Line Transversals to Translates of a Convex Body

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Abstract Let K be a convex body in the plane. Define $\lambda(K, t)$ as the smallest number satisfying the following: if $\mathcal{F}$ is any family of translates of K such that every t members of $\mathcal{F}$ have a common transversal, then all the members of $\lambda(K, t)\mathcal{F}$ have a common transversal. We give bounds for $\lambda(K, 3)$ and $\lambda(K, 4)$ for a general convex figure K . In particular, we obtain that $\lambda(K, 3) \leq 1.79$ when K is the Euclidean disc.

Keywords Line transversal · Translates · Convex body

1 Introduction

Let $\mathcal{F}$ be a family of compact convex bodies in the plane. We say that $\mathcal{F}$ has the property T if there exists a line that intersects every member of $\mathcal{F}$. Also, if k is a given positive integer we say that $\mathcal{F}$ has the property $T(k)$ if any subfamily of $\mathcal{F}$ with k members has the property T . Let K be a convex body in the plane, $\mathcal{F} = \{x_1 + K, \dots, x_n + K\}$ be a finite family of translates of K , and $\lambda > 0$, then the family $\lambda\mathcal{F}$ is defined as $\{x_1 + \lambda K, \dots, x_n + \lambda K\}$.

Given a centrally symmetric convex body K in the plane, we denote by d_K the Minkowski distance (or K -distance), that is, the distance function associated with the body K in the usual manner: given a point $x \in \mathbb{R}^2$ we define the K -norm

$$\|\cdot\|_K : \mathbb{R}^2 \rightarrow \mathbb{R}$$

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Fig. 1 Geometric meaning of the K -norm

Fig. 2 K -width of X in direction u

as $\|x\|_K = \inf\{\lambda \in \mathbb{R}^+ : x \in \lambda K\}$. Furthermore, this norm defines a metric as follows: $\forall x, y \in \mathbb{R}^2$ we define the K -distance between points x and y as $d_K(x, y) = \|x - y\|_K$.

Intuitively (see Fig. 1), the geometric meaning of the K -norm of a given point $x \in \mathbb{R}^2$ is the following: let y be the point where the ray $\overrightarrow{ox}$ intersects the boundary of K , then

$$\|x\|_K = \frac{d(o, x)}{d(o, y)},$$

where $d(z, w)$ stands for the Euclidean distance between the points z and w . Also, given a set X in the plane we define the K -width of X in direction u as $\frac{w_u(X)}{w_u(\frac{1}{2}K)}$, where $w_u(Y)$ denotes the Euclidean width of a set Y in direction u (see Fig. 2). Furthermore, the K -width of X is defined as

$$\inf \left\{ \frac{w_u(X)}{w_u(\frac{1}{2}K)} : u \in \mathbb{S}^1 \right\}.$$

In 1964 B. Grünbaum [4] raised the following question:

Let K be a convex body in the plane. What is the smallest positive number $\lambda = \lambda(K, s)$ such that if $\mathcal{F}$ is any finite family of translates of K with the property $T(s)$, then the family $\lambda\mathcal{F}$ has the property T ?

It seems that the problem is not easier if we consider only families of pairwise disjoint translates, so it is well justified to define the number $\lambda_{\text{disj}}(K, s)$ in an analogous way for disjoint families of translates of a convex body K .

Some results on Grünbaum's problem are listed below. The first two of them are due to J. Eckhoff [2]:

Theorem E1 *Let K be a convex body in the plane, then*

$$\lambda(K, 3) \leq 2.$$

Theorem E2 *Let Q be the unit square, then*

$$\lambda(Q, 3) = 2.$$

In what follows we will denote the unit disc by B . For the case of the disc, A. Heppes [5] proved the following:

Theorem H1 $\lambda_{\text{disj}}(B, 3) < 1.65$.

And the first named author [8] of this work proved:

Theorem J $\lambda(B, 4) = \frac{1+\sqrt{5}}{2}$.

There also are some asymptotic results, for instance [6]:

Theorem H2 *For every real number $w \in (0, 1]$ there is a number $N(w)$ such that for every $T(3)$ -family, $\mathcal{F}$, of at least $N(w)$ pairwise disjoint translates of a convex figure K we have that $(1 + w/2)\mathcal{F}$ has the property T .*

In this paper we give bounds of $\lambda(K, 3)$ and $\lambda(K, 4)$ for arbitrary convex bodies. We also give a proof of the following fact observed by J. Eckhoff [2]: $\lambda(K, 3) = 2$ if and only if K is a parallelogram. As far as we know, a proof of this result has never been published.

2 New Results

We begin with some interesting and useful comments. By an useful reduction noticed by H. Tverberg [11] we know that K and $\frac{1}{2}(K - K)$ share the same properties with respect to line transversals. So, in what follows we consider only translates of a centrally symmetric convex body K . Let us assume that K is centered at the origin O . It is not difficult to see that if three translates $K + a, K + b, K + c$ have a common line transversal ℓ intersecting them in that order, then the line ℓ' parallel to $[b, c]$ and equidistant from the points a, b, c , also intersects the translates in the same order. To see this proceed as follows: start to shrink (continuously and uniformly) the translates until they have no longer a line transversal intersecting them in that order. Consider the last moment they still having a transversal in the given order, then the transversal is tangent to each one of the shrunk translates $\alpha K + a, \alpha K + b, \alpha K + c$.

The Banach–Mazur distance between two centrally symmetric convex bodies K_1 and K_2 centered at the origin is the smallest positive number $\delta = \delta(K_1, K_2)$ such that there exists a linear transformation φ that satisfies $K_1 \subset \varphi(K_2) \subset \delta K_1$. The application $\delta(K_1, K_2)$ is a multiplicative metric called the Banach–Mazur metric.

Fig. 3 Line transversal to three translates

Now consider the equivalence relationship: $K_1 \sim K_2$ if there exists an affine transformation φ such that $\varphi(K_1) = K_2$. The quotient space of the set of centrally symmetric convex bodies, with this equivalence relationship, is called the Banach–Mazur compactum. This name is justified by the fact that, together with δ , this space is a compact metric set. It is well known (see [1]) that the Banach–Mazur compactum is homeomorphic to the Alexandroff compactification of a Hilbert manifold.

It is not difficult to see that for a fixed r , the application $[K] \mapsto \lambda(K, r)$, where $[K]$ is the class of sets affinely equivalent to K , is well defined and continuous on the Banach–Mazur compactum. We have, in fact, a little more than continuity.

Lemma 1 *If K_1 and K_2 are centrally symmetric convex bodies, then*

$$\lambda(K_1, r) \leq \lambda(K_2, r) \delta(K_1, K_2).$$

For one of the main results, we need some additional definitions. Let X be a set of points in the plane, we say that X has the property T_K if the K -width of X is at most 1. If r is a positive integer, we say that X has the property $T_K(r)$ if any subset of X with r points has the property T_K . Furthermore, let $\mu(K)$ be the supremum of the K -widths of the parallelograms with the $T_K(3)$ property. It is known that $\mu(B) = \sqrt{2}$ (to see [3] and [7]), however, for an arbitrary convex body K we have the following.

Theorem 1 *There exists a number $\varepsilon > 0$ such that for every convex body K ,*

$$\frac{4}{3} + \varepsilon \leq \lambda(K, 3) \leq \max\left(\frac{1 + \sqrt{1 + 4\mu(K)}}{2}, 1.76\right).$$

Using this result on the disc we obtain the upper bound 1.79, which improves substantially the previous known upper bound ($\lambda(B, 3) < 2$). However, it is conjectured (to see [2]) that $\lambda(B, 3) = \frac{1+\sqrt{5}}{2}$.

Corollary 1

$$\frac{1 + \sqrt{5}}{2} \leq \lambda(B, 3) \leq \frac{1 + \sqrt{1 + 4\sqrt{2}}}{2} \approx 1.79004.$$

The following theorem establishes that the number 2 characterizes the parallelogram.

Theorem 2 $\lambda(K, 3) = 2$ if and only if K is a parallelogram.

Note that characterizing a class of convex bodies $[K]$ by the value of $\lambda(K, r)$ is only possible if K minimizes or maximizes the value of $\lambda(K, r)$ in the Banach–Mazur compactum. This is because of the topology of this space; if the minimum and maximum of $\lambda(\cdot, r)$ are attained at K_0 and K_1 respectively, then there is a path in the Banach–Mazur compactum connecting $[K_0]$ and $[K_1]$ that does not pass through any other given $[K]$.

Finally, we establish the following unexpected theorem:

Theorem 3 Let K be a convex body in the plane, then

$$\lambda(K, 4) \leq \frac{1 + \sqrt{5}}{2}.$$

In [8] was proved that $\lambda(B, 4) = \frac{1+\sqrt{5}}{2}$, however, it is easy to see that the number $\frac{1+\sqrt{5}}{2}$ does not characterize the Euclidean disc. Indeed, consider the regular decagon, P , inscribed in the unit circle B and let K be any convex body such that $P \subset K \subset B$ (as shown in Fig. 4).

Now, considering the pentagonal example (see Fig. 5) we have that $\lambda(K, 4) \geq \frac{1+\sqrt{5}}{2}$. Then, by Theorem 3 we have that $\lambda(K, 4) = \frac{1+\sqrt{5}}{2}$.

Fig. 4 A convex body with $\lambda(K, 4) = \frac{1+\sqrt{5}}{2}$

Fig. 5 Pentagonal example

3 Proofs

Proof of Lemma 1 Let $\mathcal{F} = \{x_1 + K_1, \dots, x_n + K_1\}$ be a family of translates of K_1 with the $T(r)$ property. Let $\delta = \delta(K_1, K_2)$ and $\lambda = \lambda(K_2, r)$. We can choose an element $K \in [K_2]$ such that $K_1 \subset K \subset \delta K_1$, and therefore the family $\{x_1 + K, \dots, x_n + K\}$ has the property $T(r)$. Since $\{x_1 + \lambda K, \dots, x_n + \lambda K\}$ has the property T , then $\{x_1 + \lambda \delta K_1, \dots, x_n + \lambda \delta K_1\}$ also has the property T . This implies $\lambda(K_1, r) \leq \lambda \delta$. $\square$

Note. In what follows, by the area of a given set we understand its Euclidean area.

Proof of Theorem 1 To prove the first inequality, let S be a square and H be a regular hexagon. It is known [10] that $\delta(S, K) \leq \frac{3}{2}$ with equality if and only if $[K] = [H]$. Since $\lambda(S, 3) = 2$ (see for instance [2], and [8]), we have by Lemma 1 that if $K \notin [H]$, then $\frac{4}{3} < \frac{\lambda(S, 3)}{\delta(S, K)} \leq \lambda(K, 3)$. On the other hand, since $\lambda(B, 3) \geq \frac{1+\sqrt{5}}{2}$ and $\delta(B, H) = \frac{2}{\sqrt{3}}$, we have that if $K \in [H]$, then $\frac{4}{3} < \frac{\lambda(B, 3)}{\delta(B, H)} \leq \lambda(K, 3)$. Since the application $[K] \mapsto \lambda(K, 3)$ is continuous on the Banach–Mazur compactum, the above implies that there is a number $\varepsilon > 0$ such that

$$\frac{4}{3} + \varepsilon \leq \lambda(K, 3)$$

for all K .

To prove the second inequality, fix K and let X be a finite set of points with the $T_K(3)$ property.

Let $\triangle ABC$ be the triangle of biggest area formed by points in X and suppose that the K -width of the strip bounded by the line AC and its parallel through B is at most 1. For simplicity we write $d_K(P, MN)$ for the K -width of the strip bounded by a given line MN and its parallel through P , so with this notation we have that $d_K(B, AC) \leq 1$. Let D be the point in the plane such that $ABCD$ is a parallelogram and let E be the point in X that is farthest away (in the Euclidean sense) from the line AC and contained in the triangle $\triangle ACD$. Let F and G be points on the segments $[C, D]$ and $[A, D]$ such that E is on the segment $[F, G]$ and the line FG is parallel to AC .

Set $h_1 = d_K(B, AC)$ and $h_2 = d_K(E, AC)$. Now let A' , B' and C' be points such that $A'B'C'D$ is a parallelogram homothetic to $ABCD$ with center D and ratio $\frac{h_1}{h_2}$ (see Fig. 6).

Let H be any point in X not contained in the parallelogram $ABCD$. It is clear that H must be above the lines $C'D$ and $A'D$, otherwise, we were have that either $\triangle ABH$ or $\triangle BCH$ would have area greater than $|ABC|$. Without loss of generality, we may assume that A and H are on opposite sides of BD . We then have

$$|ABC| \geq |AHE| > |AHG| = \frac{d_K(H, AD)}{d_K(C, AD)} \cdot \frac{d_K(A, G)}{d_K(A, D)} \cdot |ACD|.$$

Notice that $\frac{d_K(H, AD)}{d_K(C, AD)} = \frac{d(H, AD)}{d(C, AD)}$ and $\frac{d_K(A, G)}{d_K(A, D)} = \frac{d(A, G)}{d(A, D)}$. Now, since $|ACD| = |ABC|$ and $\frac{d_K(A, G)}{d_K(A, D)} = \frac{h_2}{h_1}$, we have that $\frac{d_K(H, AD)}{d_K(C, AD)} \leq \frac{h_1}{h_2}$ and so H is inside the triangle $\triangle DB'C'$. Hence the whole set X is contained in the parallelogram $A'B'C'D$.

Fig. 6 All the centers are contained in $DA'B'C'$

Suppose that E is closer to G than to F . The case when E is closer to F is analogous. We know the K -width of triangle $\triangle BCE$ is at most 1, for this to happen there are three possibilities for the width of the triangle:

- (1) $d_K(C, BE) \leq 1$. In this case we have that the strip bounded by the line BE and its parallel through C contains the point D and has K -width at most 1. It follows that the triples $\{B, C, D\}$ and $\{A, B, D\}$ have the property $T_K(3)$. Hence $\{A, B, C, D\}$ has the $T_K(3)$ property and therefore the K -width of $ABCD$ is at most $\mu(K)$. We then obtain that the K -width of $A'B'C'D$ is at most $\frac{h_1}{h_2}\mu(K)$.
- (2) $d_K(B, CE) \leq 1$. In this case we have that the strip bounded by the line CE and its parallel through B contains the segment AG and has K -width at most 1. It follows that the strip bounded by the lines parallel to CE through B and D has K -width at most $\frac{h_1}{h_2}$. Therefore, the K -width of $A'B'C'D$ is at most $(\frac{h_1}{h_2})^2$.
- (3) $d_K(E, BC) \leq 1$. In this case we have $d_K(D, BC) = \frac{h_1}{h_2} \cdot d_K(F, BC) \leq \frac{h_1}{h_2} \cdot d_K(E, BC) \leq \frac{h_1}{h_2}$. Therefore the K -width of $A'B'C'D$ is at most $(\frac{h_1}{h_2})^2$.

In all three cases we have that the K -width of X is at most

$$\max\left\{\left(\frac{h_1}{h_2}\right)^2, \frac{h_1}{h_2}\mu(K)\right\}. \quad (1)$$

Since $\triangle ABC$ is the biggest area triangle in X and because of the definition of E , we have that the set X has K -width at most $h_1 + h_2$. In this way we have shown that the K -width of X is at most the minimum of (1) and $h_1 + h_2$.

The optimization problem of maximizing the minimum between (1) and $h_1 + h_2$ subject to $h_2 \leq h_1 \leq 1$ is not difficult to solve and has a solution given by

$$\max\left(\frac{1 + \sqrt{1 + 4\mu(K)}}{2}, \rho\right),$$

where $\rho \approx 1.75488$ is the real root of the polynomial $x^3 - 2x^2 + x - 1$. This completes the proof of the theorem. $\square$

Fig. 7 A parallelogram with the $T_K(3)$ -property

Proof of Corollary 1 The first inequality is given by the pentagonal example shown in Fig. 4. For the second inequality we use that $\mu(B) = \sqrt{2}$ (proved by J. Eckhoff [3]) and apply Theorem 1. $\square$

Proof of Theorem 2 It is known (see [3] and [8]) that if K is a parallelogram, then $\lambda(K, 3) = 2$. In view of Theorem 1, it is enough to show that $\mu(K) = 2$ only if K is a parallelogram. First we observe that μ is actually a maximum.

Using a standard compactness argument, it is enough to prove that every parallelogram with the $T_K(3)$ property and big enough diameter has K -width close to 1. This can be easily done using Theorem H2 in the following way: let $ABCD$ be such a parallelogram, if the diameter of $ABCD$ is big enough we may choose a large set of points on the diameter of $ABCD$ in such way that their mutual K -distances are bigger than 1 and their K -distances to the vertices of the parallelogram are also bigger than 1 (otherwise $ABCD$ has K -width 1). Clearly, these points together with the vertices of the parallelogram have the $T_K(3)$ property. Theorem H2 now implies that the K -width of $ABCD$ is $1 + w/2$, for a small enough $w > 0$. This implies that μ is actually a maximum and therefore we can always find a parallelogram with the $T_K(3)$ property and K -width equal to $\mu(K)$.

Now, let $X = \{A, B, C, D\}$ be the vertex set of a parallelogram with the property $T_K(3)$ and K -width equal to 2. Let O be the center of the parallelogram (as shown in Fig. 7). We have that the K -width of triangle $\triangle ABC$ must be realized using the vertex B , otherwise the K -width of X would be at most 1. Let O' be a point on the segment $[C, A]$ such that $d_K(B, O') = 1$. Clearly this point exists, otherwise, we would have that the K -width of $ABCD$ would be less than 1. Without loss of generality suppose that O' is contained in the segment $[C, O]$. Now, let D' be the intersection of the lines BO' and CD . If $O \neq O'$, then $d_K(B, D') < 2$ and the K -width of X would be less than 2. So, $O = O'$ and therefore $d_K(O, B) = 1$. Analogously, we have that $d_K(O, A) = 1$. Since the K -width of X is 2, $d_K(O, P) \geq 1$ for any point P in the boundary of the parallelogram. The convexity of K implies that $d_K(O, P) \leq 1$ for any point P in the boundary of the parallelogram. It follows that $d_K(O, P) = 1$ for every point P in the boundary of the parallelogram. Therefore K is the parallelogram $ABCD$. $\square$

Fig. 8 Triangle with maximum area **Fig. 9** Triangle with maximum area inscribed in a pentagon

Proof of Theorem 3 The proof of Theorem 1 given in [8] works for a general convex body K if we replace the Euclidean width by the K -width. So, we will follow the main idea in [8] with some simple modifications.

Consider a fixed convex body K . By the observations given before, we may consider that K is a centrally symmetric set. Let $\mathcal{F}$ be a finite family of translates of $\frac{1}{2}K$ with the $T(4)$ property and let X be the set consisting on the centers of the translates. We have that X has the $T_K(4)$ property. Also, let $A, B, C \in X$ be points such that the area of $\triangle ABC$ is maximal. Clearly, $X \subset \triangle A'B'C'$, where $\triangle A'B'C'$ is the triangle homothetic to $\triangle ABC$ with center of homothety at the centroid of $\triangle ABC$ and constant of homothety equal to -2 (see Fig. 8). Suppose that $d_K(A, BC) \leq 1$ and let $E \in X$ be the point in $\triangle A'B'C'$ which is further apart from the line $C'B'$. We may also assume that $d_K(E, C'B') > \tau = \frac{1+\sqrt{5}}{2}$, otherwise we are done.

Now we will give the statement of a lemma due to E.G. Straus [9] whose direct implications are an important ingredient in the proof of the theorem. We shall return to the proof of Theorem 3 soon after.

Lemma S *Let Γ be a convex body in the plane and assume that one of the sides of an inscribed triangle Δ of maximum area lies on $\text{bd } \Gamma$, then the ratio of the areas satisfies $\sqrt{5} \cdot |\Delta| \geq |\Gamma|$, with equality if and only if Γ is an affine regular pentagon.*

It is easy to see that when Γ is a convex pentagon then there is a triangle, Δ , of maximum area whose vertices are vertices of the pentagon. Hence, one of the sides of Δ lies on $\text{bd } \Gamma$ and so Straus's lemma applies. Suppose that A, B, C, D, E are the vertices of Γ and $\triangle ABD$ is a triangle with maximum area. Let P be the point where $[A, D]$ intersects $[B, E]$ and let Q be the point where $[A, C]$ intersects $[B, D]$ (as shown in Fig. 9).

Fig. 10 All the centers are in the shaded region

Claim 1 $\frac{d(E, B)}{d(P, B)} + \frac{d(A, C)}{d(A, Q)} \leq 2\tau$.

Proof of Claim 1 By the maximality of $|ABD|$ and by Lemma S we have that $|DBC| + |DAE| + |ABD| \leq \sqrt{5} \cdot |ABD|$. In other words

$$\frac{\frac{d(Q, C)}{d(A, Q)} \cdot |ABD| + \frac{d(E, P)}{d(P, B)} \cdot |ABD| + |ABD|}{|ABD|} \leq \sqrt{5},$$

that is

$$\frac{d(Q, C)}{d(A, Q)} + \frac{d(E, P)}{d(P, B)} + 1 \leq \sqrt{5}.$$

It follows that

$$\frac{d(Q, C)}{d(A, Q)} + 1 + \frac{d(E, P)}{d(P, B)} + 1 \leq 1 + \sqrt{5},$$

therefore, $\frac{d(Q, C) + d(A, Q)}{d(A, Q)} + \frac{d(E, P) + d(P, B)}{d(P, B)} = \frac{d(A, C)}{d(A, Q)} + \frac{d(E, B)}{d(P, B)} \leq 1 + \sqrt{5}$. $\square$

Notice that Claim 1 remains true if instead of Euclidean distance we consider the K -distance.

Now, we return to the proof of Theorem 3. Let $M \in [B, A']$ and $N \in [C, A']$ such that $E \in [M, N]$ and $MN \parallel BC$ (see Fig. 10). Consider the points $R, P \in [A, B']$ and $Q, S \in [C, B']$ such that $\frac{d_K(R, C'A')}{d_K(A, C'A')} + \frac{d_K(M, C'B')}{d_K(B, C'B')} = 2\tau$, and $\frac{d_K(P, C'A')}{d_K(A, C'A')} = \tau$. Similarly, we define the points $R' \in [C', A]$ and $S' \in [C', B]$ such that $R'S' \parallel AB$ and $\frac{d_K(S', B'A')}{d_K(B, B'A')} + \frac{d_K(M, C'B')}{d_K(B, C'B')} = 2\tau$. By Claim 1 we know that X must be contained in the hexagon $RR'S'MNS$. It follows that if either $d_K(B, AC) \leq 1$ or $d_K(C, AB) \leq 1$ then either the strip bounded by RS and $C'A'$ or the strip bounded by $R'S'$ and $B'A'$ has K -width less than or equal to τ . Then, we may assume now that $d_K(B, AC) > 1$ and $d_K(C, AB) > 1$.

Since $\{A, B, C, E\} \in T_K(4)$ we have that either $d_K(A, BE) \leq 1$ or $d_K(A, CE) \leq 1$. We will consider that $d_K(A, BE) \leq 1$, the other case is completely analogous. Now, let S_1 be the strip bounded by the line BE and its parallel through A and let S_2 be

Fig. 11 All the centers are between ℓ_2 and ℓ_1

the strip bounded by the two lines parallel to BE through the points R and M , respectively. Let λ_1 and λ_2 be the K -width of S_1 and S_2 , respectively; clearly $\lambda_1 \leq 1$ and we will prove that $\lambda_2 \leq \tau$ (see Fig. 11). Let $T \in [P, Q]$ be the point such that $MT \parallel AB$, and let Y and L be the points where the line MT intersects the segments $[R, S]$ and $[R, B']$, respectively.

Since $\frac{d_K(F, M)}{d_K(B, A')} = \frac{d_K(R, P)}{d_K(A, B')}$ we have that $d_K(P, L) = d_K(R, P)$, hence T is the midpoint of the segment $[L, Y]$ and then the line RT intersects the segment $[B', S]$ at its midpoint. It follows that ℓ_2 intersects the interior of $[T, Y]$. We have that

$$\frac{d_K(U, M)}{d_K(A, B)} < \frac{d_K(T, M)}{d_K(A, B)} = \frac{d_K(Q, A')}{d_K(A, B)} = \frac{1 + \sqrt{5}}{2};$$

this implies that $\frac{\lambda_2}{\lambda_1} < \frac{1 + \sqrt{5}}{2}$. Therefore, $\lambda_2 < \frac{1 + \sqrt{5}}{2}$. $\square$

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