Hyperseparoids: A Representation Theorem

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Abstract In this note, hyperseparoids are introduced; hyperseparoids are to separoids as Tverberg’s theorem is to Radon’s theorem. Also, a geometric representation theorem for acyclic k -separoids is presented which generalises that for separoids exhibited in Bracho and Strausz (Period. Math. Hung. 53:115–120, 2006).

Keywords Separoids · Tverberg’s theorem · Radon’s theorem · Abstract convexity · Relational systems · Combinatorial geometry

A *separoid* is a (finite) set S endowed with a symmetric relation $\dagger \subseteq \binom{2^S}{2}$, defined in its family of subsets, which satisfies the following simple properties (cf. [1–7]): for $A, B \subseteq S$

$$\begin{aligned} \circ \quad A \dagger B &\Rightarrow A \cap B = \emptyset, \\ \circ \circ \quad A \dagger B \quad \text{and} \quad B \subset B' \subseteq (S \setminus A) &\Rightarrow A \dagger B'. \end{aligned}$$

A related pair $A \dagger B$ is called a *Radon partition*. A pair of disjoint subsets $\alpha, \beta \subseteq S$ which are not a Radon partition are said to be *separated* and denoted $\alpha \mid \beta$ —hence the name of the structure. The separoid is *acyclic* if the empty set is separated from the total one; i.e., $\emptyset \mid S$.

Examples of separoids can be found in graphs [5], in oriented matroids [3, 6], and in combinatorial convexity [4], to mention just a few (see also [7]). However, the main example to keep in mind is a family of convex sets $S = \{K_1, \dots, K_n\}$ where a Radon partition is a disjoint pair of its subsets $A, B \subseteq S$ such that the convex hull of

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Fig. 1 Two representations of the same separoid

their unions intersect; i.e.,

$$A \dagger B \iff \langle \cup A \rangle \cap \langle \cup B \rangle \neq \emptyset \quad (\text{and } A \cap B = \emptyset).$$

Here, and in the sequel, $\langle \cdot \rangle$ denotes the convex hull operator. This is because, given a separoid S , there always exists a family of convex sets in the Euclidian $(|S| - 1)$ -dimensional space—one for each element of S —such that its separoid (as defined above) is precisely S (cf. [2]). This is a key ingredient in [1, 3, 6].

Observe that the separoid of a family of convex sets does not distinguish if three sets have a point in common, or if they simply intersect pairwise (see Fig. 1)—this phenomenon motivated us to introduce the notion of a *virtual ℓ -transversal* in [1] and of a *virtual Tverberg partition* in [3].

The aim of this note is to capture such a difference in an abstract/combinatorial setting. Thus, let me introduce the following new concept (cf. [7]): a *k -separoid* is a (finite) set S endowed with a symmetric k -relational system $\mathcal{T} \subseteq \binom{2^S}{k}$, defined in its family of subsets, which satisfies two simple properties: for $A_1, \dots, A_k \subseteq S$

- $\{A_1, \dots, A_k\} \in \mathcal{T} \implies A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j,$
- $\{A_1, \dots, A_k\} \in \mathcal{T} \quad \text{and} \quad B \subseteq (S \setminus \cup A_i) \implies \{A_1, \dots, A_k \cup B\} \in \mathcal{T}.$

The elements of such a relational system will be called *Tverberg partitions*. Clearly, separoids are 2-separoids. The k -separoid is *acyclic* if $\{A_1, \dots, A_k\} \in \mathcal{T} \implies \prod |A_i| > 0$.

The following discussion can be made in a more general context (for all $k > 1$) but, to keep things simple, I will restrict to the case $k = 3$ —the general case is totally analogous. Thus, a Tverberg partition will be denoted simply by $A \dagger B \dagger C$. Analogously, if three pairwise disjoint subsets $\alpha, \beta, \gamma \subseteq S$ of a 3-separoid are not a Tverberg partition, they will be said to be a *3-separation* and denoted by $\alpha \mid \beta \mid \gamma$.

Theorem 1 *Every acyclic 3-separoid of order $n = |S|$ can be represented by a family of convex polytopes, and their Tverberg partitions, in the $(n - 1)$ -dimensional Euclidian space.*

Proof Let S be a 3-separoid and identify S with the n -set $\{1, \dots, n\}$. For each Tverberg partition $A \dagger B \dagger C$ and each element $i \in A$, assign a point of $\mathbb{R}^n$

$$\mathbf{P}_{A \dagger B \dagger C}(i) := \mathbf{e}_i + \frac{1}{3} \left[\frac{1}{|A|} \sum_{a \in A} \mathbf{e}_a + \frac{1}{|B|} \sum_{b \in B} \mathbf{e}_b + \frac{1}{|C|} \sum_{c \in C} \mathbf{e}_c \right] - \frac{1}{|A|} \sum_{a \in A} \mathbf{e}_a, \quad (*)$$

where $\{\mathbf{e}_i\}$ denotes the canonical basis, and realize each element $i \in S$ as the convex hull of all such points:

$$i \mapsto \mathcal{K}_i := \langle \mathbf{P}_{A \dagger B \dagger C}(i) : i \in A \text{ and } A \dagger B \dagger C \rangle.$$

Observe that these convex polytopes lie in the $(n-1)$ -dimensional affine subspace of $\mathbb{R}^n$ spanned by the basis because $(*)$ is, in fact, an affine combination.

To prove that this construction is correct, two steps are needed. First, a straightforward calculation shows that the construction is made to guarantee that the Tverberg partitions are preserved; that is, consider a partition $A \dagger B \dagger C$ and observe that the vertices of the simplices $\langle \mathbf{e}_a : a \in A \rangle$, $\langle \mathbf{e}_b : b \in B \rangle$, and $\langle \mathbf{e}_c : c \in C \rangle$ “translate” to realize such a partition while intersecting precisely in their baricentre, i.e.,

$$\begin{aligned} \frac{1}{|A|} \sum_{a \in A} \mathbf{P}_{A \dagger B \dagger C}(a) &= \frac{1}{|B|} \sum_{b \in B} \mathbf{P}_{B \dagger C \dagger A}(b) = \frac{1}{|C|} \sum_{c \in C} \mathbf{P}_{C \dagger A \dagger B}(c) \\ &= \frac{1}{3} \left(\frac{1}{|A|} \sum_{a \in A} \mathbf{e}_a + \frac{1}{|B|} \sum_{b \in B} \mathbf{e}_b + \frac{1}{|C|} \sum_{c \in C} \mathbf{e}_c \right), \end{aligned}$$

and therefore

$$\left\langle \bigcup_{a \in A} \mathcal{K}_a \right\rangle \cap \left\langle \bigcup_{b \in B} \mathcal{K}_b \right\rangle \cap \left\langle \bigcup_{c \in C} \mathcal{K}_c \right\rangle \neq \emptyset.$$

On the other hand, to prove that the k -separations are also preserved, the following well-known fact is used: compact convex sets $K_1, \dots, K_k$ in $\mathbb{R}^d$ have no point in common if and only if there are open semispaces $\ell_+^1, \dots, \ell_+^k$ such that $K_i \subset \ell_+^i$ (for every $i = 1, \dots, k$) and $\bigcap \ell_+^i = \emptyset$. The case $k = 2$ is the basic separation theorem, and the general case follows by induction.

Now, consider a 3-separation $\alpha \mid \beta \mid \gamma$, and define the affine extension $\psi = \psi_{\alpha \mid \beta \mid \gamma} : \mathbb{R}^n \rightarrow \mathbb{R}^2$ of the following equations: for $j = 1, \dots, n$,

$$\psi(\mathbf{e}_j) := \begin{cases} \mathbf{u} & \text{if } j \in \alpha, \\ \mathbf{v} & \text{if } j \in \beta, \\ \mathbf{w} & \text{if } j \in \gamma, \\ \mathbf{0} & \text{otherwise,} \end{cases} \quad (**)$$

where

$$\mathbf{u} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} := \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \frac{1}{2} \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}.$$

Observe that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are the vertices of a regular triangle with centre at $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

It follows from the definitions (*) and (**)—and with a little abuse of the notation—that

$$\begin{aligned}\psi(\mathbf{P}_{A \dagger B \dagger C}(i)) &= \psi(\mathbf{e}_i) - \frac{2}{3|A|} \begin{pmatrix} |A \cap \alpha| \\ |A \cap \beta| \\ |A \cap \gamma| \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} \\ &\quad + \frac{1}{3|B|} \begin{pmatrix} |B \cap \alpha| \\ |B \cap \beta| \\ |B \cap \gamma| \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} \\ &\quad + \frac{1}{3|C|} \begin{pmatrix} |C \cap \alpha| \\ |C \cap \beta| \\ |C \cap \gamma| \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}\end{aligned}\quad (1)$$

Denote by $\psi_\alpha := \psi(\mathbf{P}_{A \dagger B \dagger C}(i))$ whenever $i \in \alpha$, and analogously with β and γ . Then, it follows from (1) that

$$\begin{aligned}\psi_\alpha \cdot \mathbf{u} &= 1 - \frac{2|A \cap \alpha| - (|A \cap \beta| + |A \cap \gamma|)}{3|A|} \\ &\quad + \frac{|B \cap \alpha| - \frac{1}{2}(|B \cap \beta| + |B \cap \gamma|)}{3|B|} \\ &\quad + \frac{|C \cap \alpha| - \frac{1}{2}(|C \cap \beta| + |C \cap \gamma|)}{3|C|} \\ &\geq 0,\end{aligned}\quad (2)$$

If we have that $\psi_\alpha \cdot \mathbf{u} > 0$ and $\psi_\beta \cdot \mathbf{v} > 0$ and $\psi_\gamma \cdot \mathbf{w} > 0$, we are done (the semispaces $\psi^{-1}(\mathbf{u}_+^\perp)$, $\psi^{-1}(\mathbf{v}_+^\perp)$, and $\psi^{-1}(\mathbf{w}_+^\perp)$ will do). So let us suppose, without loss of generality, that $\psi_\alpha \cdot \mathbf{u} = 0$.

It follows from (2) that $\psi_\alpha \cdot \mathbf{u} = 0$ if and only if $A \subseteq \alpha$ and $B \subseteq \beta \cup \gamma$ and $C \subseteq \beta \cup \gamma$. In such a case, we also have that

$$\psi_\beta \cdot \mathbf{v} = 1 + \frac{1}{3} + \frac{1}{3} \left[\left(\frac{|B \cap \beta|}{|B|} + \frac{|C \cap \beta|}{|C|} \right) - \frac{1}{2} \left(\frac{|B \cap \gamma|}{|B|} + \frac{|C \cap \gamma|}{|C|} \right) \right] \geq 1$$

and, analogously, $\psi_\gamma \cdot \mathbf{w} \geq 1$. Then we can pick any small number $0 < \epsilon < 1$, define the semispaces

$$\begin{aligned}\ell_+^\alpha &:= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{u} > -\epsilon\}, \\ \ell_+^\beta &:= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{v} > 1 - \epsilon\}, \\ \ell_+^\gamma &:= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{w} > 1 - \epsilon\},\end{aligned}$$

and their preimages $\psi^{-1}(\ell_+^\alpha)$, $\psi^{-1}(\ell_+^\beta)$, and $\psi^{-1}(\ell_+^\gamma)$ will do the work, concluding the proof. $\square$

Observe that, as the Radon partitions do not contain information on the Tverberg partitions (cf. [3]), the 3-separoid also loses some information on the 2-separoid; e.g.,

consider 5 (different) points in the line and give them the order (1, 2, 3, 4, 5). These points have the following two Tverberg partitions:

$$15 \vdash 24 \vdash 3 \quad \text{and} \quad 14 \vdash 25 \vdash 3.$$

After the representation of Theorem 1, the configuration $\{\mathcal{K}_i : i = 1, \dots, 5\}$ will lose, among others, the Radon partition $2 \vdash 13$. To correct this “anomaly” we can go one step further and define a *hyperseparoid* as a set S endowed with a collection of families of subsets $\mathcal{T} \subseteq 2^{2^S}$ which satisfies, for $A_1, \dots, A_k \subseteq S$,

$$\begin{aligned} \circ \{A_1, \dots, A_k\} \in \mathcal{T} &\Rightarrow \{A_1, \dots, A_{k-1}\} \in \mathcal{T}, \\ \circ \circ \{A_1, \dots, A_k\} \in \mathcal{T} &\Rightarrow A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j, \\ \circ \circ \circ \{A_1, \dots, A_k\} \in \mathcal{T} \quad \text{and} \quad B \subseteq (S \setminus \cup A_i) &\Rightarrow \{A_1, \dots, A_k \cup B\} \in \mathcal{T}. \end{aligned}$$

Clearly, a similar construction to that of Theorem 1 will lead to a representation of hyperseparoids by convex polytopes and their Tverberg (and Radon) partitions.

Let us close this note with a couple of challenging open questions. In the spirit of [2], when can a hyperseparoid be represented by points (instead of convex sets)? Given a hyperseparoid, can we decide in polynomial time if it can be represented by points?

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