Smooth Fano Polytopes Arising from Finite Partially Ordered Sets

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Abstract Gorenstein Fano polytopes arising from finite partially ordered sets will be introduced. Then we study the problem of which partially ordered sets yield smooth Fano polytopes.

Keywords Smooth Fano polytope · $\mathbb{Q}$ -factorial Fano polytope · Gorenstein Fano polytope · Totally unimodular matrix · Finite partially ordered set

Introduction

An integral (or lattice) polytope is a convex polytope all of whose vertices have integer coordinates. Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d .

- We say that $\mathcal{P}$ is a *Fano polytope* if the origin of $\mathbb{R}^d$ is the unique integer point belonging to the interior of $\mathcal{P}$.
- A Fano polytope $\mathcal{P}$ is called *terminal* if each integer point belonging to the boundary of $\mathcal{P}$ is a vertex of $\mathcal{P}$.
- A Fano polytope is called *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is the convex polytope which consists of those $x \in \mathbb{R}^d$ such that $\langle x, y \rangle \leq 1$ for all $y \in \mathcal{P}$, where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^d$.)
- A *$\mathbb{Q}$ -factorial Fano polytope* is a simplicial Fano polytope, i.e., a Fano polytope each of whose faces is a simplex.

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- A *smooth Fano polytope* is a Fano polytope such that the vertices of each facet form a $\mathbb{Z}$ -basis of $\mathbb{Z}^d$. (Sometimes, smooth polytopes denote simple polytopes, which are dual polytopes of simplicial polytopes.)

Thus in particular a smooth Fano polytope is $\mathbb{Q}$ -factorial, Gorenstein and terminal.

Øbro [10] succeeded in finding an algorithm which yields the classification list of the smooth Fano polytopes for given d . It is proved in Casagrande [2] that the number of vertices of a Gorenstein $\mathbb{Q}$ -factorial Fano polytope is at most $3d$ if d is even, and at most $3d - 1$ if d is odd. B. Nill and M. Øbro [9] classified the Gorenstein $\mathbb{Q}$ -factorial Fano polytopes of dimension d with $3d - 1$ vertices. Gorenstein Fano polytopes are classified when $d \leq 4$ by Kreuzer and Skarke [6, 7] and the relevance of Gorenstein Fano polytopes to Mirror Symmetry was studied by Batyrev [1]. The study of the classification of terminal or canonical Fano polytopes was done by Kasprzyk [4, 5]. The combinatorial conditions for what it implies to be terminal and canonical are explained in Reid [11].

In the present paper, given a finite partially ordered set P , we associate a terminal Fano polytope $\mathcal{Q}_P$. By using the theory of totally unimodular matrices, it turns out that these Fano polytopes are Gorenstein. Then we study the problem of which partially ordered sets yield $\mathbb{Q}$ -factorial Fano polytopes. Finally, it turns out that the Fano polytope $\mathcal{Q}_P$ is smooth if and only if $\mathcal{Q}_P$ is $\mathbb{Q}$ -factorial.

1 Fano Polytopes Arising from Finite Partially Ordered Sets

Let $P = \{y_1, \dots, y_d\}$ be a finite partially ordered set and

$$\hat{P} = P \cup \{\hat{0}, \hat{1}\},$$

where $\hat{0}$ (resp. $\hat{1}$) is a unique minimal (resp. maximal) element of $\hat{P}$ with $\hat{0} \notin P$ (resp. $\hat{1} \notin P$). Let $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. We say that $e = \{y_i, y_j\}$, where $0 \leq i, j \leq d + 1$ with $i \neq j$, is an *edge* of $\hat{P}$ if e is an edge of the Hasse diagram of $\hat{P}$. (The Hasse diagram of a finite partially ordered set can be regarded as a finite nondirected graph.) In other words, $e = \{y_i, y_j\}$ is an edge of $\hat{P}$ if y_i and y_j are comparable in $\hat{P}$, say, $y_i < y_j$, and there is no $z \in P$ with $y_i < z < y_j$.

Definition 1.1 Let $\hat{P} = \{y_0, y_1, \dots, y_d, y_{d+1}\}$ be a finite partially ordered set with $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. Let $\mathbf{e}_i$ denote the i th canonical unit coordinate vector of $\mathbb{R}^d$. Given an edge $e = \{y_i, y_j\}$ of $\hat{P}$ with $y_i < y_j$, we define $\rho(e) \in \mathbb{R}^d$ by setting

$$\rho(e) = \begin{cases} \mathbf{e}_i & \text{if } j = d + 1, \\ -\mathbf{e}_j & \text{if } i = 0, \\ \mathbf{e}_i - \mathbf{e}_j & \text{if } 1 \leq i, j \leq d. \end{cases}$$

Moreover, we write $\mathcal{Q}_P \subset \mathbb{R}^d$ for the convex hull of the finite set

$$\{\rho(e) : e \text{ is an edge of } \hat{P}\}.$$

Example 1.2 Let $P = \{y_1, y_2, y_3\}$ be the finite partially ordered set with the partial order $y_1 < y_2$. Then $\hat{P}$ together with $\rho(e)$'s and $\mathcal{Q}_P$ are drawn below:

Let P be a finite partially ordered set. A subset Q of P is called a *chain* of P if Q is a totally ordered subset of P . The *length* of a chain Q is $\ell(Q) = \sharp(Q) - 1$. A chain Q of P is *saturated* if $x, y \in Q$ with $x < y$, then there is no $z \in P$ with $x < z < y$. A *maximal chain* of $\hat{P}$ is a saturated chain Q of $\hat{P}$ with $\{\hat{0}, \hat{1}\} \subset Q$.

Lemma 1.3 *The convex polytope $\mathcal{Q}_P$ is a Fano polytope.*

Proof Let $e = \{y_i, y_j\}$ be an edge of $\hat{P}$ with $y_i < y_j$. Let c_e denote the number of maximal chains Q of $\hat{P}$ with $\{y_i, y_j\} \subset Q$. If $\{y_{i_1}, y_{i_2}, \dots, y_{i_m}\}$ is a maximal chain of $\hat{P}$ with $y_0 = y_{i_1} < y_{i_2} < \dots < y_{i_m} = y_{d+1}$, then

$$\sum_{j=1}^{m-1} \rho(\{y_{i_j}, y_{i_{j+1}}\}) = (0, \dots, 0).$$

Hence

$$\sum_e c_e \rho(e) = (0, \dots, 0),$$

where e ranges all edges of $\hat{P}$. Thus the origin of $\mathbb{R}^d$ belongs to the interior of $\mathcal{Q}_P$. Since $\mathcal{Q}_P$ is a convex polytope which is contained in the convex hull of the finite set $\{\sum_{i=1}^d \varepsilon_i \mathbf{e}_i : \varepsilon_i \in \{0, 1, -1\}\}$ in $\mathbb{R}^d$, it follows that the origin of $\mathbb{R}^d$ is the unique integer point belonging to the interior of $\mathcal{Q}_P$. Thus $\mathcal{Q}_P$ is a Fano polytope, as desired. $\square$

Lemma 1.4 *The Fano polytope $\mathcal{Q}_P$ is terminal.*

Proof Suppose that $\mathcal{Q}_P$ contains an integer point $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$ with $\alpha \neq (0, \dots, 0)$. Then, obviously, $\alpha_1, \dots, \alpha_d \in \{-1, 0, 1\}$. Let, say, $\alpha_1 = 1$. Let $e_1, \dots, e_n$ be all edges of $\hat{P}$ and $e_{i_1}, \dots, e_{i_m}$ the edges with $y_1 \in e_{i_j}$ for $j = 1, \dots, m$. If we set $e_{i_j} = \{y_{i_j}, y_{i_{j'}}\}$ with $y_{i_j} < y_{i_{j'}}$, since α belongs to the convex hull of $\{\rho(e_1), \dots, \rho(e_n)\}$, then one has

$$\sum_{j=1}^m r_{i_j} q_{i_j} = \alpha_1 = 1,$$

where $0 \leq r_{i_1}, \dots, r_{i_m} \leq 1$ and $q_{i_j} = 1$ (resp. $q_{i_j} = -1$) if $y_1 < y_{i_{j'}}$ (resp. $y_{i_j} < y_1$). By removing all r_{i_j} with $r_{i_j} = 0$, we may assume that

$$\sum_{j=1}^{m'} r_{i_j} q_{i_j} = 1,$$

where $0 < r_{i_1}, \dots, r_{i_{m'}} \leq 1$. Since $\sum_{j=1}^{m'} r_{i_j} \leq 1$, there is no j with $q_{i_j} = -1$. Hence $\sum_{j=1}^{m'} r_{i_j} = 1$. If $m' > 1$, then $0 < r_{i_1}, \dots, r_{i_{m'}} < 1$. Thus $\sum_{j=1}^{m'} r_{i_j} \rho(e_{i_j}) = \alpha \notin \mathbb{Z}^d$. Hence $m' = 1$. In other words, if $\mathcal{Q}_P$ contains an integer point $\alpha \neq (0, \dots, 0)$, then α must be one of $\rho(e_1), \dots, \rho(e_n)$ and $\rho(e_1), \dots, \rho(e_n)$ are precisely the vertices of $\mathcal{Q}_P$. $\square$

Lemma 1.5 *The Fano polytope $\mathcal{Q}_P$ is Gorenstein.*

Proof Via the theory of totally unimodular matrices [12, Chap. 19], it follows that the equation of each supporting hyperplane of $\mathcal{Q}_P$ is of the form $a_1 x_1 + \dots + a_d x_d = 1$ with each $a_i \in \mathbb{Z}$. In other words, the dual polytope of $\mathcal{Q}_P$ is integral. Hence $\mathcal{Q}_P$ is Gorenstein, as required. $\square$

Remark 1.6 There is a well-known integral convex polytope arising from a finite partially ordered set P , which is called an *order polytope* $\mathcal{O}_P$. (See [13, Chap. 3] and [14].) The authors propose to consult [3, 8, 15] and [16] on the related work on order polytopes. One can verify immediately that the primitive outer normals of each facet of $\mathcal{O}_P$ one-to-one corresponds to each vertex of $\mathcal{Q}_P$. Now $\mathcal{O}_P$ is Gorenstein if and only if P is pure, i.e., all maximal chains of $\hat{P}$ have the same length. When

P is pure, let l denote the length of each maximal chain of $\hat{P}$. Then the dilated polytope $l\mathcal{O}_P$ contains a unique integer point $\alpha \in \mathbb{Z}^d$, where d is the cardinality of P , belonging to the interior of $l\mathcal{O}_P$. Then the dual polytope of the Gorenstein Fano polytope $l\mathcal{O}_P - \alpha$ coincides with $\mathcal{Q}_P$. Thus, when P is pure, we can associate $\mathcal{Q}_P$ with the dual polytope of an order polytope $\mathcal{O}_P$.

2 When Is $\mathcal{Q}_P$ $\mathbb{Q}$ -factorial?

Let $P = \{y_1, \dots, y_d\}$ be a finite partially ordered set and $\hat{P} = P \cup \{y_0, y_{d+1}\}$, where $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. A sequence $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ is called a *path* in $\hat{P}$ if Γ is a path in the Hasse diagram of $\hat{P}$. In other words, $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ is a path in $\hat{P}$ if $y_{i_j} \neq y_{i_k}$ for all $1 \leq j < k \leq m$ and if $\{y_{i_j}, y_{i_{j+1}}\}$ is an edge of $\hat{P}$ for all $1 \leq j \leq m-1$. In particular, if $\{y_{i_1}, y_{i_m}\}$ is also an edge of $\hat{P}$, then Γ is called a *cycle*. The *length* of a path $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ is $\ell(\Gamma) = m-1$ or $\ell(\Gamma) = m$ if Γ is a cycle.

A path $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_{m+1}})$ is called *special* if

$$\sharp\{j : y_{i_j} < y_{i_{j+1}}, 1 \leq j \leq m-1\} = \sharp\{k : y_{i_k} > y_{i_{k+1}}, 1 \leq k \leq m-1\}.$$

Given a special path $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$, there exists a unique function

$$\mu_\Gamma : \{y_{i_1}, y_{i_2}, \dots, y_{i_m}\} \rightarrow \{0, 1, 2, \dots\}$$

such that

- $\mu_\Gamma(y_{i_{j+1}}) = \mu_\Gamma(y_{i_j}) + 1$ (resp. $\mu_\Gamma(y_{i_j}) = \mu_\Gamma(y_{i_{j+1}}) + 1$) if $y_{i_j} < y_{i_{j+1}}$ (resp. $y_{i_j} > y_{i_{j+1}}$);
- $\min\{\mu_\Gamma(y_{i_1}), \mu_\Gamma(y_{i_2}), \dots, \mu_\Gamma(y_{i_m})\} = 0$.

In particular, Γ is special if and only if $\mu_\Gamma(y_{i_1}) = \mu_\Gamma(y_{i_m})$.

Similarly, a special cycle is defined and given a special cycle C , there exists a unique function μ_C which is defined the same way as above.

Example 2.1 Among the two paths and three cycles drawn below, the three ones depicted on the left-hand side (one path and two cycles) are special; the remaining two ones (one path and one cycle) are not.

We say that a path $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_{m+1}})$ or a cycle $C = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ of $\hat{P}$ belongs to a facet of $\mathcal{Q}_P$ if there is a facet $\mathcal{F}$ of $\mathcal{Q}_P$ with $\rho(\{y_{i_j}, y_{i_{j+1}}\}) \in \mathcal{F}$ for all $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$.

We say that a cycle $C = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ is *very special* if C is special and if $\{y_0, y_{d+1}\} \not\subset \{y_{i_1}, y_{i_2}, \dots, y_{i_m}\}$.

Lemma 2.2

- (a) Let $C = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ be a cycle in $\hat{P}$. If C belongs to a facet of $\mathcal{Q}_P$, then C is a special cycle. In particular, C is a very special cycle or C contains a special path $(y_{i_1}, y_{i_2}, \dots, y_{i_{r+1}})$ with $y_{i_1} = y_0$ and $y_{i_{r+1}} = y_{d+1}$.
- (b) Let $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$ be a path in $\hat{P}$. If Γ belongs to a facet of $\mathcal{Q}_P$, then Γ is a special path.

Proof (a) Let $a_1x_1 + \dots + a_dx_d = 1$, where each $a_i \in \mathbb{Q}$, denote the equation of the supporting hyperplane of $\mathcal{Q}_P$ which defines the facet. Since $\{y_{i_j}, y_{i_{j+1}}\}$ are edges of $\hat{P}$ for $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$, it follows that $a_{i_j} - a_{i_{j+1}} = q_j$, where $q_j = 1$ if $y_{i_j} < y_{i_{j+1}}$ and $q_j = -1$ if $y_{i_j} > y_{i_{j+1}}$. Now,

$$\sum_{i=1}^m q_j = \sum_{i=1}^m (a_{i_j} - a_{i_{j+1}}) = 0.$$

Hence C must be special.

Suppose that $\{y_0, y_{d+1}\} \subset \{y_{i_1}, y_{i_2}, \dots, y_{i_m}\}$. Let $y_{i_1} = y_0$ and $y_{i_{r+1}} = y_{d+1}$. Since $\{y_{i_j}, y_{i_{j+1}}\}$, $1 \leq j \leq r$, are edges of $\hat{P}$, one has $-a_{i_2} = 1$, $a_{i_r} = 1$ and $a_{i_j} - a_{i_{j+1}} = q_j$ for $j = 2, 3, \dots, r-1$. On the one hand, one has

$$-a_{i_2} + \sum_{j=2}^{r-1} (a_{i_j} - a_{i_{j+1}}) + a_{i_r} = 0.$$

On the other hand, one has

$$\begin{aligned} & -a_{i_2} + \sum_{j=2}^{r-1} (a_{i_j} - a_{i_{j+1}}) + a_{i_r} \\ &= 1 + \sum_{j=2}^{r-1} q_j + 1 \\ &= -\mu_C(y_0) + \mu_C(y_{i_2}) + \sum_{j=2}^{r-1} (\mu_C(y_{i_{j+1}}) - \mu_C(y_{i_j})) + \mu_C(y_{d+1}) - \mu_C(y_{i_r}) \\ &= \mu_C(y_{d+1}) - \mu_C(y_0). \end{aligned}$$

It then follows that one must be $\mu_C(y_0) = \mu_C(y_{d+1})$. Let $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_{r+1}})$. Then it is clear that $\mu_\Gamma(y_0) = \mu_\Gamma(y_{d+1})$. Thus Γ is a special path. Hence C contains a special path Γ .

(b) A proof can be given by the similar way of a proof of (a). $\square$

Let P be a finite partially ordered set and $y, z \in \hat{P}$ with $y < z$. The *distance* of y and z in $\hat{P}$ is the smallest integer s for which there is a saturated chain $Q = \{z_0, z_1, \dots, z_s\}$ with

$$y = z_0 < z_1 < \dots < z_s = z.$$

Let $\text{dist}_{\hat{P}}(y, z)$ denote the distance of y and z in $\hat{P}$.

Theorem 2.3 *Let $P = \{y_1, \dots, y_d\}$ be a finite partially ordered set and $\hat{P} = P \cup \{y_0, y_{d+1}\}$, where $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. Then the following conditions are equivalent:*

- (i) $\mathcal{Q}_P$ is $\mathbb{Q}$ -factorial;
- (ii) $\mathcal{Q}_P$ is smooth;
- (iii) $\hat{P}$ possesses no very special cycle $C = (y_{i_1}, \dots, y_{i_m})$ such that

$$\mu_C(y_{i_a}) - \mu_C(y_{i_b}) \leq \text{dist}_{\hat{P}}(y_{i_b}, y_{i_a}) \quad (1)$$

for all $1 \leq a, b \leq m$ with $y_{i_b} < y_{i_a}$, and

$$\mu_C(y_{i_a}) - \mu_C(y_{i_b}) \leq \text{dist}_{\hat{P}}(y_0, y_{i_a}) + \text{dist}_{\hat{P}}(y_{i_b}, y_{d+1}) \quad (2)$$

for all $1 \leq a, b \leq m$, and no special path $\Gamma = (y_{i_1}, \dots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$ such that

$$\mu_{\Gamma}(y_{i_a}) - \mu_{\Gamma}(y_{i_b}) \leq \text{dist}_{\hat{P}}(y_{i_b}, y_{i_a}) \quad (3)$$

for all $1 \leq a, b \leq m$ with $y_{i_b} < y_{i_a}$.

Proof ((i) $\Rightarrow$ (iii)) If $C = (y_{i_1}, \dots, y_{i_m})$ is a cycle in $\hat{P}$ with $y_{i_{m+1}} = y_1$, then

$$\sum_{j=1}^m q_j \rho(\{y_{i_j}, y_{i_{j+1}}\}) = (0, \dots, 0),$$

where $q_j = 1$ if $y_{i_j} < y_{i_{j+1}}$ and $q_j = -1$ if $y_{i_j} > y_{i_{j+1}}$. Thus in particular $\rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m$, cannot be affinely independent if C is special.

Now, suppose that $\hat{P}$ possesses a very special cycle $C = (y_{i_1}, \dots, y_{i_m})$ which satisfies the inequalities (1) and (2). Our work is to show that $\mathcal{Q}_P$ is not simplicial. Let $v_j = \rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$. Since $v_1, \dots, v_m$ cannot be affinely independent, to show that $\mathcal{Q}_P$ is not simplicial, what we must prove is the existence of a face of $\mathcal{Q}_P$ which contains the vertices $v_1, \dots, v_m$.

Let $a_1, \dots, a_d$ be integers. Write $\mathcal{H} \subset \mathbb{R}^d$ for the hyperplane defined by the equation $a_1 x_1 + \dots + a_d x_d = 1$ and $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ for the closed half-space defined by the inequality $a_1 x_1 + \dots + a_d x_d \leq 1$. We will determine $a_1, \dots, a_d$ such that $\mathcal{H}$ is a supporting hyperplane of a face $\mathcal{F}$ of $\mathcal{Q}_P$ with $\{v_1, \dots, v_m\} \subset \mathcal{F}$ and with $\mathcal{Q}_P \subset \mathcal{H}^{(+)}$.

First Step. It follows from (2) that

$$\max_{1 \leq a \leq m} (\mu_C(y_{i_a}) - \text{dist}_{\hat{P}}(y_0, y_{i_a})) \leq \min_{1 \leq b \leq m} (\mu_C(y_{i_b}) + \text{dist}_{\hat{P}}(y_{i_b}, y_{d+1})). \quad (4)$$

By using (1), if $y_0 \in \{y_{i_1}, \dots, y_{i_m}\}$, then the left-hand side of (4) is equal to $\mu_C(y_0)$. Similarly, if $y_{d+1} \in \{y_{i_1}, \dots, y_{i_m}\}$, then the right-hand side of (4) is equal to $\mu_C(y_{d+1})$.

Now, fix an arbitrary integer a with

$$\max_{1 \leq a \leq m} (\mu_C(y_{i_a}) - \text{dist}_{\hat{P}}(y_0, y_{i_a})) \leq a \leq \min_{1 \leq b \leq m} (\mu_C(y_{i_b}) + \text{dist}_{\hat{P}}(y_{i_b}, y_{d+1})).$$

However, exceptionally, if $y_0 \in \{y_{i_1}, \dots, y_{i_m}\}$, then $a = \mu_C(y_0)$. If $y_{d+1} \in \{y_{i_1}, \dots, y_{i_m}\}$, then $a = \mu_C(y_{d+1})$. Let $a_{i_j} = a - \mu_C(y_{i_j})$ for $1 \leq j \leq m$. Then one has

$$-a_{i_j} \leq \text{dist}_{\hat{P}}(y_0, y_{i_j}), \quad a_{i_j} \leq \text{dist}_{\hat{P}}(y_{i_j}, y_{d+1}). \quad (5)$$

Moreover, it follows easily that each v_j lies on the hyperplane of $\mathbb{R}^d$ defined by the equation

$$\sum_{i_j \notin \{0, d+1\}} a_{i_j} x_{i_j} = 1.$$

Second Step. Let $A = \hat{P} \setminus (\{y_0, y_{d+1}\} \cup \{y_{i_1}, \dots, y_{i_m}\})$ and $y_i \in A$.

- Suppose that there is y_{i_j} with $y_{i_j} < y_i$ and that there is no y_{i_k} with $y_{i_k} > y_i$. Then we define a_i by setting

$$a_i = \max(\{a_{i_j} - \text{dist}_{\hat{P}}(y_{i_j}, y_i) : y_{i_j} < y_i\} \cup \{0\}).$$

- Suppose that there is no y_{i_j} with $y_{i_j} < y_i$ and that there is y_{i_k} with $y_{i_k} > y_i$. Then we define a_i by setting

$$a_i = \min(\{a_{i_k} + \text{dist}_{\hat{P}}(y_i, y_{i_k}) : y_i < y_{i_k}\} \cup \{0\}).$$

- Suppose that there is y_{i_j} with $y_{i_j} < y_i$ and that there is y_{i_k} with $y_{i_k} > y_i$. Then either

$$b_i = \max(\{a_{i_j} - \text{dist}_{\hat{P}}(y_{i_j}, y_i) : y_{i_j} < y_i\} \cup \{0\})$$

or

$$c_i = \min(\{a_{i_k} + \text{dist}_{\hat{P}}(y_i, y_{i_k}) : y_i < y_{i_k}\} \cup \{0\})$$

must be zero. In fact, if $b_i \neq 0$ and $c_i \neq 0$, then there are j and k with $a_{i_j} > \text{dist}_{\hat{P}}(y_{i_j}, y_i)$ and $-a_{i_k} > \text{dist}_{\hat{P}}(y_i, y_{i_k})$. Since $\mu_C(y_{i_k}) - \mu_C(y_{i_j}) = a_{i_j} - a_{i_k}$ and since $\text{dist}_{\hat{P}}(y_{i_j}, y_i) + \text{dist}_{\hat{P}}(y_i, y_{i_k}) \geq \text{dist}_{\hat{P}}(y_{i_j}, y_{i_k})$, it follows that

$$\mu_C(y_{i_k}) - \mu_C(y_{i_j}) > \text{dist}_{\hat{P}}(y_{i_j}, y_{i_k}).$$

This contradicts (1). Hence either $b_i = 0$ or $c_i = 0$. If $b_i \neq 0$, then we set $a_i = b_i$. If $c_i \neq 0$, then we set $a_i = c_i$. If $b_i = c_i = 0$, then we set $a_i = 0$.

- Suppose that there is no y_{i_j} with $y_{i_j} < y_i$ and that there is no y_{i_k} with $y_{i_k} > y_i$. Then we set $a_i = 0$.

Third Step. Finally, we finish determining the integers $a_1, \dots, a_d$. Let $\mathcal{H} \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $a_1x_1 + \dots + a_dx_d = 1$ and $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ the closed half-space defined by the inequality $a_1x_1 + \dots + a_dx_d \leq 1$. Since each v_j lies on the hyperplane $\mathcal{H}$, in order for $\mathcal{F} = \mathcal{H} \cap \mathcal{Q}_P$ to be a face of $\mathcal{Q}_P$, it is required to show $\mathcal{Q}_P \subset \mathcal{H}^{(+)}$. Let $\{y_i, y_j\}$ with $y_i < y_j$ be an edge of $\hat{P}$.

- Let $y_i \in \{y_{i_1}, \dots, y_{i_m}\}$ with $y_j \notin \{y_{i_1}, \dots, y_{i_m}\}$. If $y_j \neq y_{d+1}$, then

$$a_j \geq \max\{a_i - 1, 0\},$$

where $a_0 = 0$. Thus $a_i - a_j \leq 1$. If $y_j = y_{d+1}$, then by using (5) one has $a_i \leq 1$, as desired.

- Let $y_j \in \{y_{i_1}, \dots, y_{i_m}\}$ with $y_i \notin \{y_{i_1}, \dots, y_{i_m}\}$. If $y_i \neq y_0$, then

$$a_i \leq \min\{a_j + 1, 0\},$$

where $a_{d+1} = 0$. Thus $a_i - a_j \leq 1$. If $y_i = y_0$, then by using (5) one has $-a_j \leq 1$, as desired.

Let $A' = \hat{P} \setminus \{y_{i_1}, \dots, y_{i_m}\}$. Write B for the subset of A' consisting of those $y_i \in A'$ such that there is j with $y_{i_j} < y_i$. Write C for the subset of A' consisting of those $y_i \in A'$ such that there is k with $y_i < y_{i_k}$. Again, let $e = \{y_i, y_j\}$ with $y_i < y_j$ be an edge of $\hat{P}$. In each of the nine cases below, a routine computation easily yields that $\rho(e) \in \mathcal{H}^{(+)}$.

- $y_i \in B \setminus C$ and $y_j \in B \setminus C$;
- $y_i \in C \setminus B$ and $y_j \in C \setminus B$;
- $y_i \in C \setminus B$ and $y_j \in B \setminus C$;
- $y_i \in C \setminus B$ and $y_j \in B \cap C$;
- $y_i \in C \setminus B$ and $y_j \notin B \cup C$;
- $y_i \in B \cap C$ and $y_j \in B \cap C$;
- $y_i \in B \cap C$ and $y_j \in B \setminus C$;
- $y_i \notin B \cup C$ and $y_j \in B \setminus C$;
- $y_i \notin B \cup C$ and $y_j \notin B \cup C$.

For example, in the first case, a routine computation is as follows. Let $y_j \neq y_{d+1}$. Let $a_i = 0$. Then, since $a_j \geq 0$, one has $a_i - a_j \leq 1$. Let $a_i > 0$. Then, since $a_j \geq a_i - 1$, one has $a_i - a_j \leq 1$. Let $y_j = y_{d+1}$ and $a_i > 0$. Then there is j with $a_i = a_{i_j} - \text{dist}_{\hat{P}}(y_{i_j}, y_i)$. By using (5) one has $a_{i_j} \leq \text{dist}_{\hat{P}}(y_{i_j}, y_{d+1})$. Thus $a_i \leq \text{dist}_{\hat{P}}(y_{i_j}, y_{d+1}) - \text{dist}_{\hat{P}}(y_{i_j}, y_i)$. Hence $a_i \leq 1$, as required.

Fourth step. Suppose that $\hat{P}$ possesses a special path $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$ which satisfies the inequalities (3). Then one has

$$\sum_{j=1}^{m-1} q_j \rho(\{y_{i_j}, y_{i_{j+1}}\}) = (0, \dots, 0),$$

where $q_j = 1$ if $y_{i_j} < y_{i_{j+1}}$ and $q_j = -1$ if $y_{i_j} > y_{i_{j+1}}$. Thus $\rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m-1$, cannot be affinely independent. Our work is to show that $\mathcal{Q}_P$ is not simplicial.

In this case, however, the same discussion can be given as the case which $\hat{P}$ possesses a very special cycle. (We should set $a = \mu_\Gamma(y_0) (= \mu_\Gamma(y_{d+1}))$.)

((iii) $\Rightarrow$ (i)) Now, suppose that $\mathcal{Q}_P$ is not $\mathbb{Q}$ -factorial. Thus $\mathcal{Q}_P$ possesses a facet $\mathcal{F}$ which is not a simplex. Let $v_1, \dots, v_n$ denote the vertices of $\mathcal{F}$, where $n > d$, and e_j the edge of $\hat{P}$ with $v_j = \rho(e_j)$ for $1 \leq j \leq n$. Let $a_1x_1 + \dots + a_dx_d = 1$ denote the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ of $\mathcal{Q}_P$ with $\mathcal{F} = \mathcal{Q}_P \cap \mathcal{H}$ and with $\mathcal{Q}_P \subset \mathcal{H}^{(+)}$, where $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ is the closed-half space defined by the inequality $a_1x_1 + \dots + a_dx_d \leq 1$. Since $v_1, \dots, v_n$ are not affinely independent, there is $(r_1, \dots, r_n) \in \mathbb{Z}^n$ with $(r_1, \dots, r_n) \neq (0, \dots, 0)$ such that $r_1v_1 + \dots + r_nv_n = (0, \dots, 0)$. By removing r_j with $r_j = 0$, we may assume that $r_1v_1 + \dots + r_{n'}v_{n'} = (0, \dots, 0)$, where $r_j \neq 0$ for $1 \leq j \leq n'$ with $r_1 + \dots + r_{n'} = 0$. Let $e_j = \{y_{ij}, y_{i'j}\}$ with $1 \leq i_j, i'j \leq d$. If either y_{ij} or $y_{i'j}$ appears only in e_j among the edges $e_1, \dots, e_{n'}$, then $r_j = 0$. Hence both y_{ij} and $y_{i'j}$ must appear in at least two edges among $e_1, \dots, e_{n'}$. Let G denote the subgraph of the Hasse diagram of $\hat{P}$ with the edges $e_1, \dots, e_{n'}$. Then there is no end point of G in P . Thus G possesses a cycle of $\hat{P}$ or G is a path of $\hat{P}$ from y_0 to y_{d+1} . Since $v_1, \dots, v_{n'}$ are contained in the facet $\mathcal{F}$, Lemma 2.2 says that every cycle in G is very special or else G contains a special path.

Suppose that G possesses a very special cycle $C = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$. Our goal is to show that C satisfies the inequalities (1) and (2).

Let $y_{k_0} < y_{k_1} < \dots < y_{k_\ell}$ be a saturated chain of $\hat{P}$ with $\ell = \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$ such that each of y_{k_0} and y_{k_ℓ} belongs to $\{y_{i_1}, y_{i_2}, \dots, y_{i_m}\}$. We claim

$$\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell}).$$

- Let $y_0 \neq y_{k_0}$ and $y_{d+1} \neq y_{k_\ell}$. Since $\mathbf{e}_{k_j} - \mathbf{e}_{k_{j+1}} \in \mathcal{Q}_P$, one has $a_{k_j} - a_{k_{j+1}} \leq 1$ for each $0 \leq j \leq \ell - 1$. Hence $a_{k_0} - a_{k_\ell} \leq \ell$. On the other hand, $a_{k_0} - a_{k_\ell} = \mu_C(y_{k_\ell}) - \mu_C(y_{k_0})$. Thus $\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$.
- Let $y_0 = y_{k_0}$ and $y_{d+1} \neq y_{k_\ell}$. Since $-\mathbf{e}_{k_1} \in \mathcal{Q}_P$, one has $-a_{k_1} \leq 1$. Since $\mathbf{e}_{k_j} - \mathbf{e}_{k_{j+1}} \in \mathcal{Q}_P$, one has $a_{k_j} - a_{k_{j+1}} \leq 1$ for each $1 \leq j \leq \ell - 1$. Hence $a_{k_1} - a_{k_\ell} \leq \ell - 1$. Thus $-a_{k_\ell} \leq \ell$. On the other hand, $-a_{k_\ell} = \mu_C(y_{k_\ell}) - \mu_C(y_{k_0})$. Thus $\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$.
- Let $y_0 \neq y_{k_0}$ and $y_{d+1} = y_{k_\ell}$. Since $\mathbf{e}_{k_j} - \mathbf{e}_{k_{j+1}} \in \mathcal{Q}_P$, one has $a_{k_j} - a_{k_{j+1}} \leq 1$ for each $0 \leq j \leq \ell - 2$. Hence $a_{k_0} - a_{k_{\ell-1}} \leq \ell - 1$. Since $\mathbf{e}_{k_{\ell-1}} \in \mathcal{Q}_P$, one has $a_{k_{\ell-1}} \leq 1$. Hence $a_{k_0} \leq \ell$. On the other hand, $a_{k_0} = \mu_C(y_{k_\ell}) - \mu_C(y_{k_0})$. Thus $\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$.

Finally, fix arbitrary y_{ij} and y_{ik} with $\mu_C(y_{ij}) < \mu_C(y_{ik})$. Then $-a_{ik} \leq \text{dist}_{\hat{P}}(y_0, y_{ik})$ and $a_{ij} \leq \text{dist}_{\hat{P}}(y_{ij}, y_{d+1})$. We claim

$$\mu_C(y_{ik}) - \mu_C(y_{ij}) \leq \text{dist}_{\hat{P}}(y_0, y_{ik}) + \text{dist}_{\hat{P}}(y_{ij}, y_{d+1}).$$

If $y_{ij} \neq y_0$ and $y_{ik} \neq y_{d+1}$, then $a_{ij} - a_{ik} = \mu_C(y_{ik}) - \mu_C(y_{ij})$. If $y_{ij} = y_0$ and $y_{ik} \neq y_{d+1}$, then $-a_{ik} = \mu_C(y_{ik}) - \mu_C(y_{ij})$. If $y_{ij} \neq y_0$ and $y_{ik} = y_{d+1}$, then $a_{ij} = \mu_C(y_{ik}) - \mu_C(y_{ij})$. Hence the required inequality follows immediately.

Suppose that G contains a special path $\Gamma = (y_{i_1}, y_{i_2}, \dots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$. Our goal is to show that C satisfies the inequalities (3). Now the same discussion can be given as above.

((i) $\Rightarrow$ (ii)) If P is a totally ordered set, then $\mathcal{Q}_P$ is a d -simplex with the vertices, say, $-\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \dots, \mathbf{e}_{d-1} - \mathbf{e}_d, \mathbf{e}_d$. Thus in particular $\mathcal{Q}_P$ is smooth.

Now, suppose that P is not a totally ordered set. Then $\hat{P}$ possesses a cycle. Let $C = (y_{i_1}, \dots, y_{i_m})$ be a cycle in $\hat{P}$. If C is not special, then Lemma 2.2 (a) says that C cannot belong to a facet of $\mathcal{Q}_P$. If C is special, then as was shown in the proof of (i) $\Rightarrow$ (iii) it follows that $\rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$, are not affinely independent. Hence there is no facet $\mathcal{F}$ of $\mathcal{Q}_P$ with $\rho(\{y_{i_j}, y_{i_{j+1}}\}) \in \mathcal{F}$ for all $1 \leq j \leq m$.

Let $\mathcal{F}$ be an arbitrary facet of $\mathcal{Q}_P$ with d vertices $v_j = \rho(e_j)$, $1 \leq j \leq d$. Let G denote the subgraph of the Hasse diagram of $\hat{P}$ with the edges $e_1, \dots, e_d$ and $V(G)$ the vertex set of G . Since $\mathcal{F}$ is of dimension $d - 1$, it follows that, for each $1 \leq i \leq d$, there is a vertex of $\mathcal{F}$ whose i th coordinate is nonzero. Hence $P \subset V(G)$. Suppose that $P = V(G)$. Since G has d edges, it follows that G possesses a cycle, a contradiction. Hence either $y_0 \in V(G)$ or $y_{d+1} \in V(G)$.

What we must prove is that the determinant

$$\begin{vmatrix} v_1 \\ \vdots \\ v_d \end{vmatrix} \quad (6)$$

is equal to ± 1 . Let, say, $e_1 = \{y_1, y_{d+1}\}$. Thus $v_1 = (1, 0, \dots, 0)$. Now, since G is a forest, by arranging the numbering of the elements of P if necessary, one has

$$\begin{vmatrix} v_1 \\ \vdots \\ v_d \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{d1} & a_{d2} & \cdots & \cdots & a_{dd} \end{vmatrix},$$

where each $a_{ij} \in \{1, 0, -1\}$. Since the determinant (6) is nonzero, it follows that the determinant (6) is equal to ± 1 , as desired.

((ii) $\Rightarrow$ (i)) In general, every smooth Fano polytope is $\mathbb{Q}$ -factorial. $\square$

Corollary 2.4 Suppose that a finite partially ordered set P is pure. Then the following conditions are equivalent:

- (i) $\mathcal{Q}_P$ is $\mathbb{Q}$ -factorial;
- (ii) $\mathcal{Q}_P$ is smooth;
- (iii) P is a disjoint union of chains;
- (iv) The polytope $\mathcal{Q}_P$ is the free sum of smooth Fano simplices.

Proof If P is pure, then every cycle of $\hat{P}$ is special and, in addition, satisfies the inequalities (1) and (2). Moreover, every path from y_0 to y_{d+1} cannot be special. Hence $\mathcal{Q}_P$ is $\mathbb{Q}$ -factorial if and only if there is no very special cycle, i.e., every cycle of $\hat{P}$ possesses both $\hat{0}$ and $\hat{1}$. Now if there is a connected component of P which is

Table 1 Second row: The number of finite partially ordered sets with d elements. Third row: The number of finite partially ordered sets constructing smooth Fano polytopes

	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d = 8$
Posets	1	2	4	12	39	184	1082	8746
Smooth	1	2	3	6	12	31	83	266

not a chain, then P possesses a very special cycle. Thus $\mathcal{Q}_P$ is $\mathbb{Q}$ -factorial if and only if P does not possess a connected component which is not a chain. In other words, $\mathcal{Q}_P$ is $\mathbb{Q}$ -factorial if and only if P is a disjoint union of chains. Furthermore, smooth Fano simplices arising from finite partially ordered sets are constructed from only totally ordered sets. That is to say, P is a disjoint union of chains if and only if the polytope $\mathcal{Q}_P$ is the free sum of smooth Fano simplices, as desired. $\square$

Example 2.5 Among the five finite partially ordered sets drawn below, the three finite partially ordered sets depicted on the left-hand side yield a $\mathbb{Q}$ -factorial Fano polytope; the remaining two finite partially ordered sets do not yield a $\mathbb{Q}$ -factorial Fano polytope.

Let P and P' be finite partially ordered sets. Then one can verify easily that $\mathcal{Q}_P$ is isomorphic with $\mathcal{Q}_{P'}$ as a convex polytope if and only if P is isomorphic with P' or with the dual finite partially ordered set of P' as a finite partially ordered set. (For example, a proof can be given by the induction on the number of maximal chains of $\hat{P}$.)

On Table 1, the number of finite partially ordered sets with $d(\leq 8)$ elements, up to isomorphic and up to isomorphic with dual finite partially ordered sets, is written in the second row. Moreover, among those, the number of finite partially ordered sets constructing smooth Fano polytopes is written in the third row.

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