From the Icosahedron to Natural Triangulations of $\mathbb{C}P^2$ and $S^2 \times S^2$

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Received: 26 May 2010 / Revised: 28 July 2010 / Accepted: 2 August 2010 /

Published online: 24 August 2010

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Abstract We present two constructions in this paper: (a) a 10-vertex triangulation $\mathbb{C}P_{10}^2$ of the complex projective plane $\mathbb{C}P^2$ as a subcomplex of the join of the standard sphere (S_4^2) and the standard real projective plane ($\mathbb{R}P_6^2$, the decahedron), its automorphism group is A_4 ; (b) a 12-vertex triangulation $(S^2 \times S^2)_{12}$ of $S^2 \times S^2$ with automorphism group $2S_5$, the Schur double cover of the symmetric group S_5 . It is obtained by generalized bistellar moves from a simplicial subdivision of the standard cell structure of $S^2 \times S^2$. Both constructions have surprising and intimate relationships with the icosahedron. It is well known that $\mathbb{C}P^2$ has $S^2 \times S^2$ as a two-fold branched cover; we construct the triangulation $\mathbb{C}P_{10}^2$ of $\mathbb{C}P^2$ by presenting a simplicial realization of this covering map $S^2 \times S^2 \rightarrow \mathbb{C}P^2$. The domain of this simplicial map is a simplicial subdivision of the standard cell structure of $S^2 \times S^2$, different from the triangulation alluded to in (b). This gives a new proof that Kühnel's $\mathbb{C}P_9^2$ triangulates $\mathbb{C}P^2$. It is also shown that $\mathbb{C}P_{10}^2$ and $(S^2 \times S^2)_{12}$ induce the standard piecewise linear structure on $\mathbb{C}P^2$ and $S^2 \times S^2$ respectively.

Keywords Triangulated manifolds · Complex projective plane · Product of 2-spheres · Icosahedron

The research of B. Datta was supported by UGC grant UGC-SAP/DSA-IV.

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1 Introduction and Results

It is well known that the minimal triangulation $\mathbb{R}P_6^2$ of the real projective plane arises naturally from the icosahedron. Indeed, it is the quotient of the boundary complex of the icosahedron by the antipodal map. In this note, we report the surprising result that there is a small triangulation (using only 10 vertices) of the complex projective plane which is also intimately related to the icosahedron. Indeed, this simplicial complex $\mathbb{C}P_{10}^2$ occurs as a subcomplex of the simplicial join $S_4^2 * \mathbb{R}P_6^2$. Our starting point is the beautiful fact that $\mathbb{C}P^2$ is homeomorphic to the symmetrized square $(S^2 \times S^2)/\mathbb{Z}_2$ of the 2-sphere, where $\mathbb{Z}_2$ acts by co-ordinate flip. So, letting S_4^2 denote the 4-vertex triangulation of S^2 (i.e., the boundary complex of the tetrahedron), we look for a $\mathbb{Z}_2$ -stable simplicial subdivision of the product cell complex $S_4^2 \times S_4^2$, without introducing extra vertices. In order to ensure that the quotient complex (after quotienting by $\mathbb{Z}_2$) does triangulate the quotient space $(S^2 \times S^2)/\mathbb{Z}_2 = \mathbb{C}P^2$, the $\mathbb{Z}_2$ -action on this simplicial subdivision must be “pure” (cf. Definition 2 and Lemma 7 in Sect. 5). It turns out that the following $(S^2 \times S^2)_{16}$ is the unique 16-vertex triangulation satisfying these requirements.

Description of $(S^2 \times S^2)_{16}$ The vertices are x_{ij} , $1 \leq i, j \leq 4$. The full automorphism group is $A_4 \times \mathbb{Z}_2$, where A_4 acts on the indices and $\mathbb{Z}_2$ acts by $x_{ij} \leftrightarrow x_{ji}$. Modulo this group the facets (maximal simplices) are the following:

$$x_{11}x_{22}x_{33}x_{12}x_{13}, x_{11}x_{22}x_{12}x_{14}x_{34}, x_{11}x_{22}x_{14}x_{24}x_{34}, x_{11}x_{22}x_{21}x_{24}x_{31}, \\ x_{11}x_{22}x_{24}x_{31}x_{34}.$$

The full list of facets of $(S^2 \times S^2)_{16}$ may be obtained from these five basic facets by applying the group $A_4 \times \mathbb{Z}_2$. Under this group, the first three basic facets form orbits of length 24 each, while each of the last two forms an orbit of length 12, yielding a total of $3 \times 24 + 2 \times 12 = 96$ facets. It may be verified that the face vector of $(S^2 \times S^2)_{16}$ is (16, 84, 216, 240, 96).

Description of $\mathbb{C}P_{10}^2$ Quotienting the above $(S^2 \times S^2)_{16}$ by the group $\mathbb{Z}_2$ generated by the automorphism $x_{ij} \leftrightarrow x_{ji}$, we get the $\mathbb{C}P_{10}^2$ mentioned above. Its vertices are x_{ij} , $1 \leq i \leq j \leq 4$. Let α, β be the generators of the alternating group A_4 given by $\alpha = (123)$, $\beta = (12)(34)$. Then α, β act on the vertices of $\mathbb{C}P_{10}^2$ by:

$$\alpha \equiv (x_{11}x_{22}x_{33})(x_{23}x_{13}x_{12})(x_{24}x_{34}x_{14}), \quad \beta \equiv (x_{11}x_{22})(x_{33}x_{44})(x_{24}x_{13})(x_{14}x_{23}).$$

The following are the basic facets of $\mathbb{C}P_{10}^2$ modulo $A_4 = \langle \alpha, \beta \rangle$:

$$x_{11}x_{22}x_{33}x_{12}x_{13}, x_{11}x_{22}x_{12}x_{14}x_{34}, x_{11}x_{22}x_{14}x_{24}x_{34}, x_{11}x_{22}x_{12}x_{13}x_{24}, \\ x_{11}x_{22}x_{13}x_{24}x_{34}.$$

The full list of facets of $\mathbb{C}P_{10}^2$ may be obtained from these five basic facets by applying the group A_4 . Under this group, the first three basic facets form orbits of length 12 each, while each of the last two forms an orbit of length 6, yielding a total of

$3 \times 12 + 2 \times 6 = 48$ facets. The complex $\mathbb{C}P_{10}^2$ is 2-neighborly and its face vector is $(10, 45, 110, 120, 48)$.

Here we prove the following:

Theorem 1 *There are exactly two 16-vertex simplicial complexes which (i) are simplicial subdivisions of the cell complex $S_4^2 \times S_4^2$, (ii) retain the self-homeomorphism $\alpha : (x, y) \mapsto (y, x)$ of $|S_4^2| \times |S_4^2|$ as a simplicial automorphism, and (iii) the action of $\mathbb{Z}_2 = \langle \alpha \rangle$ is pure (cf. Definition 2). These two complexes are isomorphic and one of them is $(S^2 \times S^2)_{16}$.*

Corollary 2 *The complex $\mathbb{C}P_{10}^2 := (S^2 \times S^2)_{16}/\mathbb{Z}_2$ is a 10-vertex triangulation of $\mathbb{C}P^2$. Its full automorphism group is A_4 .*

Let T and $\mathcal{I}$ denote the solid tetrahedron and the icosahedron in $\mathbb{R}^3$ respectively. Thus, the cell complex $S_4^2 \times S_4^2$ alluded to above is a subcomplex of the boundary complex of the product polytope $T \times T$ in $\mathbb{R}^6$. Although we do not present the details in this paper, Theorem 1 can be strengthened (following the same line of arguments) to show that there is a unique simplicial subdivision S_{16}^5 of the cell complex $\partial(T \times T)$ which is $\mathbb{Z}_2$ -stable with a pure $\mathbb{Z}_2$ -action. To our utter surprise, it turns out that as an abstract simplicial complex, S_{16}^5 is isomorphic to the combinatorial join $S_4^2 * S_{12}^2$ of the boundary complexes of T and $\mathcal{I}$ respectively.

Remark 1 This last fact has the following geometric interpretation. Let $T \circledast \mathcal{I}$ denote the convex hull of $T \cup \mathcal{I}$, where T and $\mathcal{I}$ sit in two (three-dimensional) affine subspaces of $\mathbb{R}^6$ meeting at a point which is in the interior of both polyhedra. Then $T \circledast \mathcal{I}$ is a simplicial 6-polytope and the boundary complex of this polytope is combinatorially isomorphic to a simplicial subdivision of the boundary complex of $T \times T$. This geometric result cries out for a geometric explanation; but we have none.

By the construction, $(S^2 \times S^2)_{16}$ is a subcomplex of $S_4^2 * S_{12}^2$. Since the decahedron $\mathbb{R}P_6^2$ is the quotient of $S_{12}^2 = \partial\mathcal{I}$ by $\mathbb{Z}_2$, and $\mathbb{Z}_2$ acts trivially on S_4^2 (the latter being the combinatorial child of the “diagonal” S^2 in $S^2 \times S^2$, i.e., the S_4^2 in Fig. 1), on passing to the quotient, we find the surprising inclusion

$$\mathbb{C}P_{10}^2 \subseteq S_4^2 * \mathbb{R}P_6^2.$$

Indeed, S_4^2 and $\mathbb{R}P_6^2$ occur as induced subcomplexes of $\mathbb{C}P_{10}^2$ on a complementary pair of vertex sets. Since both S_4^2 and $\mathbb{R}P_6^2$ are classical objects, and the combinatorial join is such a well known operation on simplicial complexes, this inclusion says that $\mathbb{C}P_{10}^2$ was all along sitting there right before our eyes!

The number 10 obtained here is not optimal. It is well known (cf. [2–4, 10]) that any triangulation of $\mathbb{C}P^2$ requires at least nine vertices, and there is a unique 9-vertex triangulation $\mathbb{C}P_9^2$ of this manifold, obtained by Kühnel [11, 12]. But, our construction is natural in that it is obtained by a combinatorial mimicry of a topological construction of $\mathbb{C}P^2$. It shares this naturalness with another 10-vertex triangulation, say K_{10}^4 , of $\mathbb{C}P^2$ available in the literature, namely the “equilibrium” triangulation of Banchoff and Kühnel [7]. Here we prove the following:

Theorem 3 *The simplicial complex $\mathbb{C}P_{10}^2$ is bistellar equivalent to both $\mathbb{C}P_9^2$ and K_{10}^4 .*

Corollary 4 (a) *Kühnel's 9-vertex simplicial complex $\mathbb{C}P_9^2$ triangulates $\mathbb{C}P^2$.*
 (b) *Both $\mathbb{C}P_{10}^2$ and $\mathbb{C}P_9^2$ induce the standard pl-structure on $\mathbb{C}P^2$.*

Of course, in principle these ideas generalize to arbitrary dimensions. In general, the d -dimensional complex projective space $\mathbb{C}P^d$ is the symmetric d -th power of S^2 , i.e., the quotient of $(S^2)^d$ by the symmetric group S_d acting by co-ordinate permutations. Unfortunately, even in the next case $d = 3$, it is not possible to subdivide the cell complex $S_4^2 \times S_4^2 \times S_4^2$ into a simplicial complex, with a pure S_3 -action, without adding more vertices. Indeed, we found that we need to add 60 more vertices to obtain an $(S^2 \times S^2 \times S^2)_{124}$. On quotienting, we obtain a $\mathbb{C}P_{30}^3$ —again with full automorphism group A_4 . The details are so complicated that we decided to postpone publication. We are presently trying to see if one can apply bistellar moves to this $\mathbb{C}P_{30}^3$ to reduce the number of vertices. It is known that any triangulation of $\mathbb{C}P^3$ requires at least 17 vertices (cf. [2]).

After we submitted a preliminary version of this paper to [arXiv:1004.3157v1](https://arxiv.org/abs/1004.3157v1), 2010, Ulrich Brehm [9] communicated to us that he had the idea of obtaining $\mathbb{C}P_{10}^2$ as a quotient of a 16-vertex $S^2 \times S^2$ in the 1980's; however, he never published the details.

We obtain a second simplicial subdivision $(S^2 \times S^2)'_{16}$ of $S_4^2 \times S_4^2$.

Description of $(S^2 \times S^2)'_{16}$ This is a second simplicial subdivision of the cell complex $S_4^2 \times S_4^2$. It has the same vertex set and automorphism group A_4 . Modulo the group A_4 , its basic facets are:

$$\begin{aligned} & x_{11}x_{12}x_{13}x_{21}x_{31}, x_{11}x_{12}x_{14}x_{21}x_{31}, x_{11}x_{13}x_{14}x_{21}x_{31}, x_{12}x_{13}x_{23}x_{31}x_{32}, \\ & x_{12}x_{14}x_{21}x_{24}x_{31}, x_{12}x_{14}x_{24}x_{31}x_{34}, x_{12}x_{21}x_{24}x_{31}x_{32}, x_{12}x_{24}x_{31}x_{32}x_{34}. \end{aligned}$$

Each facets is in an orbit of length 12, yielding a total of $8 \times 12 = 96$ facets. The complex $(S^2 \times S^2)'_{16}$ has the same face vector as $(S^2 \times S^2)_{16}$, namely, (16, 84, 216, 240, 96).

We perform a finite sequence of generalized bistellar moves on $(S^2 \times S^2)'_{16}$ and obtain the following 12-vertex triangulation $(S^2 \times S^2)_{12}$ of $S^2 \times S^2$.

Description of $(S^2 \times S^2)_{12}$ The vertices are x_{ij} , $1 \leq i \neq j \leq 4$. Its automorphism group $2S_5$ is generated by the two automorphisms $h = (x_{12}x_{14}x_{21}x_{24}x_{31}) \times (x_{13}x_{42}x_{43}x_{32}x_{34})$ and $g = (x_{12}x_{21}x_{24}x_{42}x_{14}x_{41}x_{43}x_{34}x_{13}x_{31}x_{32}x_{23})$. Modulo this group, $(S^2 \times S^2)_{12}$ is generated by the following two basic facets:

$$x_{12}x_{14}x_{21}x_{24}x_{31}, x_{12}x_{13}x_{14}x_{21}x_{31}.$$

The first basic facet is in an orbit of size 12, while the second is in an orbit of size 60, yielding a total of 72 facets. Its face vector is (12, 60, 160, 180, 72).

Theorem 5 *The simplicial complex $(S^2 \times S^2)_{12}$ is a triangulation of $S^2 \times S^2$. Its full automorphism group is $2S_5$, the non-split extension of $\mathbb{Z}_2$ by S_5 .*

The complex $(S^2 \times S^2)_{12}$ has many remarkable properties. Its automorphism group is transitive on its vertices and edges. All its vertices have degree 10 and all its edges have degree 8. Indeed, the link of each edge is isomorphic to the 2-sphere S^2_8 obtained from the boundary complex of the octahedron by starring two vertices in a pair of opposite faces. Also, all triangles of $(S^2 \times S^2)_{12}$ are of degree 3 or 5. The automorphism group is transitive on its triangles of each degree. The degree 3 triangles constitute a weak pseudomanifold whose strong components are two icosahedra. Thus, we find a pair I_1, I_2 of icosahedra sitting canonically inside the 2-skeleton of $(S^2 \times S^2)_{12}$. These two icosahedra are “antimorphic” in the sense that the identity map is an antimorphism between them (cf. Definition 1 below). The structure of $(S^2 \times S^2)_{12}$ is completely described in terms of this antimorphic pair of icosahedra. The full automorphism group $2S_5$ of $(S^2 \times S^2)_{12}$ is a double cover of the common automorphism group of these two icosahedra.

Again, the number 12 here is not optimal. In [12], Kühnel and Laßmann have shown that any triangulation of $S^2 \times S^2$ needs at least 11 vertices, and in [14], Lutz finds (via computer search) several 11-vertex triangulations of $S^2 \times S^2$, all with trivial automorphism groups. Surprisingly, even though $(S^2 \times S^2)_{12}$ is not minimal, it does not admit any proper bistellar moves. Thus, there is no straightforward way to obtain a minimal triangulation of $S^2 \times S^2$ starting from $(S^2 \times S^2)_{12}$.

In [17], Sparla proved two remarkable inequalities on the Euler characteristic χ of a combinatorial 4-manifold M satisfying certain conditions. His first result is that if there is a centrally symmetric simplicial polytope P of dimension $d \geq 6$ such that $M \subseteq \partial P$ and $\text{skel}_2(M) = \text{skel}_2(P)$, then $10(\chi - 2) \geq 4^3 \binom{(d-1)/2}{3}$. Equality holds here if and only if P is a cross polytope (i.e., dual of a hypercube). His second result is: if M has $2d$ vertices and admits a fixed point free involution then $10(\chi - 2) \leq 4^3 \binom{(d-1)/2}{3}$. Equality holds if and only if M also satisfies the hypothesis of the first result for a cross polytope P . Notice that, in view of the Dehn–Sommerville equations, equality in either inequality determines the face vector of M in terms of d alone. To obtain an example of equality (in both results) with $d = 6$, Sparla searched for (and found) a 4-manifold with the predicted face vector under the assumption of an automorphism group $A_5 \times \mathbb{Z}_2$. To determine the topological type of the resulting 12-vertex 4-manifold, he had to compute its intersection form and then appeal to Freedman’s classification of simply connected smooth 4-manifolds. We believe that our approach to Sparla’s complex not only elucidates its true genesis, but also reveals its rich combinatorial structure and contributes to an elementary determination of its topological type. Note, however, that Sparla’s approach reveals yet another remarkable property of $(S^2 \times S^2)_{12}$. It provides a tight rectilinear embedding of $S^2 \times S^2$ in $\mathbb{R}^6$.

Remark 2 If X is a triangulated 4-manifold on at most 12 vertices, then its vertex links are homology 3-spheres on at most 11 vertices, and hence (cf. [5]) are combinatorial spheres. Thus all triangulated 4-manifolds on at most 12 vertices are combinatorial manifolds. (More generally, this argument yields: All triangulated d -manifolds

on at most $d + 8$ vertices are combinatorial manifolds.) In particular, both $\mathbb{C}P_{10}^2$ and $(S^2 \times S^2)_{12}$ are combinatorial manifolds. Actually, an old result of Bing [8] says that all the vertex links of any triangulated 4-manifold are simply connected triangulated 3-manifolds. Therefore, in view of Perelman's theorem (Poincaré conjecture) [15], all triangulated 4-manifolds are combinatorial manifolds, irrespective of the number of vertices.

Remark 3 In [1], Akhmedov and Park have shown that $S^2 \times S^2$ has countably infinite number of distinct smooth structures. Since there is an one to one correspondence between the smooth structures and pl-structures on a 4-manifold (cf. [16, p. 167]), it follows that $S^2 \times S^2$ has infinitely many distinct pl-structures. Since $(S^2 \times S^2)_{16}$ and $(S^2 \times S^2)'_{16}$ are simplicial subdivisions of $S_4^2 \times S_4^2$, it follows that the pl-structures given by $(S^2 \times S^2)_{16}$ and $(S^2 \times S^2)'_{16}$ are standard. Again, $(S^2 \times S^2)_{12}$ is combinatorially equivalent to $(S^2 \times S^2)'_{16}$ (cf. Remark 5) and hence gives the same pl-structure as $(S^2 \times S^2)'_{16}$. So, all the triangulations of $S^2 \times S^2$ discussed here give the standard pl-structure on $S^2 \times S^2$.

2 Preliminaries

All simplicial complexes considered here are finite and the empty set is a simplex (of dimension -1) of every simplicial complex. We now recall some definitions here.

For a finite set V with $d + 2$ ($d \geq 0$) elements, the set ∂V (respectively, $\bar{V}$) of all the proper (resp. all the) subsets of V is a simplicial complex and triangulates the d -sphere S^d (resp. the $(d + 1)$ -ball). The complex ∂V is called the *standard d -sphere* and is also denoted by $S_{d+2}^d(V)$ (or simply by S_{d+2}^d). The complex $\bar{V}$ is called the *standard $(d + 1)$ -ball* and is also denoted by $D_{d+2}^{d+1}(V)$ (or simply by D_{d+2}^{d+1}). (Generally, we write $X = X_n^d$ to indicate that X has n vertices and dimension d .)

For simplicial complexes X, Y with disjoint vertex sets, their *join* $X * Y$ is the simplicial complex whose simplices are all the disjoint unions $A \cup B$ with $A \in X, B \in Y$.

If σ is a simplex of a simplicial complex X then the *link* of σ in X , denoted by $\text{lk}_X(\sigma)$, is the simplicial complex whose simplices are the simplices τ of X such that $\tau \cap \sigma = \emptyset$ and $\sigma \cup \tau$ is a simplex of X . The number of vertices in the link of σ is called the *degree* of σ . Also, the *star* of σ , denoted by $\text{star}_X(\sigma)$ or $\text{star}(\sigma)$, is the subcomplex $\bar{\sigma} * \text{lk}_X(\sigma)$ of X .

For a simplicial complex X , $|X|$ denotes the *geometric carrier*. It may be described as the subspace of $[0, 1]^{V(X)}$ (where $V(X)$ is the vertex set of X) consisting of all functions $f: V(X) \rightarrow [0, 1]$ satisfying (i) $\text{Support}(f) \in X$ and (ii) $\sum_{x \in V(X)} f(x) = 1$. If a space Y is homeomorphic to $|X|$ then we say that X *triangulates* Y . If $|X|$ is a topological manifold (respectively, d -sphere) then X is called a *triangulated manifold* (resp. *triangulated d -sphere*). If $|X|$ is a pl manifold (with the pl-structure induced by X) then X is called a *combinatorial manifold*. For $1 \leq d \leq 4$, X is a combinatorial d -manifold if and only if the vertex links are triangulated $(d - 1)$ -spheres.

The *face vector* of a d -dimensional simplicial complex is the vector $(f_0, f_1, \dots, f_d)$, where f_i is the number of i -dimensional simplices in the complex.

If X is a d -dimensional pure simplicial complex (i.e., every maximal simplex is d -dimensional) and $D, \hat{D}$ are triangulations of the d -ball such that (i) $\partial D = \partial \hat{D} = \hat{D} \cap X$, and (ii) $D \subseteq X$, then the simplicial complex $\hat{X} := (X \setminus D) \cup \hat{D}$ is said to be obtained from X by a *generalized bistellar move* (GBM) with respect to the pair $(D, \hat{D})$. Clearly, in this case, $\hat{X}$ and X triangulate the same topological space and if u is a vertex in ∂D then $\text{lk}_{\hat{X}}(u)$ is obtained from $\text{lk}_X(u)$ by a GBM (cf. [6]).

In particular, let A be a simplex of X whose link in X is a standard sphere ∂B . Suppose also that $B \not\subseteq X$. Then, we may perform the GBM with respect to the pair of balls $(A * \partial B, B * \partial A)$. Such an operation is called a *bistellar move*, and will be denoted by $A \mapsto B$. Also, if C is any simplex of X and x is a new symbol, then we may perform the GBM on X with respect to the pair $(\bar{C} * \text{lk}_X(C), (\bar{x} * \partial C) * \text{lk}_X(C))$. The resulting simplicial complex $\hat{X}$ is said to be obtained from X by *starring* the vertex x in the simplex C . In case C is a facet, this is a bistellar move—the only sort of bistellar move which increases the number of vertices. All other kinds of bistellar moves are said to be *proper*.

Two pure simplicial complexes are called *bistellar equivalent* if one is obtained from the other by a finite sequence of bistellar moves. If X is obtained from Y by the bistellar move $A \mapsto B$ then the complex Z obtained from Y by starring a new vertex u in B is a subdivision of both X and Y . This implies that bistellar equivalent complexes induce same pl-structure on their common geometric carrier.

The group $\mathbb{Z}_2$ acts on $S^2 \times S^2$ by co-ordinate flip. The following proposition is well known to algebraic geometers (cf. [13]):

Proposition 6 *The quotient space $(S^2 \times S^2)/\mathbb{Z}_2$ is homeomorphic to the complex projective plane $\mathbb{C}P^2$.*

3 Relations with the Icosahedron

Emergence of the Icosahedron Let T_0 be the tetrahedron with vertex set $V = \{x_1, x_2, x_3, x_4\}$. Then, viewed abstractly, the boundary complex of the product polytope $T_0 \times T_0$ has vertex set $V \times V$, and faces $A \times B$, where A and B range over all the subsets of V . The product cell complex for $S_4^2 \times S_4^2 = (\partial T_0) \times (\partial T_0)$ is the subcomplex consisting of cells $A \times B$, where A and B range over all the proper subsets of V . We use the notation x_{ij} to denote the vertex (x_i, x_j) of $T_0 \times T_0$. For $i \neq j$, $k \neq l$, $x_{ij}x_{kl}$ forms an edge of $T_0 \times T_0$ if and only if it is one of the solid edges of the icosahedron in Fig. 1. (This picture is a Schlegel diagram obtained by projecting the boundary of the icosahedron on one of its faces. Thus, there is only one “hidden” face (namely, $x_{41}x_{42}x_{43}$) in the picture. What is important for us is the label given to the vertices.)

Notice that the broken edges in the icosahedron are precisely the edges $x_{ij}x_{kl}$ where $\{i, j, k, l\}$ is an even permutation of $\{1, 2, 3, 4\}$.

To obtain the appropriate triangulation of $S^2 \times S^2$, we join x_{ii} to all vertices for all i and also introduce the broken edges of the icosahedron. Thus viewed, one sees the simplicial subdivision $(S^2 \times S^2)_{16}$ of the cell complex $(\partial T_0) \times (\partial T_0)$ as a subcomplex of $(\partial T) * (\partial \mathcal{I})$, where T is the tetrahedron with vertex set $\{x_{ii} : 1 \leq i \leq 4\}$ and $\mathcal{I}$ is the icosahedron depicted in Fig. 1.

Fig. 1 Tetrahedron, Icosahedron and Decahedron

Notice also that the $\mathbb{Z}_2$ -action $x_{ij} \leftrightarrow x_{ji}$ fixes the vertices of T and acts on $\mathcal{I}$ as the antipodal map. Thus, going modulo $\mathbb{Z}_2$, we find $\mathbb{C}P_{10}^2$ as a subcomplex of the 5-dimensional simplicial complex $S_4^2 * \mathbb{R}P_6^2$, where S_4^2 is the 4-vertex 2-sphere given by the boundary complex of T and $\mathbb{R}P_6^2$ is the (minimal) triangulation of the real projective plane (with vertices of the same name being identified) given in Fig. 1.

From our nomenclature for the vertices, the inclusion $\mathbb{C}P_{10}^2 \subseteq S_4^2 * \mathbb{R}P_6^2$ is obvious, as is the fact that $(\partial T) * (\partial \mathcal{I})$ is a simplicial subdivision of the boundary complex of $T_0 \times T_0$.

Finally, note that $\Delta_i = \{x_{ij} : j \neq i\}$ and $\Delta^i = \{x_{ji} : j \neq i\}$ are triangles of the icosahedron, and $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ and $\{\Delta^1, \Delta^2, \Delta^3, \Delta^4\}$ are antipodal quadruples (consisting of triangles) partitioning the vertex set of the icosahedron. It is easy to see that there are exactly five such pairs in the icosahedron, and the automorphism group $A_5 \times \mathbb{Z}_2$ of $\mathcal{I}$ acts transitively on them. The stabilizer of each such pair is $A_4 \times \mathbb{Z}_2$, and A_4 acts regularly on the vertex set of $\mathcal{I}$. Our choice of nomenclature for the vertices of $\mathcal{I}$ amounts to choosing one such antipodal pair of quadruples. This is because we have $\Delta_i \cap \Delta^j = \emptyset$ if $i = j$ and $= \{x_{ij}\}$ if $i \neq j$. Viewed dually, one sees Kepler's regular tetrahedra embedded in the dodecahedron. Namely, the centers of Δ_i , $1 \leq i \leq 4$ (as well as of Δ^i , $1 \leq i \leq 4$) are the vertices of a regular tetrahedron inscribed in the dual dodecahedron.

The 12-vertex triangulation $(S^2 \times S^2)_{12}$ of $S^2 \times S^2$ is obtained from $(S^2 \times S^2)'_{16}$ by a sequence of bistellar moves (cf. proof of Theorem 5). However, its most elegant description requires the introduction of the following definition.

Definition 1 Let I_1 and I_2 be two copies of the icosahedron. A bijection $f: V(I_1) \rightarrow V(I_2)$ is said to be an *antimorphism* if, for all vertices x, y of I_1 , we have (a) x and y are at distance one in I_1 if and only if $f(x)$ and $f(y)$ are at distance two in I_2 , and (b) x and y are at distance two in I_1 if and only if $f(x)$ and $f(y)$ are at distance one

Fig. 2 An antimorphic pair of icosahedra

in I_2 . (It follows that x and y are at distance 3 (antipodal) in I_1 if and only if $f(x)$ and $f(y)$ are at distance 3 (antipodal) in I_2 .) Here distance refers to the usual graphical distance on the respective edge graph. In case $V(I_1) = V(I_2)$ and the identity map is an antimorphism between I_1 and I_2 , then we say that I_1 and I_2 are *antimorphic*. Thus, the two icosahedra in Fig. 2 below are antimorphic (the map, taking each vertex of the left icosahedron in Fig. 2 to the vertex of the same name in the right icosahedron, is an antimorphism).

Another description of $(S^2 \times S^2)_{12}$ Take an antimorphic pair of icosahedra, say I_1 and I_2 (with common vertex set V). It turns out that I_1 and I_2 have the identical automorphism group $A_5 \times \mathbb{Z}_2$ (not merely isomorphic; cf. Lemma 9 below). Also, there is a bijection φ from the triangles of I_1 to the triangles of I_2 such that for each triangle $\Delta = abc$ of I_1 , $\varphi(\Delta) = ijk$ is the only triangle of I_2 for which aij , bjk and cik are triangles of I_2 (cf. Lemma 9). Now, the vertex set of $(S^2 \times S^2)_{12}$ is $V (= V(I_1) = V(I_2))$ and it has two types of facets. (i) For each vertex x , the neighbors of x in I_1 form facets. (ii) For each triangle Δ of I_1 and each vertex y in $\Delta' = \varphi(\Delta)$, $(\Delta \cup \Delta') \setminus \{y\}$ is a facet. Thus $(S^2 \times S^2)_{12}$ has 12 facets of the first type and $20 \times 3 = 60$ facets of the second type. From the description, it is clear that the common automorphism group $A_5 \times \mathbb{Z}_2$ of I_1 and I_2 is an automorphism group of $(S^2 \times S^2)_{12}$. It turns out that its full automorphism group is $2S_5$ generated by the two automorphisms $g = (x_{12}x_{21}x_{24}x_{42}x_{14}x_{41}x_{43}x_{34}x_{13}x_{31}x_{32}x_{23})$ and $h = (x_{12}x_{14}x_{21}x_{24}x_{31})(x_{13}x_{42}x_{43}x_{32}x_{34})$. The automorphism g interchanges I_1 and I_2 .

Remark 4 It should be emphasized that the existence of an antimorphic pair of icosahedra (exploited in the above construction of $(S^2 \times S^2)_{12}$) is a minor miracle, and only an empirically verified fact. Its deeper geometric significance, if any, remains to be understood.

4 A Self-dual CW Decomposition of $\mathbb{C}P^2$

Here we have taken the cell complex $\partial T_0 \times \partial T_0$, and triangulated it to obtain the simplicial complex $(S^2 \times S^2)_{16}$ and finally quotiented this simplicial complex by $\mathbb{Z}_2$ to obtain $\mathbb{C}P^2_{10}$. This procedure reflects our obsession with simplicial complexes. However, one may straightaway quotient the cell complex by $\mathbb{Z}_2$ to obtain a (non-regular) CW decomposition of $\mathbb{C}P^2$. This CW complex is self-dual in the sense that its face-vector $(10, 24, 31, 24, 10)$ exhibits a curious palindromic symmetry. We proceed to describe it in some details. Consider the $\mathbb{Z}_2$ -action on $\mathbb{R}^6 \equiv \mathbb{R}^3 \times \mathbb{R}^3$ given by $(x, y) \leftrightarrow (y, x)$. Let $\eta: \mathbb{R}^6 \rightarrow \mathbb{R}^6/\mathbb{Z}_2$ be the quotient map. We know that $\eta(S^2 \times S^2) = \mathbb{C}P^2$. We give a CW decomposition W of the space $\eta(\partial T_0 \times \partial T_0)$.

For $0 \leq i \leq 4$, let W^i denote the set of i -cells in W . For $i \neq 2$ the i -cells in W are the images (under the map η) of i -cells in $\partial T_0 \times \partial T_0$. A 2-cell in W is the image of a 2-cell F in $\partial T_0 \times \partial T_0$ which is not of the form $E \times E$ for some edge E in ∂T_0 . More explicitly

$$\begin{aligned} W^0 &= V(\mathbb{C}P^2_{10}), \\ W^1 &= \{\eta(E) : E \text{ is an edge of } \partial T_0 \times \partial T_0\}, \\ W^2 &= \{\eta(|x_{ij}x_{ik}x_{il}|) : 1 \leq j < k < l \leq 4, 1 \leq i \leq 4\} \\ &\quad \cup \{\eta(|x_i x_j| \times |x_k x_l|) : i < j, k < l \text{ and either } i < k \text{ or } i = k \text{ and } j < l\}, \\ W^3 &= \{\eta(A) : A \text{ is a 3-cell of } \partial T_0 \times \partial T_0\} \quad \text{and} \\ W^4 &= \{\eta(B) : B \text{ is a 4-cell of } \partial T_0 \times \partial T_0\}. \end{aligned}$$

Then, W^1 contains 24 cells, W^2 contains $16 + 15 = 31$ cells, W^3 contains $4 \times 6 = 24$ cells and W^4 contains 10 cells. Clearly, each 1-cell in W is regular (i.e., homeomorphic to a closed interval). Since all the 2-cells are homeomorphic images of the corresponding 2-cells in $\partial T_0 \times \partial T_0$, it follows that all the 2-cells in W are regular.

For $0 \leq i \leq 4$, let $X_i = \bigcup_{\beta \in W^0 \cup \dots \cup W^i} \beta$. Then $\partial \alpha \subseteq X_{i-1}$ if $\alpha \in W^i$ for $i \neq 3$. Let γ be a 3-cell in W . If $\gamma = \eta(|x_i x_j x_k| \times |x_i x_j|)$, $i < j < k$, then γ is obtained from $|x_i x_j x_k| \times |x_i x_j|$ by identifying $|x_{ii} x_{jj} x_{ij}|$ with $|x_{ii} x_{jj} x_{ji}|$ (by the identification given by $x_{ij} \leftrightarrow x_{ji}$). Thus, γ is a regular 3-cell and $\partial \gamma = \eta(|x_i x_k| \times |x_i x_j|) \cup \eta(|x_j x_k| \times |x_i x_j|) \cup \eta(|x_{ii} x_{ji} x_{ki}|) \cup \eta(|x_{ij} x_{jj} x_{kj}|)$. (Now, it is clear why we do not have to take $\eta(|x_i x_j| \times |x_i x_j|)$ in W^2 . In fact, $\eta(|x_i x_j| \times |x_i x_j|)$ is inside of γ .) Therefore, $\partial \gamma \subseteq X_2$. Same things are true if $\gamma = \eta(|x_i x_j x_k| \times |x_i x_k|)$ or $\eta(|x_i x_j x_k| \times |x_j x_k|)$. On the other hand, if $\gamma = \eta(F \times E)$, where E is an edge and F is a 2-simplex and $E \not\subseteq F$, then γ is homeomorphic to $F \times E$ and hence is a regular 3-cell. In this case, it follows from the definition of W^2 that $\partial \gamma \subseteq X_2$. Thus W is a CW complex.

If σ is a 4-cell in W then, either $\sigma = \eta(|x_i x_j x_k| \times |x_i x_j x_k|)$, for some $i < j < k$ or $\sigma = \eta(|x_i x_j x_k| \times |x_i x_j x_l|)$, where $\{i, j, k, l\}$ is an even permutation of $\{1, 2, 3, 4\}$. In the first case, σ is homeomorphic to $|x_{ii} x_{jj} x_{kk} x_{ij} x_{ik}| \cup |x_{ii} x_{jj} x_{kk} x_{ij} x_{jk}| \cup |x_{ii} x_{jj} x_{kk} x_{ik} x_{jk}|$ and hence σ is a regular 4-cell. In the second case, σ is obtained

from $|x_i x_j x_k| \times |x_i x_j x_l|$ by identifying $|x_{ii} x_{jj} x_{ij}|$ with $|x_{ii} x_{jj} x_{ji}|$ (by the identification given by $x_{ij} \leftrightarrow x_{ji}$). So, σ is not a regular cell. Thus W^4 contains four regular 4-cells and six singular 4-cells.

Since each cell in W is the quotient of a cell in $S_4^2 \times S_4^2$, $(S^2 \times S^2)_{16}$ is a simplicial subdivision of $S_4^2 \times S_4^2$ and $\mathbb{C}P_{10}^2$ is the quotient of $(S^2 \times S^2)_{16}$, it follows that $\mathbb{C}P_{10}^2$ is a simplicial subdivision of W .

5 Proofs

Definition 2 Let G be a group of simplicial automorphisms of a simplicial complex X with vertex set $V(X)$. We shall say that the action of G on X is *pure* if it satisfies: (a) whenever u, v are distinct vertices from the same G -orbit, uv is a non-edge of X , and (b) for each G -orbit $\theta \subseteq V(X)$ and each $\alpha \in X$, the stabilizer G_α of α in G acts transitively on $\theta \cap V(\text{lk}_X(\alpha))$.

Lemma 7 Let G be a group of simplicial automorphisms of a simplicial complex X . Let $q: V(X) \rightarrow V(X)/G$ denote the quotient map, and $X/G := \{q(\alpha) : \alpha \in X\}$. If the action of G on X is pure then X/G is a simplicial complex which triangulates $|X|/G$ (where the action of G on $V(X)$ is extended to an action of G on $|X|$ piecewise linearly, i.e., affinely on the geometric carrier of each simplex). That is, we have $|X/G| = |X|/G$.

Proof The condition (a) ensures that the quotient map q is one-one on each simplex of X . The simplicial map $q: X \rightarrow X/G$ induces a piecewise linear continuous map $|q|$ from $|X|$ onto $|X/G|$.

Claim The fibers of $q: X \rightarrow X/G$ are precisely the G -orbits on simplices of X (that is, if $\alpha, \alpha' \in X$ are such that $q(\alpha) = q(\alpha')$ then there exists $g \in G$ such that $g(\alpha) = \alpha'$).

We prove the claim by induction on $k = \dim(\alpha) = \dim(\alpha')$. The claim is trivial for $k = -1$. So, assume $k \geq 0$, and the claim is true for all smaller dimensions. Choose a simplex $\beta \subseteq \alpha$ of dimension $k - 1$, and let $\beta' \subseteq \alpha'$ be such that $q(\beta') = q(\beta)$. By induction hypothesis, β' and β are in the same G -orbit. Therefore, applying a suitable element of G , we may assume, without loss of generality, that $\beta' = \beta$. Let $\alpha = \beta \cup \{x\}$, $\alpha' = \beta \cup \{x'\}$. Then $q(x) = q(x')$, i.e., x and x' are in the same G -orbit. Now, by assumption (b), there is a $g \in G_\beta$ such that $g(x) = x'$. Then $g(\alpha) = \alpha'$. This proves the claim.

In the presence of condition (a), the claim ensures that the fibers of $|q|$ are precisely the G -orbits on points of $|X|$. Hence $|q|$ induces the required homeomorphism between $|X|/G$ and $|X/G|$. $\square$

Up to isomorphism, there are exactly two 6-vertex 2-spheres, namely, S_1 and S_2 given in Fig. 3. We need the following lemma to prove Theorem 1.

Fig. 3 (a) 6-vertex 2-spheres. (b) Triangular prism

Fig. 4 Simplicial subdivision of the triangular prism

Lemma 8 Let C be the triangular prism given in Fig. 3(b) (i.e., C is the product of a 2-simplex and an edge). Up to isomorphism, there exists a unique 6-vertex simplicial subdivision $\hat{C}$ of C . The facets (tetrahedra) in $\hat{C}$ are $a_1b_1b_2b_3, a_1a_2b_2b_3, a_1a_2a_3b_3$. Moreover, $\partial\hat{C}$ is isomorphic to S_2 of Fig. 3(a) and determines $\hat{C}$ uniquely.

Proof Let $\hat{C}$ be a 6-vertex subdivision of C . Then there exists a 3-simplex σ in $\hat{C}$ which contains the 2-simplex $b_1b_2b_3$. Without loss of generality, we may assume that $\sigma = a_1b_1b_2b_3$. Then C is the union of σ and the pyramid P given in Fig. 4. Since we are not allowed to introduce new vertices, clearly the rectangular base of P must be triangulated using two triangles, in one of two isomorphic ways, and the remaining tetrahedra in $\hat{C}$ must have the apex of P as a vertex and one of these two triangles as base. Thus, without loss of generality, $P = a_1a_2b_2b_3 \cup a_1a_2a_3b_3$. This proves the first part.

The last part follows from the fact that the facets of $\hat{C}$ are the maximal cliques in the 1-skeleton of $\partial\hat{C}$. $\square$

Proof of Theorem 1 Let X be a 16-vertex simplicial subdivision of $S_4^2 \times S_4^2$ satisfying (i), (ii) and (iii).

For $i \neq j$, consider the 2-cell $x_ix_j \times x_ix_j$. By (iii), $x_ix_jx_j$ can not be an edge in X . This implies that $x_{ii}x_{jj}, x_{ii}x_{ji}x_{jj}, x_{ii}x_{ij}x_{jj} \in X$ and $x_ix_j \times x_ix_j = x_{ii}x_{ji}x_{jj} \cup x_{ii}x_{ij}x_{jj}$ (cf. Fig. 5(a)).

For i, j, k distinct, consider the 2-cell $x_ix_j \times x_ix_k$. Since X satisfies (iii), both x_{ij} and x_{ji} cannot be in $\text{lk}_X(x_{ik})$. Now, $x_{ik}x_{ij}$ is an edge in the cell complex $S_4^2 \times S_4^2$ and hence is an edge in X . Thus, $x_{ik}x_{ji}$ can not be an edge in X . This implies that $x_{ii}x_{jk}, x_{ii}x_{ji}x_{jk}, x_{ii}x_{ik}x_{jk} \in X$ and $x_ix_j \times x_ix_k = x_{ii}x_{ji}x_{jk} \cup x_{ii}x_{ik}x_{jk}$ (cf. Fig. 5(b)).

Fig. 5 Simplicial subdivisions of rectangular 2-cells of $S_4^2 \times S_4^2$

Fig. 6 Simplicial subdivisions of 3-cells of $S_4^2 \times S_4^2$

Consider the 2-cell $x_1x_3 \times x_2x_4$. Clearly, $x_1x_3 \times x_2x_4 = x_{12}x_{32}x_{34} \cup x_{12}x_{14}x_{34}$ or $= x_{12}x_{32}x_{14} \cup x_{32}x_{14}x_{34}$.

Case 1. $x_1x_3 \times x_2x_4 = x_{12}x_{32}x_{34} \cup x_{12}x_{14}x_{34}$ (cf. Fig. 5(c)). So, $x_{12}x_{34} \in X$. Then, by (ii), $x_{21}x_{43} \in X$ and, by (iii), $x_{12}x_{43}, x_{21}x_{34} \notin X$. This implies that $x_2x_3 \times x_1x_4 = x_{21}x_{31}x_{24} \cup x_{31}x_{24}x_{34}$ (cf. Fig. 5(d)). So, $x_{31}x_{24} \in X$. Then, by (ii), $x_{13}x_{42} \in X$ and, by (iii), $x_{13}x_{24}, x_{31}x_{42} \notin X$. This implies that $x_1x_2 \times x_3x_4 = x_{13}x_{23}x_{14} \cup x_{23}x_{14}x_{24}$ (cf. Fig. 5(e)). So, $x_{14}x_{23} \in X$. Then, by (ii), $x_{41}x_{32} \in X$ and, by (iii), $x_{14}x_{32}, x_{41}x_{23} \notin X$. These give the 2-skeleton of X . Observe that we have already 84 edges as mentioned in the construction of $(S^2 \times S^2)_{16}$ and, since X satisfies (iii), all the 36 remaining 2-sets are non-edges in X .

Observe that any 3-cell in $S_4^2 \times S_4^2$ is the product of a 2-simplex and an edge. For i, j, k distinct, consider the 3-cell $x_ix_jx_k \times x_ix_j$. Since $x_{ii}x_{jj}, x_{ii}x_{ki}$ and $x_{jj}x_{ki}$ are edges, by Lemma 8, $x_ix_jx_k \times x_ix_j = x_{ii}x_{ij}x_{kj}x_{jj} \cup x_{ii}x_{ki}x_{kj}x_{jj} \cup x_{ii}x_{ki}x_{ji}x_{jj}$ is the unique subdivision of $x_ix_jx_k \times x_ix_j$ (cf. Fig. 6(a)). Similarly, $x_ix_j \times x_ix_jx_k = x_{ii}x_{ji}x_{jk}x_{jj} \cup x_{ii}x_{ik}x_{jk}x_{jj} \cup x_{ii}x_{ik}x_{ij}x_{jj}$ is the unique subdivision of $x_ix_j \times x_ix_jx_k$ (cf. Fig. 6(b)).

For i, j, k, l distinct, consider the 3-cell $x_ix_jx_k \times x_ix_l$. Here $x_{ii}x_{jl}$ and $x_{ii}x_{kl}$ are edges. By interchanging j and k (if required) we may assume that $\{i, j, k, l\}$ is an even permutation of $\{1, 2, 3, 4\}$. Then $x_{ki}x_{jl}$ is an edge and hence, by Lemma 8, $x_ix_jx_k \times x_ix_l = x_{ii}x_{il}x_{kl}x_{jl} \cup x_{ii}x_{ki}x_{kl}x_{jl} \cup x_{ii}x_{ki}x_{ji}x_{jl}$ is the unique subdivision of $x_ix_jx_k \times x_ix_l$ (cf. Fig. 6(c)). Similarly, for the 3-cell $x_ix_l \times x_ix_jx_k$, we may assume that $\{i, j, k, l\}$ is an even permutation of $\{1, 2, 3, 4\}$. Then $x_{ik}x_{lj}$ is an edge and hence, by Lemma 8, $x_ix_l \times x_ix_jx_k = x_{ii}x_{li}x_{lk}x_{lj} \cup x_{ii}x_{ik}x_{lk}x_{lj} \cup x_{ii}x_{ik}x_{ij}x_{lj}$ is the unique subdivision of $x_ix_l \times x_ix_jx_k$ (cf. Fig. 6(d)). These give the 3-skeleton of X .

For i, j, k distinct, consider the 4-cell $A = x_ix_jx_k \times x_ix_jx_k$. The boundary ∂A of A consists of six 3-cells. From above, it follows that $S^1(\{x_{ii}, x_{jj}, x_{kk}\}) * C_6 \subseteq$

X is the subdivision of ∂A , where C_6 is the 6-cycle $C_6(x_{ij}, x_{ik}, x_{jk}, x_{ji}, x_{ki}, x_{kj})$. Let $D \subseteq X$ be the subdivision of A . Then, D is a 9-vertex 4-ball with boundary $\partial D = S^1(\{x_{ii}, x_{jj}, x_{kk}\}) * C_6$. Clearly, C_6 is an induced subcomplex of X . Therefore, each 4-simplex in B must contain $x_{ii}x_{jj}x_{kk}$. Thus, $x_{ii}x_{jj}x_{kk}$ is a simplex in $D \setminus \partial D$. Therefore, $\text{lk}_D(x_{ii}x_{jj}x_{kk})$ is a cycle and hence $= C_6$. These imply that $D = \overline{x_{ii}x_{jj}x_{kk}} * C_6$.

Now, consider the 4-cell $B = x_i x_j x_k \times x_i x_j x_l$, where i, j, k, l are distinct. By interchanging i and j (if required) we may assume that $\{i, j, k, l\}$ is an even permutation of $\{1, 2, 3, 4\}$. The boundary ∂B of B consists of six 3-cells. From above, it follows that the subdivision of ∂B in X is a 9-vertex triangulated 3-sphere and obtained from $S^1_3(\{x_{ii}, x_{jj}, x_{kl}\}) \times C_5$ by starring the vertex x_{ji} in the 3-simplex $\alpha := x_{ii}x_{jj}x_{jl}x_{ki}$, where C_5 is the 5-cycle $C_5(x_{ij}, x_{il}, x_{jl}, x_{ki}, x_{kj})$. Since $x_{ji}x_{ij}$, $x_{ji}x_{il}$, $x_{ji}x_{kj}$ and $x_{ji}x_{kl}$ are non-edges, it follows that $\sigma := x_{ii}x_{jj}x_{ji}x_{jl}x_{ki}$ is the only possible 4-simplex containing x_{ji} inside B . So, $\sigma \in X$. Then $B = \sigma \cup P$, where P is a 4-cell such that $P \cap \sigma = \alpha$ and $S^1_3(\{x_{ii}, x_{jj}, x_{kl}\}) * C_5 \subseteq X$ is the subdivision of ∂P in X (i.e., P is the 4-cell whose geometric carrier is $(|B| \setminus |\sigma|) \cup |\alpha|$). Let Q be the simplicial subdivision of P in X . So, $\partial Q = S^1_3(\{x_{ii}, x_{jj}, x_{kl}\}) * C_5$. Since C_5 is induced in X , it follows that any 4-simplex in Q must contain $x_{ii}x_{jj}x_{kl}$. Since $x_{ii}x_{jj}x_{kl} \in Q \setminus \partial Q$, $\text{lk}_Q(x_{ii}x_{jj}x_{kl})$ is a cycle and hence $= C_5$. These imply that $Q = \overline{x_{ii}x_{jj}x_{kl}} * C_5$. Then $B = (\overline{x_{ii}x_{jj}x_{kl}} * C_5) \cup \bar{\sigma}$.

Now, we have subdivided all the 4-cells in $S^2_4 \times S^2_4$. It is routine to check that the resulting simplicial complex X is identical with the complex $(S^2 \times S^2)_{16}$ defined in Sect. 1.

Case 2. $x_1x_3 \times x_2x_4 = x_{12}x_{32}x_{14} \cup x_{32}x_{14}x_{34}$. By the same method as in Case 1, one can show that X is uniquely determined and is isomorphic to $(S^2 \times S^2)_{16}$ via the map f given by the transposition $(1, 2)$ on the suffixes, i.e., $f \equiv (x_{11}x_{22})(x_{13}x_{23})(x_{14}x_{24})(x_{31}x_{32})(x_{41}x_{42})$. This completes the proof. $\square$

Proof of Corollary 2 From Proposition 6, Lemma 7 and Theorem 1, it is immediate that $\mathbb{C}P^2_{10}$ triangulates $\mathbb{C}P^2$.

Since the automorphism group $A_4 = \langle \alpha, \beta \rangle$ of $(S^2 \times S^2)_{16}$ commutes with $\mathbb{Z}_2$, it descends to an automorphism group $A_4 = \langle \bar{\alpha}, \bar{\beta} \rangle$ of $\mathbb{C}P^2_{10}$. We need to show that there are no other automorphisms.

It is easy to check that the four vertices x_{ii} , $1 \leq i \leq 4$, are the only ones with 2-neighborly links. Therefore, the full automorphism group must fix this set of four vertices. Since A_4 is 2-transitive on this 4-set, it suffices to show that there is no non-trivial automorphism γ fixing both x_{11} and x_{22} . Suppose the contrary. Then γ is a non-trivial automorphism of $\text{lk}(x_{11}x_{22})$. But $\text{lk}(x_{11}x_{22})$ is the 8-vertex triangulated 2-sphere given in Fig. 7.

From the picture, it is apparent that $\text{lk}(x_{11}x_{22})$ has only one non-trivial automorphism, namely $(x_{13}, x_{24})(x_{14}, x_{23})(x_{33}, x_{44})$. Therefore, $\gamma = (x_{13}, x_{24})(x_{14}, x_{23})(x_{33}, x_{44})$ and hence γ fixes the 3-simplex $x_{11}x_{33}x_{44}x_{34}$. Then γ must either fix or interchange the two vertices x_{13} and x_{14} in the link of this 3-simplex, a contradiction. This completes the proof. $\square$

Fig. 7 $\text{lk}_{\mathbb{CP}_{10}^2}(x_{11}x_{22})$

Proof of Theorem 3 Consider the following sequence of bistellar moves on $\mathbb{CP}_{10}^2$ (performed one after the other):

- (i) $x_{22}x_{33}x_{44} \mapsto x_{23}x_{24}x_{34}$, (ii) $x_{11}x_{33}x_{44} \mapsto x_{13}x_{14}x_{34}$,
- (iii) $x_{11}x_{22}x_{44} \mapsto x_{12}x_{14}x_{24}$, (iv) $x_{14}x_{33}x_{44} \mapsto x_{12}x_{13}x_{34}$,
- (v) $x_{22}x_{34}x_{44} \mapsto x_{13}x_{23}x_{24}$, (vi) $x_{23}x_{33}x_{44} \mapsto x_{12}x_{24}x_{34}$,
- (vii) $x_{12}x_{22}x_{44} \mapsto x_{13}x_{14}x_{24}$, (viii) $x_{33}x_{44} \mapsto x_{12}x_{13}x_{24}x_{34}$,
- (ix) $x_{22}x_{44} \mapsto x_{13}x_{14}x_{23}x_{24}$.

At the end of these moves, we get a 10-vertex triangulation K of $\mathbb{CP}^2$.

On K we perform the following sequence of bistellar moves one after another.

- (x) $x_{11}x_{24}x_{44} \mapsto x_{12}x_{14}x_{23}$, (xi) $x_{11}x_{13}x_{44} \mapsto x_{14}x_{23}x_{34}$,
- (xii) $x_{11}x_{44} \mapsto x_{12}x_{14}x_{23}x_{34}$, (xiii) $x_{44}x_{14}x_{24} \mapsto x_{12}x_{13}x_{23}$,
- (xiv) $x_{44}x_{14} \mapsto x_{34}x_{12}x_{13}x_{23}$, (xv) $x_{44} \mapsto x_{24}x_{34}x_{12}x_{13}x_{23}$.

(Note that the last three bistellar moves together is same as the GBM with respect to $(\overline{x_{44}} * S_3^1(\{x_{14}, x_{24}, x_{34}\}) * S_3^1(\{x_{12}, x_{13}, x_{23}\}), S_3^1(\{x_{14}, x_{24}, x_{34}\}) * \overline{x_{12}x_{13}x_{23}})$.) The last bistellar move deletes the vertex x_{44} and hence obtain a 9-vertex triangulation L of $\mathbb{CP}^2$. (Observe that $A_1 = \{x_{11}, x_{23}, x_{24}\}$, $A_2 = \{x_{14}, x_{33}, x_{12}\}$, $A_3 = \{x_{34}, x_{22}, x_{13}\}$ is an amicable partition of L whose layer is of first type (cf. [4]).)

Let $\mathbb{CP}_9^2$ be as described in [11] with vertex set $\{1, 2, \dots, 9\}$. Consider the map $\varphi: L \rightarrow \mathbb{CP}_9^2$ given by: $\varphi(x_{11}) = 1$, $\varphi(x_{23}) = 2$, $\varphi(x_{24}) = 3$, $\varphi(x_{34}) = 4$, $\varphi(x_{22}) = 5$, $\varphi(x_{13}) = 6$, $\varphi(x_{14}) = 7$, $\varphi(x_{33}) = 8$, $\varphi(x_{12}) = 9$. It is easy to see that φ is an isomorphism. Thus, $\mathbb{CP}_{10}^2$ is bistellar equivalent to $\mathbb{CP}_9^2$.

Now, on K we perform the following sequence of bistellar moves:

- (xvi) $x_{11}x_{22}x_{33} \mapsto x_{12}x_{13}x_{23}$, (xvii) $x_{22}x_{33}x_{24} \mapsto x_{14}x_{23}x_{34}$,
- (xviii) $x_{22}x_{33}x_{13} \mapsto x_{12}x_{14}x_{23}$, (xix) $x_{22}x_{33} \mapsto x_{12}x_{14}x_{23}x_{34}$.

We obtain a 10-vertex triangulation M of $\mathbb{CP}^2$. Let K_{10}^4 be as described in [7] with vertex set $\{X, Y, Z, 0, 1, \dots, 6\}$. Consider the map $\psi: M \rightarrow K_{10}^4$ given by $\psi(x_{33}) = X$, $\psi(x_{22}) = Y$, $\psi(x_{44}) = Z$, $\psi(x_{11}) = 0$, $\psi(x_{13}) = 1$, $\psi(x_{12}) = 2$, $\psi(x_{23}) = 3$, $\psi(x_{14}) = 4$, $\psi(x_{34}) = 5$, $\psi(x_{24}) = 6$. It is easy to see that ψ is an isomorphism. Thus, $\mathbb{CP}_{10}^2$ is bistellar equivalent to K_{10}^4 . This completes the proof. $\square$

Proof of Corollary 4 Part (a) follows from Corollary 2 and Theorem 3.

In [7], explicit coordinates for simplices of K_{10}^4 in the Fubini–Study metric were given. This shows that the induced pl-structure on $\mathbb{C}P^2$ by K_{10}^4 is the standard one. Part (b) now follows from Theorem 3. $\square$

Lemma 9 *Let I_1 and I_2 be an antimorphic pair of icosahedra. Then we have:*

- (a) $\text{Aut}(I_1) = \text{Aut}(I_2) = A_5 \times \mathbb{Z}_2$.
- (b) *For each triangle Δ of I_1 , there is a unique triangle Δ' of I_2 such that each of the three triangles of I_2 sharing an edge with Δ' has its third vertex in Δ . Further, the map $\varphi: \Delta \mapsto \Delta'$ is a bijection from the triangles of I_1 to the triangles of I_2 . There is a similarly defined bijection ψ from the triangles of I_2 to the triangles of I_1 , and*
- (c) *Every isomorphism $f: I_1 \rightarrow I_2$ intertwines φ and ψ .*

(Warning: The maps φ and ψ are **not** induced by any vertex-to-vertex map!)

Proof Recall that I_1 and I_2 have the same vertex set and the same pairs of antipodal vertices. Thus, they have the same antipodal map (sending each vertex x to its antipode $\bar{x}$). Now, the full automorphism group of the icosahedron is generated by its rotation group A_5 and the antipodal map. So, to prove Part (a), it suffices to show that I_1 and I_2 share the same rotation group. For each pair $x, \bar{x}$ of antipodes, I_i has a rotation symmetry $\alpha_{x,\bar{x}}^i$ which fixes x and $\bar{x}$ and rotates the remaining vertices along the 5-cycles $\text{lk}_{I_i}(x)$ and $\text{lk}_{I_i}(\bar{x})$. The rotation group of I_i is generated by these automorphisms of order five. But, $\text{lk}_{I_2}(x)$ (respectively, $\text{lk}_{I_2}(\bar{x})$) is the graph theoretic complement of the pentagon $\text{lk}_{I_1}(\bar{x})$ (respectively, $\text{lk}_{I_1}(x)$). Therefore, $\alpha_{x,\bar{x}}^2$ is the square of $\alpha_{x,\bar{x}}^1$. This proves Part (a).

Notice that if $f_1, f_2: I_1 \rightarrow I_2$ are two antimorphisms, then $f_1 \circ f_2^{-1} \in \text{Aut}(I_2)$ and $f_2^{-1} \circ f_1 \in \text{Aut}(I_1)$. Thus, the antimorphism is unique up to right multiplication by elements of $\text{Aut}(I_1)$ (or left multiplication by elements of $\text{Aut}(I_2)$). Therefore, there is no loss of generality in taking the antimorphic pair of icosahedra as the one given in Fig. 2.

Since the common automorphism group is transitive on the triangles of I_1 (and of I_2), it is enough to look at the triangle $\Delta = x_{12}x_{13}x_{14}$ of I_1 . From Fig. 2, we see that the links in I_2 of two vertices of Δ have exactly two vertices in common. Namely, we have $V(\text{lk}_{I_2}(x_{12})) \cap V(\text{lk}_{I_2}(x_{13})) = \{x_{21}, x_{32}\}$, $V(\text{lk}_{I_2}(x_{12})) \cap V(\text{lk}_{I_2}(x_{14})) = \{x_{24}, x_{41}\}$, $V(\text{lk}_{I_2}(x_{13})) \cap V(\text{lk}_{I_2}(x_{14})) = \{x_{31}, x_{43}\}$. Therefore, any triangle Δ' of I_2 satisfying the requirement must be contained in the vertex set $\{x_{21}, x_{32}, x_{24}, x_{41}, x_{31}, x_{43}\}$. But one sees that this set of six vertices contains a unique triangle in I_2 , namely $\Delta' = x_{21}x_{31}x_{41}$. Thus the map $\varphi: \Delta \rightarrow \Delta'$ is well defined. Similarly, there is a well defined map ψ from the triangles of I_2 to the triangles of I_1 . The map $\psi \circ \varphi$ is the antipodal map on the triangles of I_1 to themselves. Similarly, $\varphi \circ \psi$ is the antipodal map on triangles of I_2 . Hence φ (as well as ψ) is a bijection. This proves Part (b).

To prove Part (c), let f be any isomorphism from I_1 to I_2 . Since I_1 and I_2 are antimorphic, it is immediate that f also defines an isomorphism from I_2 to I_1 . Let

Δ be any triangle of I_1 and let $\Delta' = \varphi(\Delta)$. By definition, there are three triangles $\Delta'_1, \Delta'_2, \Delta'_3$ of I_2 each of which shares a vertex with Δ and an edge with Δ' . Then $f(\Delta)$ and $f(\Delta')$ are triangles of I_2 and I_1 , respectively. Also, $f(\Delta'_1), f(\Delta'_2), f(\Delta'_3)$ are three triangles of I_1 each of which shares a vertex with $f(\Delta)$ and an edge with $f(\Delta')$. Therefore, by definition of ψ , $\psi(f(\Delta)) = f(\Delta') = f(\varphi(\Delta))$. $\square$

Proof of Theorem 5 As in the proof of Theorem 1, one may verify that $(S^2 \times S^2)'_{16}$ is a simplicial subdivision of $S^2_4 \times S^2_4$, and hence it triangulates $S^2 \times S^2$. We apply the following sequence of bistellar moves to $(S^2 \times S^2)'_{16}$ to create a second 16-vertex triangulation $(S^2 \times S^2)_{16}$ of $S^2 \times S^2$:

$$\begin{aligned} x_{12}x_{13}x_{14} &\mapsto x_{23}x_{34}x_{42}, & x_{21}x_{23}x_{24} &\mapsto x_{14}x_{31}x_{43}, \\ x_{31}x_{32}x_{34} &\mapsto x_{12}x_{24}x_{41}, & x_{41}x_{42}x_{43} &\mapsto x_{13}x_{21}x_{32}. \end{aligned}$$

Since this set of bistellar moves is stable under the automorphism group A_4 of $(S^2 \times S^2)'_{16}$, it follows that $(S^2 \times S^2)_{16}$ inherits the group A_4 . Also, both complexes have $\text{lk}(x_{11}) = S^1_3(\{x_{12}, x_{13}, x_{14}\}) * S^1_3(\{x_{21}, x_{31}, x_{41}\})$. However, while $(S^2 \times S^2)'_{16}$ has both $x_{12}x_{13}x_{14}$ and $x_{21}x_{31}x_{41}$ as triangles, we have chosen the bistellar moves judiciously to ensure that $(S^2 \times S^2)_{16}$ does not have the triangle $x_{12}x_{13}x_{14}$. Therefore, we may apply the following four GBM's (one after the other) to $(S^2 \times S^2)_{16}$ to delete the four vertices x_{ii} , $1 \leq i \leq 4$:

$$\begin{aligned} &(\text{st}(x_{11}), D_3^2(\{x_{12}, x_{13}, x_{14}\}) * S_3^1(\{x_{21}, x_{31}, x_{41}\})), \\ &(\text{st}(x_{22}), D_3^2(\{x_{21}, x_{23}, x_{24}\}) * S_3^1(\{x_{12}, x_{32}, x_{42}\})), \\ &(\text{st}(x_{33}), D_3^2(\{x_{31}, x_{32}, x_{34}\}) * S_3^1(\{x_{13}, x_{23}, x_{43}\})), \\ &(\text{st}(x_{44}), D_3^2(\{x_{41}, x_{42}, x_{43}\}) * S_3^1(\{x_{14}, x_{24}, x_{34}\})). \end{aligned}$$

The resulting complex X is therefore a 12-vertex triangulation of $S^2 \times S^2$. So, to confirm the first statement of this theorem, it suffices to show that X is isomorphic to the complex $(S^2 \times S^2)_{12}$ described in Sect. 3. Indeed, with the antimorphic pair of icosahedra (and their vertex names) as in Fig. 2, we shall show that we actually have $X = (S^2 \times S^2)_{12}$.

Notice that X inherits the automorphism group A_4 from $(S^2 \times S^2)_{16}$, and modulo this group, the following six are basic facets of X :

$$\begin{aligned} &x_{12}x_{14}x_{21}x_{24}x_{31}, x_{12}x_{13}x_{14}x_{21}x_{31}, x_{12}x_{23}x_{31}x_{13}x_{32}, \\ &x_{12}x_{31}x_{34}x_{14}x_{24}, x_{24}x_{31}x_{32}x_{12}x_{21}, x_{24}x_{31}x_{32}x_{12}x_{41}. \end{aligned}$$

Each basic facet is in an A_4 -orbit of size 12, yielding a total of $6 \times 12 = 72$ facets of X . Since $(S^2 \times S^2)_{12}$ also has 72 facets and since the group A_4 (acting on subscripts) is a subgroup of the automorphism group $A_5 \times \mathbb{Z}_2$ of $(S^2 \times S^2)_{12}$, it suffices to observe that all six basic facets of X listed above are also facets of $(S^2 \times S^2)_{12}$. Indeed, the first facet $x_{12}x_{14}x_{21}x_{24}x_{31}$ is in $(S^2 \times S^2)_{12}$ since these five vertices are the

neighbors of x_{23} in I_1 (and of x_{41} in I_2). In each of the remaining five basic facets of X , the first three vertices constitute a triangle Δ of I_1 with the last two vertices in the corresponding triangle $\Delta' = \varphi(\Delta)$ of I_2 (cf. Lemma 9). (For instance, $\Delta = x_{12}x_{13}x_{14}$ is a triangle of I_1 , with corresponding triangle $\Delta' = x_{21}x_{31}x_{41}$ of I_2 . Therefore, the second basic facet of X is a facet of $(S^2 \times S^2)_{12}$.) This shows that $(S^2 \times S^2)_{12} = X$, so that $(S^2 \times S^2)_{12}$ triangulates $S^2 \times S^2$.

To compute the full automorphism group of $(S^2 \times S^2)_{12}$, notice that it has exactly 40 triangles of degree 3 (the rest are of degree 5), namely the twenty triangles of I_1 and the twenty triangles of I_2 . Consider the graph whose vertices are these forty triangles, two of them being adjacent if and only if they share an edge. This graph has exactly two connected components, of size 20 each, namely the triangles of I_1 and I_2 . This shows that any automorphism f of $(S^2 \times S^2)_{12}$ either fixes both I_1 and I_2 or interchanges them. So, $\text{Aut}(I_1) = \text{Aut}(I_2) = A_5 \times \mathbb{Z}_2$ is a subgroup of index at most two in the full automorphism group of $(S^2 \times S^2)_{12}$.

Let $f: I_1 \rightarrow I_2$ be any isomorphism. Since I_1 and I_2 are antimorphic, it is immediate that f is also an isomorphism from I_2 to I_1 . Since the five neighbors in I_1 of any vertex are also the neighbors in I_2 of the antipodal vertex, it is immediate that f maps each of the 12 facets of the first kind in $(S^2 \times S^2)_{12}$ to a facet of the same kind. Also, for any triangle Δ of I_1 , the construction of $(S^2 \times S^2)_{12}$ shows that $\text{lk}(\Delta) = S_3^1(\varphi(\Delta))$, and also, for any triangle Δ' of I_2 , $\text{lk}(\Delta') = S_3^1(\psi(\Delta'))$. Since f intertwines φ and ψ (Lemma 9), we also have $\text{lk}(f(\Delta)) = S_3^1(\psi(f(\Delta))) = S_3^1(f(\varphi(\Delta))) = f(S_3^1(\varphi(\Delta))) = f(\text{lk}(\Delta))$. Similarly, for any triangle Δ' of I_2 , $\text{lk}(f(\Delta')) = f(\text{lk}(\Delta'))$. Thus, f also maps all sixty facets of the second type in $(S^2 \times S^2)_{12}$ to facets of the same type. Thus, any isomorphism between I_1 and I_2 is also an automorphism of $(S^2 \times S^2)_{12}$. Therefore, the full automorphism group G of $(S^2 \times S^2)_{12}$ has $H = A_5 \times \mathbb{Z}_2$ as an index two subgroup. Thus, G is of order 240. Indeed, G consists of the 120 common automorphisms of I_1 and I_2 , and the 120 isomorphisms between I_1 and I_2 . In particular, take $g = (x_{12}x_{21}x_{24}x_{42}x_{14}x_{41}x_{43}x_{34}x_{13}x_{31}x_{32}x_{23})$, which is an isomorphism between I_1 and I_2 . Note that g^6 is the common antipodal map of I_1 and I_2 , hence it is in the center of G . Thus, $G/\langle g^6 \rangle$ is the extension of A_5 by the involution $\alpha = g \pmod{g^6}$. But A_5 has only one non-trivial extension by an involution, namely S_5 . So, G is an extension of a central involution by S_5 . It can not be the split extension $S_5 \times \mathbb{Z}_2$ since this has no element of order 12. Therefore, G is the unique non-split extension $2S_5$ of $\mathbb{Z}_2$ by S_5 . $\square$

Remark 5 If the link of a vertex u in a triangulated 4-manifold X is $S_3^1(\{x, y, z\}) * S_3^1(\{a, b, c\})$ and xyz is not a simplex in X then the GBM $(\text{st}_X(u), D_3^2(\{x, y, z\}) * S_3^1(\{a, b, c\}))$ is equivalent to the sequence of the following three bistellar moves: $uab \mapsto xyz$, $ua \mapsto cxyz$, $u \mapsto bcxyz$. Thus, from the proof of Theorem 5, $(S^2 \times S^2)_{12}$ can be obtained from $(S^2 \times S^2)'_{16}$ by a sequence of bistellar moves only.

Acknowledgements The authors thank the anonymous referee for many useful comments which led to substantial improvements in the presentation of this paper. The authors are thankful to Siddhartha Gadgil, Frank Lutz, Wolfgang Kühnel and Alberto Verjovsky for useful conversations and references. The authors thank Ipsita Datta for her help in the proof of Theorem 3.

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