exists. Suppose, for a contradiction, that μ_z does not exist. Then, every cluster containing z also contains one of u and v ; therefore, any such a cluster contains both u and v , by the assumptions of the lemma. Hence, $\sigma(z) = \sigma(u, v, z) = \sigma(z')$, and $\sigma(u, v, z)$ is the smallest containing cluster of at least two vertices, namely z and z' . We prove that there exists an edge between two vertices whose smallest containing cluster is $\sigma(z)$, thus contradicting the fact that C satisfies Property O3, and hence proving that μ_z exists. Consider the edges incident to z' .

First, suppose that a cluster containing both u and v and not containing z exists (see Fig. 22). Let $\mu_{u,v}$ be the biggest cluster containing both u and v and not containing z . Then, if no neighbor of z' has $\sigma(z')$ as smallest containing cluster, each edge e_l incident to z' intersects the boundary $B(\mu_{u,v})$ of $\mu_{u,v}$ in a point p_l belonging to the polygonal line $l(\mu_{u,v})$ that is the part of $B(\mu_{u,v})$ lying in the interior of $o(G)$. Order the points p_l as they are encountered when walking on $l(\mu_{u,v})$ from one endpoint to the other one. Denote by $p_1, p_2, \dots, p_m$ such points. By the maximality of G and by the fact that G contains internal vertices, every vertex of G has degree at least three. Hence, $m \geq 3$ and there exists a face delimited by a 3-cycle containing edges e_1 and e_m . However, the third edge e^* of such a cycle crosses twice $l(\mu_{u,v})$ and hence $B(\mu_{u,v})$, contradicting the c -planarity of the considered embedding. Second, suppose that a cluster containing both u and v and not containing z does not exist (see Fig. 23). Then, if μ_v exists and if no neighbor of z' has $\sigma(z')$ as smallest containing cluster, each edge e_l incident to z' is also incident either to a vertex belonging to μ_u or to a vertex belonging to μ_v ; if μ_v does not exist and if no neighbor of z' has $\sigma(z')$ as smallest containing cluster, each edge e_l incident to z' is also incident to a vertex belonging to μ_u .

Fig. 23 If $z' \neq z$, if $\sigma(z') = \sigma(u, v, z)$, and if a cluster containing both u and v and not containing z does not exist, then μ_z exists

By the c -planarity of the embedding, the order of the edges incident to z' is a sequence $(e_1, e_2, \dots, e_i, e_{i+1}, \dots, e_m)$ so that: (i) e_j is incident to a vertex in μ_u , for $j = 1, 2, \dots, i$, (ii) $e_i = (z', u')$, (iii) $e_{i+1} = (z', v')$, and (iv) e_j is incident to a vertex in μ_v , for $j = i + 1, i + 2, \dots, m$. Observe that, if μ_v does not exist, $e_{i+1} = (z', v)$ and $m = i + 1$. By the maximality of G and by the fact that G contains internal vertices, every vertex of G has degree at least three. Hence, $m \geq 3$ and there exists a face delimited by a 3-cycle containing edges e_1 and e_m . However, the third edge e^* of such a cycle is an edge between a vertex in μ_u and a vertex in μ_v (if μ_v exists) or is an edge between a vertex in μ_u and v (if μ_v does not exist), contradicting the fact that (u', v') is the last edge in E' . This completes the proof of the claim that μ_z exists.

Next, we claim that there exists an edge incident to z' and to a vertex belonging to μ_z . The proof of such a claim can be done analogously to the proof that μ_z exists, namely by supposing, for a contradiction, that no edge incident to z' is incident to a vertex belonging to μ_z , and by showing that this implies that the considered embedding is not c -planar (if a cluster containing both u and v and not containing z exists), or that (u', v') is not the last edge in E' (if a cluster containing both u and v and not containing z does not exist).

Then, there exists an edge (z', z_2) incident to z' such that z_2 belongs to μ_z . Let $(z_2, z_3, \dots, z_Z)$ be a chordless path such that $z_Z = z$ and z_{i+1} is a descendant of z_i , for $2 \leq i \leq Z - 1$. Such a path exists by Corollary 1 (notice that $\sigma(z_2)$ contains z and contains neither u nor v). Consider path $(z_1, z_2, z_3, \dots, z_Z)$. For each chord (z_1, z_j) , with $j > 2$, replace the subpath of the current path by edge (z_1, z_j) . This results in a chordless path $\mathcal{P}_z$. By construction, paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfy Condition 1 of the lemma.

- Suppose that $\sigma(z')$ contains u and v and does not contain z . In the following we show how to find a path $\mathcal{P}_z^1 = (z_1, z_2, \dots, z_{Z^*})$ such that: (i) $z_1 = z'$, (ii) $\sigma(z_i)$ contains u and v and does not contain z , for $1 \leq i \leq Z^* - 2$, (iii) $\sigma(z_{Z^*-1})$ is

either $\sigma(u, v, z)$ or a cluster containing u and v and not containing z , (iv) zZ^* belongs to μ_z , and (v) $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$, for each $1 \leq i \leq Z^* - 2$.

Assume that $\mathcal{P}_z^1$ has already been determined till a vertex z_i . We claim that there exists an edge incident to z_i and to a vertex z_{i+1} such that either $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$ containing u and v and not containing z , or $\sigma(z_{i+1}) = \sigma(u, v, z)$, or $\sigma(z_{i+1})$ contains z and does not contain u and v .

Suppose, for a contradiction, that every edge incident to z_i is incident to a vertex belonging to μ_u , or to a vertex belonging to μ_v , or to a vertex belonging to a cluster that contains u and v , that does not contain z , and that is a descendant of $\sigma(z_i)$. Observe that vertex z_i is not adjacent to any vertex v_j such that $\sigma(v_j) = \sigma(z_i)$, by Property O3 of Definition 1.

First, consider the case in which $\sigma(z_i) \neq \sigma(u, v)$, that is, there exists a child $\mu(u, v, \neg z_i)$ of $\sigma(z_i)$ in T containing u and v and not containing z . Then, each edge incident to z_i is also incident to a vertex belonging to $\mu(u, v, \neg z_i)$. Analogously as above, a contradiction can be reached by proving that there exists an edge e^* that crosses twice the boundary $B(\mu(u, v, \neg z_i))$ of $\mu(u, v, \neg z_i)$.

Second, consider the case in which $\sigma(z_i) = \sigma(u, v)$. Then, a contradiction can be reached by proving that: (i) if all the edges incident to z_i are also incident to vertices belonging to μ_u (or if all the edges incident to z_i are also incident to vertices belonging to μ_v), then there exists an edge that crosses twice the boundary $B(\mu_u)$ of μ_u (resp. the boundary $B(\mu_v)$ of μ_v); (ii) if some edges incident to z_i are also incident to vertices in μ_u and some edges incident to z_i are also incident to vertices in μ_v , then (u', v') is not the last edge in E' .

This proves the claim, namely that there exists an edge incident to z_i and to a vertex z_{i+1} such that either $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$ containing u and v and not containing z , or $\sigma(z_{i+1}) = \sigma(u, v, z)$, or $\sigma(z_{i+1})$ contains z and does not contain u and v . The repetition of such an argument eventually leads to the choice of a vertex zZ^* that belongs to μ_z or to the choice of a vertex zZ^*-1 such that $\sigma(zZ^*-1) = \sigma(u, v, z)$. If zZ^* belongs to μ_z , then $\mathcal{P}_z^1$ has already been determined satisfying the desired properties. If $\sigma(zZ^*-1) = \sigma(u, v, z)$ then the same argument as above shows that there exists an edge (zZ^*-1, zZ^*) such that zZ^* belongs to μ_z , thus obtaining a path $\mathcal{P}_z^1$ satisfying the desired properties.

Let $\mathcal{P}_z^2 = (zZ^*, zZ^*+1, \dots, zZ)$ be a chordless path such that $zZ = z$, and z_{i+1} is a descendant of z_i , for $Z^* \leq i \leq Z - 1$. Such a path exists by Corollary 1 (notice that $\sigma(zZ^*)$ contains z and contains neither u nor v).

Finally, consider the path obtained by concatenating $\mathcal{P}_z^1$ and $\mathcal{P}_z^2$. Such a path may eventually have chords. For each chord (z_i, z_j) , with $j > i + 1$, replace the subpath of the current path by edge (z_i, z_j) . This results in a chordless path $\mathcal{P}_z$. By construction, paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfy Condition 1 of the lemma. $\square$

A pseudo-code description of the algorithm for finding three paths satisfying Condition 1 of Lemma 12 or two paths satisfying Condition 2 of Lemma 12 is presented in Algorithm 5.

As in Lemma 12, let $C(G, T)$ be a maximal outerclustered graph, let u , v , and z be the vertices incident to $o(G)$, and suppose that: (i) $\sigma(u) \neq \sigma(v)$, $\sigma(u) \neq \sigma(z)$, and

Algorithm 5 TWO OR THREE PATHS IN MAXIMAL OUTERCLUSTERED GRAPHS**Input:** $C(G, T)$, as in Lemma 12.**Output:** Paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfying Condition 1 of Lemma 12, or paths $\mathcal{P}_u$ and $\mathcal{P}_v$ satisfying Condition 2 of Lemma 12

```

1:  $\mu_u \leftarrow$  biggest cluster containing  $u$  and not containing  $v$ ;
2:  $\mu_v \leftarrow$  biggest cluster containing  $v$  and not containing  $u$  (if it exists);
3: if  $\mu_v$  exists then
4:    $E' \leftarrow$  set of edges whose end-vertices are one in  $\mu_u$  and one in  $\mu_v$ ;
5: else
6:    $E' \leftarrow$  set of edges incident to  $v$  whose other end-vertex is in  $\mu_u$ ;
7: end if
8:  $(u', v') \leftarrow$  last of the edges in  $E'$  incident to  $l(\mu_u)$  starting from  $(u, v)$ ;
9:  $z' \leftarrow$  vertex forming a face with  $(u', v')$ ;
10:  $\mathcal{P}_u \leftarrow$  edge  $(z', u')$  plus the path obtained by Algorithm 3 on  $C$ ,  $u$ , and  $u'$ ;
11:  $\mathcal{P}_v \leftarrow$  edge  $(z', v')$  plus the path obtained by Algorithm 3 on  $C$ ,  $v$ , and  $v'$ ;
12: if  $z' = z$  then
13:   return  $\mathcal{P}_u$  and  $\mathcal{P}_v$ ;
14: else
15:   if  $\sigma(z')$  contains  $z$  and contains neither  $u$  nor  $v$  then
16:      $\mathcal{P}_z \leftarrow$  the path obtained by Algorithm 3 on  $C$ ,  $z$ , and  $z'$ ;
17:     return  $\mathcal{P}_u$ ,  $\mathcal{P}_v$ , and  $\mathcal{P}_z$ ;
18:   end if
19:   if  $\sigma(z') = \sigma(u, v, z)$  then
20:      $(z', z_2) \leftarrow$  edge such that  $z_2$  is in a cluster containing  $z$  and neither  $u$  nor  $v$ ;
21:      $(z_2, z_3, \dots, z_Z) \leftarrow$  path obtained by Algorithm 3 on  $C$ ,  $z$ , and  $z_2$ ;
22:      $\mathcal{P}_z \leftarrow$  the path obtained by Algorithm 4 on  $G$  and  $(z', z_2, z_3, \dots, z_Z)$ ;
23:     return  $\mathcal{P}_u$ ,  $\mathcal{P}_v$ , and  $\mathcal{P}_z$ ;
24:   end if
25:   if  $\sigma(z')$  contains  $u$  and  $v$  and does not contain  $z$  then
26:      $z_j \leftarrow z'$ ;
27:     while  $\sigma(z_j)$  contains  $u$  and  $v$  and does not contain  $z$  do
28:        $z_i \leftarrow z_j$ ;  $(z_i, z_{i+1}) \leftarrow$  edge such that  $\sigma(z_{i+1})$  is not a descendant of  $\sigma(z_i)$ ;
29:       add  $(z_i, z_{i+1})$  to the current path;  $z_j \leftarrow z_{i+1}$ ;
30:     end while
31:     if  $\sigma(z_i) = \sigma(u, v, z)$  then
32:        $(z_i, z_{i+1}) \leftarrow$  edge such that  $\sigma(z_{i+1})$  contains  $z$  and neither  $u$  nor  $v$ ;
33:       add  $(z_i, z_{i+1})$  to the current path;
34:     end if
35:      $\mathcal{P}_z^1 = (z_1, z_2, \dots, z_{Z^*}) \leftarrow$  the current path;
36:      $\mathcal{P}_z^2 = (z_{Z^*}, z_{Z^*+1}, \dots, z_Z) \leftarrow$  path obtained by Algorithm 3 on  $C$ ,  $z$ , and  $z_{Z^*}$ ;
37:      $\mathcal{P}_z \leftarrow$  the path obtained by Algorithm 4 on  $G$  and  $\mathcal{P}_z^1$  plus  $\mathcal{P}_z^2$ ;
38:     return  $\mathcal{P}_u$ ,  $\mathcal{P}_v$ , and  $\mathcal{P}_z$ ;
39:   end if
40: end if

```

Fig. 24 Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ when Condition 1 of Lemma 12 holds. Thick edges show paths $\mathcal{P}_u$, $\mathcal{P}_v$, $\mathcal{P}_z$, and edges (u, v) , (u, z) , (v, z) , and (u_2, v_2)

$\sigma(v) \neq \sigma(z)$; (ii) if there exists a cluster containing exactly two vertices incident to $\sigma(G)$, then such vertices are u and v ; (iii) if the smallest containing cluster of one of u and v contains the other one, then $\sigma(v)$ contains u ; and (iv) G has internal vertices.

Suppose that Condition 1 of Lemma 12 holds (see Fig. 24). Denote by $C_{u,v}$, by $C_{u,z}$, and by $C_{v,z}$ the clustered graphs whose underlying graphs $G_{u,v}$, $G_{u,z}$, and $G_{v,z}$ are the subgraphs of G induced by the vertices incident to and internal to cycle $\mathcal{C}_{u,v} \equiv (u, v) \cup (\mathcal{P}_u \setminus \{u_1\}) \cup (u_2, v_2) \cup (\mathcal{P}_v \setminus \{v_1\})$, incident to and internal to cycle $\mathcal{C}_{u,z} \equiv (u, z) \cup \mathcal{P}_u \cup \mathcal{P}_z$, and incident to and internal to cycle $\mathcal{C}_{v,z} \equiv (v, z) \cup \mathcal{P}_v \cup \mathcal{P}_z$, and whose inclusion trees $T_{u,v}$, $T_{u,z}$, and $T_{v,z}$ are the subtrees of T induced by the clusters containing vertices of $G_{u,v}$, of $G_{u,z}$, and of $G_{v,z}$, respectively.

Lemma 13 $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ are linearly-ordered outerclustered graphs.

Proof We prove the statement for $C_{u,v}$, the proofs for $C_{u,z}$ and $C_{v,z}$ being analogous. Refer to Fig. 25. The outer face of $G_{u,v}$ is delimited by a simple cycle $\mathcal{C}_{u,v}$. We prove that the boundary of each cluster μ containing vertices of $C_{u,v}$ and not containing all the vertices of $C_{u,v}$ intersects $\mathcal{C}_{u,v}$ exactly twice. By construction, each cluster containing both u and v contains all the vertices of $G_{u,v}$ and each cluster containing neither u nor v does not contain any vertex of $G_{u,v}$. Finally, each cluster μ containing u and not containing v (the arguments for each cluster μ containing v and not containing u being analogous) intersects edge (u, v) exactly once, intersects path $(\mathcal{P}_u \setminus \{u_1\}) \cup (u_2, v_2)$ exactly once, and does not intersect path $\mathcal{P}_v \setminus \{v_1\}$. It follows that μ intersects $\mathcal{C}_{u,v}$ twice. By Lemma 1, $C_{u,v}$ is a biconnected internally-triangulated outerclustered graph.

Consider the sequence of clusters $\Sigma = \mu_1, \mu_2, \dots, \mu_k$ such that: (i) $\mu_1 = \sigma(u)$, (ii) μ_{i+1} is the parent of μ_i in $T_{u,v}$, for $i = 1, 2, \dots, h-2$, (iii) μ_{h-1} is the biggest cluster containing u and not containing v , (iv) μ_h is $\sigma(u, v)$, (v) μ_{h+1} is the biggest cluster containing v and not containing u , (vi) μ_{i+1} is the only child of μ_i in $T_{u,v}$ containing v , for $i = h+1, h+2, \dots, k-1$, and (vii) $\mu_k = \sigma(v)$. Notice that, if

Fig. 25 A closer look to $C_{u,v}$. Cluster sequence Σ is shown. (a) A cluster containing v and not containing u exists. (b) A cluster containing v and not containing u does not exist

a cluster containing v and not containing u does not exist, then $\mu_h = \mu_k$. In the following we prove that $C_{u,v}$ is linearly-ordered according to Σ .

We prove that $C_{u,v}$ satisfies Property LO1 of Definition 2. Each vertex x in $G_{u,v}$ either belongs to μ_{h-1} , or to μ_{h+1} , or is such that $\sigma(x) = \sigma(u, v)$. Namely, if μ_v exists, then all the vertices incident to $o(G_{u,v})$ either belong to μ_{h-1} or to μ_{h+1} , by construction; hence, all the vertices of $G_{u,v}$ belong to the smallest cluster containing both μ_{h-1} and μ_{h+1} , that is $\sigma(u, v)$, and do not belong to any cluster containing neither u nor v . If μ_v does not exist, then all the vertices incident to $o(G_{u,v})$, except for v , belong to μ_{h-1} and v belongs to $\sigma(u, v)$; hence, all the vertices of $G_{u,v}$ belong to the smallest cluster containing both μ_{h-1} and v , that is $\sigma(u, v)$, and do not belong to any cluster containing neither u nor v . Further, Σ includes all and only the clusters containing at least one out of u and v , with the exception of the clusters containing both u and v and ancestors of $\sigma(u, v)$; however, each cluster containing both u and v , and ancestor of $\sigma(u, v)$ is different from $\sigma(x)$, for any vertex x of $G_{u,v}$, since x is also contained in $\sigma(u, v)$. Since, for each internal vertex x of G , $\sigma(x)$ is a cluster containing at least one out of u and v , then $\sigma(x) = \mu_i$, for some $1 \leq i \leq k$.

We prove that $C_{u,v}$ satisfies Property LO2. Edge (u, v) and the path obtained by concatenating $\mathcal{P}_u \setminus \{u_1\}$, (u_2, v_2) , and $\mathcal{P}_v \setminus \{v_1\}$ are monotone paths delimiting $o(G_{u,v})$.

We prove that $C_{u,v}$ satisfies Property LO3. By construction, μ_{i+1} is the parent of μ_i , for $i = 1, 2, \dots, h-2$, and μ_{i+1} is a child of μ_i , for $i = h+1, h+2, \dots, k-1$. Hence, in order to prove that $C_{u,v}$ satisfies Property LO3, it suffices to show that μ_h is the parent of μ_{h-1} and that μ_h is the parent of μ_{h+1} . By construction, μ_{h-1} (resp. μ_{h+1}) is the biggest cluster containing u and not containing v (resp. containing v and not containing u). Hence, the parent of μ_{h-1} (resp. of μ_{h+1}) is $\sigma(u, v)$, that by definition is μ_h . $\square$

Suppose that Condition 2 of Lemma 12 holds (see Fig. 26).

Denote by $C_{u,v}$, by $C_{u,z}$, and by $C_{v,z}$ the clustered graphs whose underlying graphs $G_{u,v}$, $G_{u,z}$, and $G_{v,z}$ are the subgraphs of G induced by the vertices incident to and internal to cycle $\mathcal{C}_{u,v} \equiv (u, v) \cup (\mathcal{P}_u \setminus \{u_1\}) \cup (u_2, v_2) \cup (\mathcal{P}_v \setminus \{v_1\})$, incident to and internal to cycle $\mathcal{C}_{u,z} \equiv (u, z) \cup \mathcal{P}_u$, and incident to and internal to cycle $\mathcal{C}_{v,z} \equiv (v, z) \cup \mathcal{P}_v$, and whose inclusion trees $T_{u,v}$, $T_{u,z}$, and $T_{v,z}$ are the subtrees of T induced by the clusters containing vertices of $G_{u,v}$, of $G_{u,z}$, and of $G_{v,z}$,

Fig. 26 Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ when Condition 2 of Lemma 12 holds. Thick edges show paths $\mathcal{P}_u$, $\mathcal{P}_v$, and edges (u, v) , (u, z) , (v, z) , and (u_2, v_2)

respectively. We have the following lemma, whose proof is analogous to the one of Lemma 13.

Lemma 14 $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ are linearly-ordered outerclustered graphs.

We are now ready to exhibit the main theorem of this section.

Theorem 2 Let $C(G, T)$ be a maximal outerclustered graph. Then, for every triangular-convex-separated drawing $\Gamma(C_o)$ of C_o , there exists an internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.

Proof Let u , v , and z be the vertices incident to $o(G)$. Suppose that G has internal vertices, otherwise C_o and C are the same graph, and the statement trivially follows.

If $\sigma(u) = \sigma(v)$, then we claim that C is a linearly-ordered outerclustered graph. Refer to Fig. 27. Observe that C is a maximal outerclustered graph by hypothesis, hence G is biconnected and internally-triangulated.

Define the sequence of clusters $\Sigma = \mu_1, \mu_2, \dots, \mu_k$ such that: (i) $\mu_1 = \sigma(u) = \sigma(v)$, (ii) μ_{i+1} is the parent of μ_i in T , for $i = 1, 2, \dots, h-2$, (iii) μ_{h-1} is the biggest cluster containing u and v and not containing z , if any such a cluster exists, (iv) $\mu_h = \sigma(u, v, z)$, (v) μ_{h+1} is the biggest cluster containing z and not containing u and v , if any such a cluster exists, (vi) μ_{i+1} is the only child of μ_i in T containing z , for $i = h+1, h+2, \dots, k-1$, and (vii) $\mu_k = \sigma(z)$. We claim that C is linearly-ordered according to Σ .

We prove that C satisfies Property LO1 of Definition 2. By construction, Σ includes all the clusters containing at least one out of u , v , and z , with the exception of the clusters that contain all of u , v , and z and that are ancestors of $\sigma(u, v, z)$; however, each cluster containing all of u , v , and z and ancestor of $\sigma(u, v, z)$ is different from $\sigma(x)$, for any vertex x of G , since x is also contained in $\sigma(u, v, z)$. Since, for

Fig. 27 If $\sigma(u) = \sigma(v)$, then C is a linearly-ordered outerclustered graph

each vertex x of G , $\sigma(x)$ is a cluster containing at least one out of u , v , and z , by definition of outerclustered graph, then $\sigma(x) = \mu_i$, for some $1 \leq i \leq k$.

We prove that C satisfies Property LO2. Path (u, v, z) and edge (u, z) are monotone paths delimiting $o(G)$.

We prove that C satisfies Property LO3. By construction, μ_{i+1} is the parent of μ_i , for $i = 1, 2, \dots, h-2$, and μ_{i+1} is a child of μ_i , for $i = h+1, h+2, \dots, k-1$. Hence, in order to prove that C satisfies Property LO3, it suffices to show that μ_h is the parent of μ_{h-1} and that μ_h is the parent of μ_{h+1} . By construction, μ_{h-1} (resp. μ_{h+1}) is the biggest cluster containing u and v and not containing z (resp. containing z and not containing u and v). Hence, the parent of μ_{h-1} (resp. of μ_{h+1}) is $\sigma(u, v, z)$, that by definition is μ_h .

Analogously, if $\sigma(u) = \sigma(z)$ or if $\sigma(v) = \sigma(z)$, C is a linearly-ordered outerclustered graph. By Lemma 4, a triangular-convex-separated drawing of C_o is also a convex-separated drawing of C_o , hence in such cases the theorem directly follows from Theorem 1.

Now suppose that $\sigma(u) \neq \sigma(v)$, $\sigma(u) \neq \sigma(z)$, and $\sigma(v) \neq \sigma(z)$. Suppose also that, if there exists a cluster containing exactly two vertices incident to $o(G)$, then such vertices are u and v . The cases in which such vertices are u and z , or v and z can be treated analogously. Suppose finally that if the smallest containing cluster of one of u and v contains the other one, then $\sigma(v)$ contains u . The case in which $\sigma(u)$ contains v can be treated analogously.

By Lemma 12, there exist three paths $\mathcal{P}_u = (u_1, u_2, \dots, u_U)$, $\mathcal{P}_v = (v_1, v_2, \dots, v_V)$, and $\mathcal{P}_z = (z_1, z_2, \dots, z_Z)$ satisfying Condition 1 of Lemma 12, or there exist two paths $\mathcal{P}_u = (u_1, u_2, \dots, u_U)$ and $\mathcal{P}_v = (v_1, v_2, \dots, v_V)$ satisfying Condition 2 of Lemma 12.

Suppose that Condition 1 of Lemma 12 holds.

By Lemma 13, $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$ are linearly-ordered outerclustered graphs.

Fig. 28 Construction of drawings $\Gamma(C_{u,v}, T_{u,v})$, $\Gamma(C_{u,z}, T_{u,z})$, and $\Gamma(C_{v,z}, T_{v,z})$ when Condition 1 of Lemma 12 holds

Refer to Fig. 28. Suppose that $U > 2$ and that $V > 2$. Consider a child $\sigma'(z_1)$ of $\sigma(z_1)$ such that: (i) if $\sigma(z_1)$ contains z and does not contain u and v , then $\sigma'(z_1)$ is the unique child of $\sigma(z_1)$; (ii) if $\sigma(z_1) = \sigma(u, v, z)$, then $\sigma'(z_1)$ is the unique child of $\sigma(z_1)$ containing z ; (iii) if $\sigma(z_1)$ contains u and v and does not contain z , then $\sigma'(z_1)$ is the unique child of $\sigma(z_1)$ containing u . The existence of such clusters in each of the three cases can be proved as in the proof of Lemma 12. Notice that, in any case, one of the two polygonal lines obtained as intersection of the triangle representing (u, v, z) and $R(\sigma(z_1), \sigma'(z_1))$ lies on edge (u, z) . Consider a point $p(z_1)$ of $\text{int}(R(\sigma(z_1), \sigma'(z_1)))$ arbitrarily close to edge (u, z) . Place $u_1 = v_1 = z_1$ at $p(z_1)$.

Let $\sigma'(z_i)$ be any child of $\sigma(z_i)$ such that the boundary of $\sigma'(z_i)$ has intersection with (u, z) , for each $2 \leq i \leq Z - 1$. Notice that such a child is unique, except for the case in which $\sigma(z_i) = \sigma(u, v, z)$, in which $\sigma(z_i)$ may have two children satisfying the required properties, namely the biggest cluster containing u and not containing z and the biggest cluster containing z and not containing u . Draw a straight-line segment $\overline{p(z_1)z}$ and place z_i at any point of the segment $\text{int}(R(\sigma(z_i), \sigma'(z_i))) \cap \overline{p(z_1)z}$, for each $2 \leq i \leq Z - 1$.

Denote by $T(u, v, p(z_1))$ the triangle having u , v , and $p(z_1)$ as vertices. Denote also by $H(l(z, p(z_1)), u)$ (by $H(l(z, p(z_1)), v)$) the open half-plane delimited by the line through z and $p(z_1)$, and containing u (resp. containing v). Consider the unique child $\sigma'(u_2)$ of $\sigma(u_2)$. Notice that one of the two polygonal lines obtained as intersection of the triangle representing (u, v, z) and $R(\sigma(u_2), \sigma'(u_2))$ lies on edge (u, z) . Consider a point $p(u_2)$ in $H(l(z, p(z_1)), u) \cap \text{int}(T(u, v, p(z_1))) \cap \text{int}(R(\sigma(u_2), \sigma'(u_2)))$ arbitrarily close to edge $(u, p(z_1))$. Place u_2 at $p(u_2)$.

Let $\sigma'(u_i)$ be the unique child of $\sigma(u_i)$, for each $3 \leq i \leq U - 1$. Draw a straight-line segment $\overline{p(u_2)u}$ and place u_i at any point of the segment $\text{int}(R(\sigma(u_i), \sigma'(u_i))) \cap \overline{p(u_2)u}$, for each $3 \leq i \leq U - 1$.

Denote by $T(p(u_2), v, p(z_1))$ the triangle having $p(u_2)$, v , and $p(z_1)$ as vertices. Denote also by $H(l(u, p(u_2)), v)$ the open half-plane delimited by the line through

u and $p(u_2)$, and containing v . Consider the unique child $\sigma'(v_2)$ of $\sigma(v_2)$. Consider any point $p(v_2)$ in $H(l(z, p(z_1)), v) \cap H(l(u, p(u_2)), v) \cap \text{int}(T(p(u_2), v, p(z_1))) \cap \text{int}(R(\sigma(v_2), \sigma'(v_2)))$. Observe that, since $p(z_1)$ and $p(u_2)$ are arbitrarily close to edge (u, z) , both half-planes $H(l(z, p(z_1)), v)$ and $H(l(u, p(u_2)), v)$ entirely contain triangle $T(u, v, z)$, except for an arbitrarily small strip close to edge (u, z) . This guarantees that $H(l(z, p(z_1)), v) \cap H(l(u, p(u_2)), v) \cap \text{int}(T(p(u_2), v, p(z_1))) \cap \text{int}(R(\sigma(v_2), \sigma'(v_2)))$ is a convex non-empty region. Then, place v_2 at $p(v_2)$.

Let $\sigma'(v_i)$ be the unique child of $\sigma(v_i)$, for each $3 \leq i \leq V - 1$. Draw a straight-line segment $p(v_2)v$ and place v_i at any point of the segment $\text{int}(R(\sigma(v_i), \sigma'(v_i))) \cap p(v_2)v$, for each $3 \leq i \leq V - 1$. Straightforward modifications make the described algorithm work also for the cases in which $U = 2$ and/or $V = 2$. Such modifications are described in Algorithm 7 below.

Denote by $\Gamma(C_{u,v}, T_{u,v})$, by $\Gamma(C_{u,z}, T_{u,z})$, and by $\Gamma(C_{v,z}, T_{v,z})$ the constructed drawings of $(C_{u,v}, T_{u,v})$, of $(C_{u,z}, T_{u,z})$, and of $(C_{v,z}, T_{v,z})$, respectively. We have the following:

Lemma 15 $\Gamma(C_{u,v}, T_{u,v})$, $\Gamma(C_{u,z}, T_{u,z})$, and $\Gamma(C_{v,z}, T_{v,z})$ are convex-separated drawings of the outer faces of $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$, respectively.

Proof We prove the statement for $\Gamma(C_{u,v}, T_{u,v})$, the proof for $\Gamma(C_{u,z}, T_{u,z})$, and $\Gamma(C_{v,z}, T_{v,z})$ being analogous. Denote by P , $P_{u,v}$, $P_{u,z}$, and $P_{v,z}$ the polygons representing cycles $\mathcal{C}$, $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$, respectively. The drawing is straight-line and rectangular by construction. The absence of edge crossings easily descends from the construction. The absence of region-region crossings descends from the fact that no cluster is drawn by the algorithm.

We prove that $\Gamma(C_{u,v}, T_{u,v})$ has no edge-region crossings. Suppose, for a contradiction, that there is an edge-region crossing between an edge e and a cluster v . If both end-vertices of e belong to v then, by the convexity of v , e is internal to v ; if one end-vertex of e belongs to v then, by the convexity of v , e crosses v exactly once; hence, it can be assumed that both the end-vertices of e do not belong to v . Any cluster containing both u and v contains all the vertices of $G_{u,v}$, hence it contains all the drawing of $C_{u,v}$ and does not cross e . Hence, it can be assumed that v contains u and does not contain v , or vice versa. Suppose that v contains u and does not contain v , the other case being analogous. Consider the parent μ of v in $T_{u,v}$. Such a parent exists since otherwise v would be the root of T , contradicting the fact that v does not contain v . By definition of triangular-convex-separated drawing, there exists a convex region $R(\mu, v)$ with the properties described in Definition 4; one of the sides of such a region separates v from the rest of the drawing, thus avoiding an edge-region crossing between e and v . More precisely, since $C(G, T)$ is an outerclustered graph, v has exactly two incident edges $e_1(v)$ and $e_2(v)$ incident to $o(G)$. Denote by $u(e_1(v))$ and $u(e_2(v))$ the end-vertices of $e_1(v)$ and $e_2(v)$ belonging to v . Denote by $p(l_1)$ the endpoint of $l_1(\mu, v)$ that lies on $e_1(v)$ (if both endpoints of $l_1(\mu, v)$ lie on $e_1(v)$, then $p(l_1)$ is the one that is closer to $u(e_1(v))$). Note that an endpoint of $l_1(\mu, v)$ lying on $e_1(v)$ exists as Γ satisfies Property CS3. Analogously define $p(l_2)$. Then, segment $p(l_1)p(l_2)$ splits P into two disjoint convex polygons P' and P'' , where P' contains

Algorithm 6 DRAWING MAXIMAL OUTERCLUSTERED GRAPHS**Input:** $C(G, T)$ and $\Gamma(C_o)$.**Output:** An internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.

```

1: if  $G$  has no internal vertex then
2:    $\Gamma(C) \leftarrow \Gamma(C_o)$ ;
3: else
4:   if  $\sigma(u) = \sigma(v)$  or  $\sigma(u) = \sigma(z)$  or  $\sigma(v) = \sigma(z)$  then
5:      $\Gamma(C) \leftarrow$  the drawing obtained by applying Algorithm 2 on  $C$  and  $\Gamma(C_o)$ ;
6:   else
7:     apply Algorithm 5 on  $C$ , obtaining either three paths  $\mathcal{P}_u, \mathcal{P}_v$ , and  $\mathcal{P}_z$ , or two
       paths  $\mathcal{P}_u$  and  $\mathcal{P}_v$ ;
8:     for  $2 \leq i \leq U - 1$  do
9:        $\sigma'(u_i) \leftarrow$  the unique child of  $\sigma(u_i)$ ;
10:    end for
11:    for  $2 \leq i \leq V - 1$  do
12:       $\sigma'(v_i) \leftarrow$  the unique child of  $\sigma(v_i)$ ;
13:    end for
14:    if Algorithm 5 on  $C$  returns three paths  $\mathcal{P}_u = (u_1, \dots, u_U)$ ,  $\mathcal{P}_v =$ 
       $(v_1, \dots, v_V)$ , and  $\mathcal{P}_z = (z_1, z_2, \dots, z_Z)$  then
15:      for  $1 \leq i \leq Z - 1$  do
16:        if  $U > 2$  then
17:           $\sigma'(z_i) \leftarrow$  any child of  $\sigma(z_i)$  whose boundary has intersection with
             $(u, z)$ ;
18:        else
19:           $\sigma'(z_i) \leftarrow$  any child of  $\sigma(z_i)$  whose boundary has intersection with
             $(v, z)$ ;
20:        end if
21:      end for
22:       $\Gamma(C) \leftarrow$  the drawing obtained by applying Algorithm 7 on
         $C$ ,  $\mathcal{P}_u$ ,  $\sigma'(u_2), \dots, \sigma'(u_{U-1})$ ,  $\mathcal{P}_v$ ,  $\sigma'(v_2), \dots, \sigma'(v_{V-1})$ ,  $\mathcal{P}_z$ ,
         $\sigma'(z_1), \dots, \sigma'(z_{Z-1})$ , and  $\Gamma(C_o)$ ;
23:    end if
24:    if Algorithm 5 on  $C$  returns two paths  $\mathcal{P}_u = (u_1, \dots, u_U)$  and  $\mathcal{P}_v =$ 
       $(v_1, \dots, v_V)$  then
25:       $\Gamma(C) \leftarrow$  the drawing obtained by applying Algorithm 8 on  $C$ ,  $\mathcal{P}_u$ ,
         $\sigma'(u_2), \dots, \sigma'(u_{U-1})$ ,  $\mathcal{P}_v$ ,  $\sigma'(v_2), \dots, \sigma'(v_{V-1})$ , and  $\Gamma(C_o)$ ;
26:    end if
27:  end if
28: end if
29: return  $\Gamma(C)$ ;

```

all and only the vertices belonging to v and P'' contains all and only the vertices not belonging to v , as $\Gamma(C_o)$ satisfies Property CS3. By convexity, e is internal to P'' , while the part of v inside P is internal to P' . Hence, e does not cross v .

Algorithm 7 DRAWING MAXIMAL OUTERCLUSTERED GRAPHS WITH THREE PATHS

Input: C , $\mathcal{P}_u = (u_1, \dots, u_U)$, $\sigma'(u_2), \dots, \sigma'(u_{U-1})$, $\mathcal{P}_v = (v_1, \dots, v_V)$, $\sigma'(v_2), \dots, \sigma'(v_{V-1})$, $\mathcal{P}_z = (z_1, z_2, \dots, z_Z)$, $\sigma'(z_1), \dots, \sigma'(z_{Z-1})$, and $\Gamma(C_o)$;

Output: An internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.

```

1: if  $U > 2$  then
2:    $p(z_1) \leftarrow$  point of  $\text{int}(R(\sigma(z_1), \sigma'(z_1)))$  close to  $(u, z)$ ; place  $z_1$  at  $p(z_1)$ ;
3:    $H(l(z, p(z_1)), u), H(l(z, p(z_1)), v) \leftarrow$  the open half-planes delimited by the
   line through  $z$  and  $p(z_1)$ , and containing  $u$  and  $v$ , respectively;
4:    $p(u_2) \leftarrow$  point of  $H(l(z, p(z_1)), u) \cap \text{int}(T(u, v, p(z_1))) \cap$ 
    $\text{int}(R(\sigma(u_2), \sigma'(u_2)))$  close to  $(u, p(z_1))$ ; place  $u_2$  at  $p(u_2)$ ;
5:   apply Algorithm 9 on  $\mathcal{P}_u \setminus \{u_1\}$ ,  $R(\sigma(u_3), \sigma'(u_3))$ ,  $R(\sigma(u_4), \sigma'(u_4))$ ,  $\dots$ ,
    $R(\sigma(u_{U-1}), \sigma'(u_{U-1}))$ , and  $p(u_2)u$ ;
6:   if  $V > 2$  then
7:      $H(l(u, p(u_2)), v) \leftarrow$  the open half-plane delimited by the line through  $u$ 
     and  $p(u_2)$  and containing  $v$ ;
8:      $p(v_2) \leftarrow$  point of  $H(l(z, p(z_1)), v) \cap H(l(u, p(u_2)), v) \cap$ 
      $\text{int}(T(p(u_2), v, p(z_1))) \cap \text{int}(R(\sigma(v_2), \sigma'(v_2)))$ ; place  $v_2$  at  $p(v_2)$ ;
9:     apply Algorithm 9 on  $\mathcal{P}_v \setminus \{v_1\}$ ,  $R(\sigma(v_3), \sigma'(v_3))$ ,  $R(\sigma(v_4), \sigma'(v_4))$ ,  $\dots$ ,
      $R(\sigma(v_{V-1}), \sigma'(v_{V-1}))$ , and  $p(v_2)v$ ;
10:   end if
11: else
12:   if  $V > 2$  then
13:      $p(z_1) \leftarrow$  point of  $\text{int}(R(\sigma(z_1), \sigma'(z_1)))$  close to  $(v, z)$ ; place  $z_1$  at  $p(z_1)$ ;
14:      $H(l(z, p(z_1)), v) \leftarrow$  the open half-plane delimited by the line through  $z$  and
      $p(z_1)$ , and containing  $v$ ;
15:      $p(v_2) \leftarrow$  point of  $H(l(z, p(z_1)), v) \cap \text{int}(T(u, v, p(z_1))) \cap$ 
      $\text{int}(R(\sigma(v_2), \sigma'(v_2)))$  close to  $(v, p(z_1))$ ; place  $v_2$  at  $p(v_2)$ ;
16:     apply Algorithm 9 on  $\mathcal{P}_v \setminus \{v_1\}$ ,  $R(\sigma(v_3), \sigma'(v_3))$ ,  $R(\sigma(v_4), \sigma'(v_4))$ ,  $\dots$ ,
      $R(\sigma(v_{V-1}), \sigma'(v_{V-1}))$ , and  $p(v_2)v$ ;
17:   end if
18: end if
19: if  $U = 2$  and  $V = 2$  then
20:    $p(z_1) \leftarrow$  point of  $\text{int}(R(\sigma(z_1), \sigma'(z_1)))$ ; place  $z_1$  at  $p(z_1)$ ;
21: end if
22: apply Algorithm 9 on  $\mathcal{P}_z$ ,  $R(\sigma(z_2), \sigma'(z_2))$ ,  $R(\sigma(z_3), \sigma'(z_3))$ ,  $\dots$ ,
    $R(\sigma(z_{Z-1}), \sigma'(z_{Z-1}))$ , and  $p(z_1)z$ ;
23: draw each edge of  $G$  between two vertices belonging to  $\mathcal{P}_u$ , to  $\mathcal{P}_v$ , or to  $\mathcal{P}_z$  as a
   straight-line segment;
24: apply Algorithm 2 on each resulting linearly-ordered outerclustered graph com-
   pleting a drawing of the corresponding outer face;
25:  $\Gamma(C) \leftarrow$  the resulting internally-convex-separated drawing of  $C$ ;
26: return  $\Gamma(C)$ ;

```

We prove that $\Gamma(C_{u,v}, T_{u,v})$ satisfies Property CS1 of Definition 3. The angles $\widehat{u_2uv}$ and $\widehat{v_2vu}$ incident to u and v inside $P_{u,v}$ are strictly less than 180° , since they are respectively less than angles $\widehat{zu\bar{v}}$ and $\widehat{\bar{z}v\bar{u}}$, that are angles of P , which is a triangle. By construction, v_2 is contained inside triangle $T(p(u_2), v, p(z_1))$, hence $\widehat{u_2v_2v}$ is the angle of a triangle having u_2 , v_2 , and v as vertices, hence it is less than 180° . Finally, angle $\widehat{uu_2v_2}$ is less than 180° , since by construction v_2 is placed in the half-plane $H(l(u, p(u_2)), v)$ delimited by the line through u and $p(u_2)$, and containing v .

We prove that $\Gamma(C_{u,v}, T_{u,v})$ satisfies Property CS2. Observe that $\sigma(u)$ and $\sigma(v)$ are the first and the last cluster in Σ , respectively, and that the angles $\widehat{u_2uv}$ and $\widehat{v_2vu}$ incident to u and v inside $P_{u,v}$ are strictly less than 180° , as proved above.

We prove that $\Gamma(C_{u,v}, T_{u,v})$ satisfies Property CS3. The existence of regions $R(\mu, v)$ inside $P_{u,v}$ descends from the existence of regions $R(\mu, v)$ inside P , where v is any child of μ in $T_{u,v}$. Namely, the drawn edges cut each convex region $R(\mu, v)$ into two or three convex regions. Such regions satisfy the properties that have to be satisfied by $R(\mu, v)$ inside $P_{u,v}$, as can be easily deduced from the fact that the same properties are satisfied by $R(\mu, v)$ inside P . $\square$

Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ are, in general, not triconnected since there could exist chords: (i) in $C_{u,v}$ between any vertex in $\mathcal{P}_u \setminus \{u_1\}$ and any vertex in $\mathcal{P}_v \setminus \{v_1\}$; (ii) in $C_{u,z}$ between vertex u_1 and any vertex in $\mathcal{P}_u \setminus \{u_1\}$, and between any vertex in $\mathcal{P}_u \setminus \{u_1\}$ and any vertex in $\mathcal{P}_z \setminus \{z_1\}$; (iii) in $C_{v,z}$ between vertex v_1 and any vertex in $\mathcal{P}_v \setminus \{v_1\}$, and between any vertex in $\mathcal{P}_v \setminus \{v_1\}$ and any vertex in $\mathcal{P}_z \setminus \{z_1\}$. By Lemma 2, each of such chords splits a linearly-ordered outerclustered graph into two smaller linearly-ordered outerclustered graphs. Further, by construction the end-vertices of each of such chords are not collinear with any other vertex of the cycle. Hence, by Lemma 3, inserting the chords as straight-line segments into drawings $\Gamma(C_{u,v}, T_{u,v})$, $\Gamma(C_{u,z}, T_{u,z})$, and $\Gamma(C_{v,z}, T_{v,z})$, that are convex-separated by Lemma 15, splits them into convex-separated drawings. When all the chords have been added, the underlying graphs of the resulting clustered graphs are all tri-connected and internally-triangulated. Hence, Theorem 1 applies and an internally-convex-separated drawing of each of such linearly-ordered outerclustered graphs can be constructed inside the corresponding outer face, thus obtaining an internally-convex-separated drawing of C .

Now suppose that Condition 2 of Lemma 12 holds.

By Lemma 14, $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$ are linearly-ordered outerclustered graphs.

Refer to Fig. 29. Suppose that $U > 2$ and that $V > 2$. Consider the unique child $\sigma'(u_i)$ of $\sigma(u_i)$, for each $2 \leq i \leq U - 1$. Notice that one of the two polygonal lines obtained as intersection of the triangle representing (u, v, z) and $R(\sigma(u_2), \sigma'(u_2))$ lies on edge (u, z) . Consider a point $p(u_2)$ of $\text{int}(R(\sigma(u_2), \sigma'(u_2)))$ arbitrarily close to (u, z) . Place u_2 at $p(u_2)$. Draw a straight-line segment $p(u_2)u$ and place u_i at any point of the segment $\text{int}(R(\sigma(u_i), \sigma'(u_i))) \cap p(u_2)u$, for each $3 \leq i \leq U - 1$.

Denote by $T(p(u_2), v, z)$ the triangle having $p(u_2)$, v , and z as vertices. Denote also by $H(l(u, p(u_2)), v)$ the open half-plane delimited by the line through u and $p(u_2)$, and containing v .

Let $\sigma'(v_i)$ be the unique child of $\sigma(v_i)$, for each $2 \leq i \leq V - 1$. Consider any point $p(v_2)$ in $H(l(u, p(u_2)), v) \cap \text{int}(T(p(u_2), v, z)) \cap \text{int}(R(\sigma(v_2), \sigma'(v_2)))$ and

Fig. 29 Construction of drawings $\Gamma(C_{u,v}, T_{u,v})$, $\Gamma(C_{u,z}, T_{u,z})$, and $\Gamma(C_{v,z}, T_{v,z})$ when Condition 2 of Lemma 12 holds

place v_2 at $p(v_2)$. Draw a straight-line segment $\overline{p(v_2)v}$ and place v_i at any point of the segment $\text{int}(R(\sigma(v_i), \sigma'(v_i)) \cap \overline{p(v_2)v})$, for each $3 \leq i \leq V-1$. Straightforward modifications make the described algorithm work also for the cases in which $U=2$ or $V=2$ (notice that $U=2$ and $V=2$ do not hold simultaneously when Condition 2 of Lemma 12 holds, otherwise G would not have internal vertices). Such modifications are described in Algorithm 8 below.

Denote by $\Gamma(C_{u,v}, T_{u,v})$, by $\Gamma(C_{u,z}, T_{u,z})$, and by $\Gamma(C_{v,z}, T_{v,z})$ the constructed drawings of $(C_{u,v}, T_{u,v})$, of $(C_{u,z}, T_{u,z})$, and of $(C_{v,z}, T_{v,z})$, respectively. We have the following lemma, whose proof is analogous to the proof of Lemma 15.

Lemma 16 $\Gamma(C_{u,v}, T_{u,v})$, $\Gamma(C_{u,z}, T_{u,z})$, and $\Gamma(C_{v,z}, T_{v,z})$ are convex-separated drawings of the outer faces of $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$, respectively.

Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ are, in general, not triconnected, since there could exist chords: (i) in $C_{u,v}$ between any vertex in $\mathcal{P}_u \setminus \{u_1\}$ and any vertex in $\mathcal{P}_v \setminus \{v_1\}$; (ii) in $C_{u,z}$ between vertex u_1 and any vertex in $\mathcal{P}_u \setminus \{u_1\}$; (iii) in $C_{v,z}$ between vertex v_1 and any vertex in $\mathcal{P}_v \setminus \{v_1\}$. By Lemma 2, each of such chords splits a linearly-ordered outerclustered graph into two smaller linearly-ordered outerclustered graphs. Further, by construction, the end-vertices of each of such chords are not collinear with any other vertex of the cycle. Hence, by Lemma 3, inserting the chords as straight-line segments into drawings $\Gamma(C_{u,v}, T_{u,v})$, $\Gamma(C_{u,z}, T_{u,z})$, and $\Gamma(C_{v,z}, T_{v,z})$, that are convex-separated by Lemma 16, splits them into convex-separated drawings. When all chords have been added, the underlying graphs of the resulting clustered graphs are all triconnected and internally-triangulated. Hence, Theorem 1 applies and an internally-convex-separated drawing of each of such linearly-ordered outerclustered graphs can be constructed inside the corresponding outer face, thus obtaining an internally-convex-separated drawing of C . $\square$

Algorithm 8 DRAWING MAXIMAL OUTERCLUSTERED GRAPHS WITH TWO PATHS

Input: C , $\mathcal{P}_u = (u_1, \dots, u_U)$, $\sigma'(u_2), \dots, \sigma'(u_{U-1})$, $\mathcal{P}_v = (v_1, \dots, v_V)$, $\sigma'(v_2), \dots, \sigma'(v_{V-1})$, and $\Gamma(C_o)$;

Output: An internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.

```

1: if  $U > 2$  then
2:    $p(u_2) \leftarrow$  point of  $\text{int}(R(\sigma(u_2), \sigma'(u_2)))$  close to  $(u, z)$ ; place  $u_2$  at  $p(u_2)$ ;
3:   apply Algorithm 9 on  $\mathcal{P}_u \setminus \{u_1\}$ ,  $R(\sigma(u_3), \sigma'(u_3))$ ,  $R(\sigma(u_4), \sigma'(u_4))$ ,  $\dots$ ,
      $R(\sigma(u_{U-1}), \sigma'(u_{U-1}))$ , and  $\overline{p(u_2)u}$ ;
4:   if  $V > 2$  then
5:      $H(l(u, p(u_2)), v) \leftarrow$  the open half-plane delimited by the line through  $u$ 
       and  $p(u_2)$ , and containing  $v$ ;
6:      $p(v_2) \leftarrow$  point of  $H(l(u, p(u_2)), v) \cap \text{int}(T(p(u_2), v, z)) \cap$ 
        $\text{int}(R(\sigma(v_2), \sigma'(v_2)))$ ; place  $v_2$  at  $p(v_2)$ ;
7:     apply Algorithm 9 on  $\mathcal{P}_v \setminus \{v_1\}$ ,  $R(\sigma(v_3), \sigma'(v_3))$ ,  $R(\sigma(v_4), \sigma'(v_4))$ ,  $\dots$ ,
        $R(\sigma(v_{V-1}), \sigma'(v_{V-1}))$ , and  $\overline{p(v_2)v}$ ;
8:   end if
9:   else
10:     $p(v_2) \leftarrow$  point of  $\text{int}(R(\sigma(v_2), \sigma'(v_2)))$ ; place  $v_2$  at  $p(v_2)$ ;
11:    apply Algorithm 9 on  $\mathcal{P}_v \setminus \{v_1\}$ ,  $R(\sigma(v_3), \sigma'(v_3))$ ,  $R(\sigma(v_4), \sigma'(v_4))$ ,  $\dots$ ,
       $R(\sigma(v_{V-1}), \sigma'(v_{V-1}))$ , and  $\overline{p(v_2)v}$ ;
12:   end if
13: draw each edge of  $G$  between two vertices belonging to  $\mathcal{P}_u$  or to  $\mathcal{P}_v$  as a straight-
    line segment;
14: apply Algorithm 2 on each resulting linearly-ordered outerclustered graph com-
    pleting a drawing of the corresponding outer face;
15:  $\Gamma(C) \leftarrow$  the resulting internally-convex-separated drawing of  $C$ ;
16: return  $\Gamma(C)$ ;

```

Algorithm 9 DRAWING PATH IN MAXIMAL OUTERCLUSTERED GRAPHS

Input: $\mathcal{P}_x = (x_1, x_2, \dots, x_X)$, $R(\sigma(x_2), \sigma'(x_2))$, $R(\sigma(x_3), \sigma'(x_3))$, $\dots$,
 $R(\sigma(x_{X-1}), \sigma'(x_{X-1}))$, and $\overline{p(x_1)x_X}$.

```

1: for  $2 \leq i \leq X - 1$  do
2:   place  $x_i$  at any point of  $\text{int}(R(\sigma(x_i), \sigma'(x_i))) \cap \overline{p(x_1)x_X}$ ;
3: end for

```

A pseudo-code description of the algorithm for drawing a maximal outerclustered graph C (supposing that, if there exists a cluster containing exactly two vertices incident to $o(G)$, then such vertices are u and v , and that, if the smallest containing cluster of one of u and v contains the other one, then $\sigma(v)$ contains u) is presented in Algorithms 6–9.

Fig. 30 Illustration for Case 1 of the algorithm for drawing clustered graphs

5 Drawing Clustered Graphs

In this section we prove that every clustered graph $C(G, T)$ admits an internally-convex-separated drawing $\Gamma(C)$ completing an arbitrary triangular-convex-separated drawing $\Gamma(C_o)$ of C_o . Recall that a drawing $\Gamma(C_o)$ of C_o consists of a drawing of the outer face $o(G)$ of G and of a drawing of each cluster containing a vertex incident to $o(G)$. We assume that $\Gamma(C_o)$ is any triangular-convex-separated drawing $\Gamma(C_o)$ of C_o and we show how to compute a straight-line rectangular drawing of C completing $\Gamma(C_o)$. Such a result is achieved by means of an inductive algorithm, where the induction is on the number of vertices plus the number of clusters of C . In the base case, C is a maximal outerclustered graph and the statement follows from Theorem 2. Consider any maximal clustered graph $C(G, T)$.

Case 1: *There exists a minimal cluster containing exactly one internal vertex and no external vertex.* Refer to Fig. 30. Let μ be a minimal cluster containing exactly one vertex v internal to G and containing no vertex incident to $o(G)$. Remove μ from T obtaining a clustered graph $C'(G, T')$. Observe that C_o and C'_o are the same graph, since μ does not contain vertices incident to $o(G)$. The number of vertices plus the number of clusters of C' is one less than the number of vertices plus the number of clusters of C . Hence, the inductive hypothesis applies and, for an arbitrary triangular-convex-separated drawing $\Gamma(C_o)$, there exists an internally-convex-separated drawing $\Gamma(C')$ of C' completing $\Gamma(C_o)$. In $\Gamma(C')$ a small disk D can be drawn with the following properties: it is centered at v , it does not intersect the boundary of any cluster, it does not contain any vertex of G different from v , and it has intersection only with the edges incident to v . For each edge e_i incident to v , choose two points p_i^1 and p_i^2 inside D , where p_i^1 is closer to v than p_i^2 . For each two edges e_i and e_{i+1} consecutively incident to v denote by f_i the face of G incident to edges e_i and e_{i+1} , and denote by $R(\mu, i)$ the quadrilateral having p_i^1, p_i^2, p_{i+1}^1 , and p_{i+1}^2 as vertices. Finally, insert a drawing of μ in $\Gamma(C')$ as a rectangle containing v and contained inside the polygon $(p_1^1, p_2^1, \dots, p_k^1, p_1^1)$, thus obtaining a drawing $\Gamma(C)$.

Lemma 17 $\Gamma(C)$ is an internally-convex-separated drawing of C .

Proof $\Gamma(C)$ has no edge crossing, by induction. $\Gamma(C)$ has no edge-region crossing since any cluster different from μ has no crossing with any edge by induction, and since μ does not intersect any edge not incident to v , because it completely lies inside D . $\Gamma(C)$ has no region-region crossing since the boundaries of any two clusters different from μ have no crossing by induction, and since the boundary of μ does not intersect the boundary of any cluster, because it completely lies inside D . The drawing is straight-line and rectangular by construction. Further, every internal face of G not incident to v is triangular-convex-separated by induction. Finally, for each face f_i , define region $R(v, \mu)$, where v is the parent of μ in the inclusion tree T of $C(G, T)$, as coincident with region $R(\mu, i)$. Then, $R(v, \mu)$ satisfies Property TCS1 of a triangular-convex-separated drawing, due to the fact that such a region is completely contained inside D , and that D is completely contained inside v . $\square$

Case 2: *There exists a separating 3-cycle.* Suppose that G contains a separating 3-cycle (u', v', z') . Let $C^1(G^1, T^1)$ be the clustered graph defined as follows. G^1 is the subgraph of G induced by all the vertices external to (u', v', z') , by u' , by v' , and by z' . T^1 is the subtree of T whose clusters contain at least one vertex of G^1 . Observe that C_o and C_o^1 are the same graph. Further, let $C^2(G^2, T^2)$ be the clustered graph defined as follows. G^2 is the subgraph of G induced by all the vertices internal to (u', v', z') , by u' , by v' , and by z' . T^2 is the subtree of T whose clusters contain at least one vertex of G^2 . Since (u', v', z') is a separating 3-cycle, the number of vertices plus the number of clusters of each of C^1 and C^2 is strictly less than the number of vertices plus the number of clusters of C . Hence, the inductive hypothesis applies and, for an arbitrary triangular-convex-separated drawing $\Gamma(C_o)$, there exists an internally-convex-separated drawing $\Gamma(C^1)$ of C^1 completing $\Gamma(C_o)$. Cycle (u', v', z') is a face f of G^1 . By definition of internally-convex-separated drawing of a graph, the drawing $\Gamma(C_f)$ of C_f in $\Gamma(C^1)$ is a triangular-convex-separated drawing. Observe that C_f and C_o^2 are the same graph. Hence, the inductive hypothesis applies again and an internally-convex-separated drawing $\Gamma(C^2)$ can be constructed completing $\Gamma(C_o^2)$. Plugging $\Gamma(C^2)$ into $\Gamma(C^1)$ provides a drawing $\Gamma(C)$ of C .

Lemma 18 $\Gamma(C)$ is an internally-convex-separated drawing of C .

Proof $\Gamma(C)$ has no edge crossing. Namely, any edge belonging to G^1 (resp. to G^2) does not cross any edge belonging to G^1 (resp. to G^2) by induction. Further, any edge belonging to G^1 and not belonging to G^2 does not cross any edge belonging to G^2 and not belonging to G^1 since such edges are separated by cycle (u', v', z') . $\Gamma(C)$ has no edge-region crossing. Namely, any edge belonging to G^1 (resp. to G^2) does not cross the boundary of any cluster of T^1 (resp. of T^2) by induction. Further, any edge belonging to G^1 (resp. to G^2) and not belonging to G^2 (resp. to G^1) does not cross the boundary of any cluster belonging to T^2 (resp. to T^1) and not belonging to T^1 (resp. to T^2), since such an edge and such a cluster are separated by cycle (u', v', z') . $\Gamma(C)$ has no region-region crossing. Namely, the boundary of any cluster belonging to T^1 (resp. to T^2) does not cross the boundary of any cluster belonging to T^1 (resp. to T^2) by induction. Further, the boundary of any cluster belonging to T^1 (resp. to T^2) and not belonging to T^2 (resp. to T^1) does not cross the boundary of

Fig. 31 Illustration for Case 3 of the algorithm for drawing clustered graphs

any cluster belonging to T^2 (resp. to T^1) and not belonging to T^1 (resp. to T^2), since such clusters are separated by cycle (u', v', z') . $\Gamma(C)$ is straight-line and rectangular by construction. Further, the drawing of any internal face f of G is triangular-convex-separated since it is triangular-convex-separated in $\Gamma(C^1)$ (if f is also a face of G^1) or in $\Gamma(C^2)$ (if f is also a face of G^2). $\square$

Case 3: There exists no separating 3-cycle and there exists an edge (u', v') not incident to $o(G)$ such that $\sigma(u') = \sigma(v')$. Refer to Fig. 31. Suppose that G contains an edge (u', v') with $\sigma(u') = \sigma(v')$ such that u' is internal and suppose that there exists no separating 3-cycle containing edge (u', v') . Since G is internally-triangulated, u' and v' have exactly two common neighbors z'_1 and z'_2 . Contract edge (u', v') to a vertex w' , that is, replace vertices u' and v' with a single vertex w' that is connected to all the vertices u' and v' are connected to. Vertex w' belongs to cluster $\sigma(u')$ and to all the ancestors of $\sigma(u')$ in T . The resulting clustered graph $C'(G', T')$ is easily shown to be a maximal c -planar clustered graph. In particular, the absence of separating 3-cycles in G guarantees that G' is simple and internally-triangulated. Observe that C_o and C'_o are the same graph. Hence, the inductive hypothesis applies and, for an arbitrary triangular-convex-separated drawing $\Gamma(C_o)$, there exists an internally-convex-separated drawing $\Gamma(C')$ of C' completing $\Gamma(C_o)$. Then, consider a small disk D centered at w' and consider any line l from w' to an interior point of the segment between z'_1 and z'_2 . Replace w' with u' and v' so that such vertices lie on l and inside D . Connect u' and v' to their neighbors, obtaining a drawing $\Gamma(C)$ of C .

Lemma 19 $\Gamma(C)$ is an internally-convex-separated drawing of C .

Proof In [14] the following property of planar straight-line drawings has been proved. In a planar straight-line drawing, for any vertex w' , there exists a disk D centered at w' such that moving w' to any point inside D leaves the straight-line drawing planar. The proof of this property takes into account the set of points from

which all the neighbors of w' are *visible*, i.e., straight-line segments can be drawn without causing crossings. This property of planar straight-line drawings has been exploited in [9, 10, 14] to argue that, in a graph G with no separating 3-cycle, an edge (u', v') that has been contracted to a single vertex w' (obtaining a graph G') can suitably replace w' so that the resulting straight-line drawing of G is planar. Here the continuity arguments used in [14] to prove the existence of D are still valid; however, the visibility between any point p of D and any neighbor z' of w' means that it is possible to draw a straight-line segment from p to z' not crossing any edge of G and not crossing twice the boundary of the same cluster; further, D has to be so small that it does not intersect the boundary of any cluster.

Then, the placement of u' and v' guarantees that $\Gamma(C)$ has no edge crossing and no edge-region crossing. Further, $\Gamma(C)$ has no region-region crossing, by induction. $\Gamma(C)$ is a straight-line rectangular drawing, by construction. Finally, for each internal face f of G not incident to u' and v' regions $R(\mu, \nu)$ can be drawn as in $\Gamma(C')$, since f has the same drawing in $\Gamma(C)$ and in $\Gamma(C')$; for each internal face f of G incident to u' and not to v' , or vice versa, regions $R(\mu, \nu)$ can be drawn similarly to $\Gamma(C')$, since the drawings of f in $\Gamma(C)$ and in $\Gamma(C')$ differ for an arbitrary small displacement of an incident vertex (in C' , face f is incident to vertex w' , that replaces the one out of u' and v' that is incident to f in C); faces (u', v', z'_1) and (u', v', z'_2) are so thin that no vertex of any rectangle representing a cluster lies inside such faces, hence regions $R(\mu, \nu)$ can easily be drawn. $\square$

It remains to prove that the case in which C is a maximal outerclustered graph is the base case.

Lemma 20 *Suppose that none of Cases 1, 2, and 3 applies. Then C is a maximal outerclustered graph.*

Proof That G is a maximal plane graph is easily proved by the observation that each of Cases 1, 2, and 3 transforms a clustered graph whose underlying graph is a maximal plane graph into one or two clustered graphs whose underlying graphs are maximal plane graphs. We now prove that C is an outerclustered graph.

Suppose, for a contradiction, that none of Cases 1, 2, and 3 applies, and that C is not an outerclustered graph. By the maximality of G , $o(G)$ is delimited by a 3-cycle. By the c -planarity of C , each cluster that contains some but not all the vertices incident to $o(G)$ intersects $o(G)$ exactly twice, thus proving Property O2 of Definition 1.

Suppose that C contains a cluster μ not containing any vertex incident to $o(G)$. Then, C contains a minimal cluster μ' not containing any vertex incident to $o(G)$, namely $\mu' = \mu$ if μ is minimal, and μ' is any minimal cluster descendant of μ , if μ is not minimal. If μ' contains exactly one vertex v , then μ' is a minimal cluster containing only v , and Case 1 applies. If μ' contains more than one vertex, then, by the c -connectivity of C , there exists at least one edge (u, v) such that $\sigma(u) = \sigma(v) = \mu'$. If (u, v) is an edge of a separating 3-cycle, then Case 2 applies. Otherwise, Case 3 applies. This proves Property O1.

Suppose that C contains an edge (u, v) such that $\sigma(u) = \sigma(v)$ and suppose that at least one out of u and v is an internal vertex of G . If edge (u, v) belongs to a

Fig. 32 The triangular-convex-separated drawing of the outer face of a clustered graph C constructed by Algorithm 11

separating 3-cycle, then Case 2 applies. Otherwise, Case 3 applies. This proves Property O3. $\square$

Hence, we get the following:

Theorem 3 *Let $C(G, T)$ be a maximal c -planar clustered graph. Then, for every triangular-convex-separated drawing $\Gamma(C_o)$ of C_o , there exists an internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.*

A pseudo-code description of the algorithm for drawing a clustered graph C , given an arbitrary triangular-convex-separated drawing $\Gamma(C_o)$ of C_o is presented in Algorithm 10. A pseudo-code description of the algorithm for drawing a clustered graph C is presented in Algorithm 11. An example of the triangular-convex-separated drawing of the outer face of a clustered graph C constructed by Algorithm 11 is depicted in Fig. 32.

6 Conclusions

In this paper we have shown that every c -planar clustered graph admits a c -planar straight-line rectangular drawing. Actually, the algorithms we proposed do not exploit at all the fact that clusters are drawn as rectangles (except for the construction of a triangular-convex-separated drawing of the outer face of the input graph). Hence, such algorithms can be modified in order to construct a c -planar straight-line drawing of a given clustered graph for an arbitrary assignment of convex shapes to the clusters.

The algorithm we described in this paper uses real coordinates, hence it requires exponential area to be implemented in a system with a finite resolution rule. However, this drawback is unavoidable, since it has been proved by Feng et al. [16] that there

Algorithm 10 DRAWING CLUSTERED GRAPHS WITH GIVEN OUTER FACE

Input: A c -planar embedding of a clustered graph $C(G, T)$ and a triangular-convex-separated drawing $\Gamma(C_o)$ of C_o .

Output: An internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_o)$.

```

1: if a minimal cluster  $\mu$  exists containing exactly one internal vertex  $v$  and no
   external vertex then
2:    $C' \leftarrow$  the graph obtained from  $C$  by removing  $\mu$ ;
3:    $\Gamma(C') \leftarrow$  the drawing obtained by applying Algorithm 10 on  $C'$  and  $\Gamma(C_o)$ ;
4:    $\Gamma(C) \leftarrow$  the drawing obtained by inserting  $\mu$  in  $\Gamma(C')$  as a small rectangle
      containing  $v$ ;
5: else
6:   if there exists a separating 3-cycle  $f = (u', v', z')$  then
7:      $G^1 \leftarrow$  the subgraph of  $G$  induced by the vertices external to or belonging
       to  $(u', v', z')$ ;
8:      $T^1 \leftarrow$  the subtree of  $T$  whose clusters contain vertices of  $G^1$ ;
9:      $C^1 \leftarrow (G^1, T^1)$ ;
10:     $G^2 \leftarrow$  the subgraph of  $G$  induced by the vertices internal to or belonging to
       $(u', v', z')$ ;
11:     $T^2 \leftarrow$  the subtree of  $T$  whose clusters contain vertices of  $G^2$ ;
12:     $C^2 \leftarrow (G^2, T^2)$ ;
13:     $\Gamma(C^1) \leftarrow$  the drawing obtained by applying Algorithm 10 on  $C^1$  and
       $\Gamma(C_o)$ ;
14:     $\Gamma(C_f) \leftarrow$  the drawing of  $C_f$  in  $\Gamma(C^1)$ ;
15:     $\Gamma(C^2) \leftarrow$  the drawing obtained by applying Algorithm 10 on  $C^2$  and
       $\Gamma(C_f)$ ;
16:     $\Gamma(C) \leftarrow$  the drawing obtained by plugging  $\Gamma(C^2)$  into  $\Gamma(C^1)$ ;
17:   else
18:     if an edge  $(u', v')$  exists not incident to  $o(G)$  such that  $\sigma(u') = \sigma(v')$  then
19:        $z'_1$  and  $z'_2 \leftarrow$  the common neighbors of  $u'$  and  $v'$ ;
20:        $G' \leftarrow$  the graph obtained from  $G$  by contracting  $(u', v')$  to a vertex  $w'$ ;
21:        $T' \leftarrow$  the tree obtained from  $T$  by assigning  $w'$  to  $\sigma(u')$ ;
22:        $C' \leftarrow (G', T')$ ;
23:        $\Gamma(C') \leftarrow$  the drawing obtained by applying Algorithm 10 on  $C'$  and
         $\Gamma(C_o)$ ;
24:        $l \leftarrow$  a line from  $w'$  to an interior point of  $\overline{z'_1 z'_2}$  in  $\Gamma(C')$ ;
25:        $\Gamma(C) \leftarrow$  the drawing obtained from  $\Gamma(C')$  by replacing  $w'$  with  $u'$  and
         $v'$  so that  $u'$  and  $v'$  lie on  $l$  inside a small disk centered at  $w'$ ;
26:     else
27:        $\Gamma(C) \leftarrow$  the drawing obtained by applying Algorithm 6 on  $C$  and  $\Gamma(C_o)$ ;
28:     end if
29:   end if
30: end if
31: return  $\Gamma(C)$ ;

```

Algorithm 11 DRAWING CLUSTERED GRAPHS**Input:** A c -planar embedding of a clustered graph $C(G, T)$.**Output:** A straight-line rectangular drawing $\Gamma(C)$ of C .

```

1: add dummy edges to  $G$  until  $C$  is maximal;
2:  $u, v, z \leftarrow$  the vertices incident to  $o(G)$ ;
3: if a cluster  $\mu$  containing exactly two out of  $u, v$ , and  $z$  exists then
4:    $u, v \leftarrow$  be the vertices out of  $u, v$ , and  $z$  belonging to  $\mu$ ;
5: end if
6:  $\epsilon \leftarrow 1/n^2$ 
7: Place  $u$  at  $(0, 0)$ ; place  $v$  at  $(2, 0)$ ; place  $z$  at  $(1, 1)$ ;
8:  $i, j \leftarrow 1$ ;
9: while a not drawn cluster exists containing  $u$  and not containing  $v$  do
10:   $\mu \leftarrow$  smallest cluster in  $T$  containing  $u$ , not containing  $v$ , and not drawn;
11:  Draw  $\mu$  with corners  $(i\epsilon, i\epsilon), (i\epsilon, -i\epsilon), (-i\epsilon, -i\epsilon), (-i\epsilon, i\epsilon)$ ;
12:   $i \leftarrow i + 1$ ;
13: end while
14: while a not drawn cluster exists containing  $v$  and not containing  $u$  do
15:   $\mu \leftarrow$  smallest cluster in  $T$  containing  $v$ , not containing  $u$ , and not drawn;
16:  Draw  $\mu$  with corners  $(2 + j\epsilon, j\epsilon), (2 + j\epsilon, -j\epsilon), (2 - j\epsilon, -j\epsilon), (2 - j\epsilon, j\epsilon)$ ;
17:   $j \leftarrow j + 1$ ;
18: end while
19:  $k \leftarrow \max\{i, j\}$ ;
20: while a not drawn cluster exists containing  $u$  and  $v$  and not containing  $z$  do
21:   $\mu \leftarrow$  smallest cluster in  $T$  containing  $u$  and  $v$ , not containing  $z$ , and not drawn;
22:  Draw  $\mu$  with corners  $(2 + k\epsilon, k\epsilon), (2 + k\epsilon, -k\epsilon), (-k\epsilon, -k\epsilon), (-k\epsilon, k\epsilon)$ ;
23:   $k \leftarrow k + 1$ ;
24: end while
25:  $l \leftarrow 1$ ;
26: while a not drawn cluster exists containing  $z$  and not containing  $u$  and  $v$  do
27:   $\mu \leftarrow$  smallest cluster in  $T$  containing  $z$ , not containing  $u$  and  $v$ , and not drawn;
28:  Draw  $\mu$  with corners  $(1 + l\epsilon, 1 + l\epsilon), (1 + l\epsilon, 1 - l\epsilon), (1 - l\epsilon, 1 - l\epsilon), (1 - l\epsilon, 1 + l\epsilon)$ ;
29:   $l \leftarrow l + 1$ ;
30: end while
31:  $m \leftarrow \max\{k, l\}$ ;
32: while a not drawn cluster exists containing  $u, v$ , and  $z$  do
33:   $\mu \leftarrow$  smallest cluster in  $T$  containing  $u, v$ , and  $z$ , and not drawn;
34:  Draw  $\mu$  with corners  $(2 + m\epsilon, 1 + m\epsilon), (2 + m\epsilon, -m\epsilon), (-m\epsilon, -m\epsilon), (-m\epsilon, 1 + m\epsilon)$ ;
35:   $m \leftarrow m + 1$ ;
36: end while
37:  $\Gamma(C_o) \leftarrow$  the resulting drawing of  $C_o$ ;
38:  $\Gamma(C) \leftarrow$  the drawing obtained by applying Algorithm 10 on  $C$  and  $\Gamma(C_o)$ ;
39: remove dummy edges from  $\Gamma(C)$ ;
40: return  $\Gamma(C)$ ;

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exist clustered graphs requiring exponential area in any straight-line drawing in which clusters are represented by convex regions.

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