An Eberhard-Like Theorem for Pentagons and Heptagons

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Received: 22 May 2009 / Revised: 5 May 2010 / Accepted: 9 May 2010 / Published online: 27 May 2010
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Abstract Eberhard proved that for every sequence (p_k) , $3 \leq k \leq r$, $k \neq 6$, of non-negative integers satisfying Euler's formula $\sum_{k \geq 3} (6 - k)p_k = 12$, there are infinitely many values p_6 such that there exists a simple convex polyhedron having precisely p_k faces of size k for every $k \geq 3$, where $p_k = 0$ if $k > r$. In this paper we prove a similar statement when nonnegative integers p_k are given for $3 \leq k \leq r$, except for $k = 5$ and $k = 7$ (but including p_6). We prove that there are infinitely many values p_5, p_7 such that there exists a simple convex polyhedron having precisely p_k faces of size k for every $k \geq 3$. We derive an extension to arbitrary closed surfaces, yielding maps of

The research of A. Georgakopoulos was supported in part by a GIF grant and in part by an FWF grant. This work was conceived during a visit of A. Georgakopoulos at the Simon Fraser University; we thank the SFU for supporting this visit.

The research of B. Mohar was supported in part by the Research Grant P1-0297 of ARRS, by an NSERC Discovery Grant and by the Canada Research Chairs program. On leave from IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

The research of R. Šámal was supported in part by PIMS postdoctoral fellowship at the Simon Fraser University, by Institute for Theoretical Computer Science (ITI), and by Department of Applied Mathematics. Partially supported by grant GA ČR P201/10/P337.

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arbitrarily high face-width. Our proof suggests a general method for obtaining results of this kind.

Keywords Eberhard theorem · Simple polyhedron · Planar graph · Face-width

1 Introduction

Consider a cubic (i.e., 3-regular) plane graph, and let p_k ($k \geq 1$) denote the number of its k -gonal faces. It is a simple corollary of Euler's formula that

$$\sum_{k \geq 1} (6 - k)p_k = 12. \quad (1)$$

It is natural to ask for which sequences $(p_k)_{k \geq 1}$ satisfying (1) there exists a cubic plane graph whose face sizes comply with the sequence (p_k) . This question is even more interesting when additional restrictions on the graph are given. The most important case is to consider graphs of three-dimensional convex polyhedra, so-called *polyhedral graphs*. By Steinitz's theorem, this is the same as requiring the graphs to be 3-connected. An important subcase is where the polyhedra are *simple*, in other words, where the corresponding graphs are cubic.

The general problem about the existence of polyhedral graphs with given face sizes is still wide open. However, there are many special cases that have been solved. For example [9, Theorem 13.4.1], it is known that there exists a simple polyhedron with six quadrangular faces and p_6 faces of size six if and only if $p_6 \neq 1$; and there exists a simple polyhedron with twelve pentagonal faces and p_6 faces of size six (a “fullerene” graph) if and only if $p_6 \neq 1$. A similar case of four triangular faces and p_6 faces of size 6 has infinitely many exceptions: such a polyhedron exists if and only if p_6 is even. We refer to [9] for a complete overview. The most fundamental result in this area is the following classical theorem of Eberhard [3], stating that there is always a solution, provided that we are allowed to replace p_6 (whose value does not affect the satisfaction of (1)) by a large enough integer.

Theorem 1.1 (Eberhard [3]) *For every sequence (p_k) , $3 \leq k \leq r$, $k \neq 6$, of nonnegative integers satisfying (1), there are infinitely many values p_6 such that there exists a simple convex polyhedron having precisely p_k faces of size k for every $k \geq 3$, where $p_k = 0$ if $k > r$.*

Eberhard's proof is not only long and messy, but also some of its parts may not satisfy today's standards of rigor. Grünbaum [9] gave a simpler complete proof utilizing graphs and Steinitz's Theorem. This result was strengthened by Fisher [5], who proved that there is always a value of p_6 that satisfies $p_6 \leq p_3 + p_4 + p_5 + \sum_{k \geq 7} p_k$.

Grünbaum also considered a 4-valent analogue of Eberhard's theorem. Fisher [6] proved a similar result for 5-valent polyhedra, establishing existence for all sequences of face sizes with $p_4 \geq 6$ that comply with Euler's formula.

Various other generalizations of Eberhard's theorem have been discovered. Papers by Jendrol' [10, 11] give a good overview and bring some of today's most general

results in this area. Some other relevant works include [1, 2, 4, 8, 13]. Several papers treat extensions of Eberhard's theorem to the torus [7, 12, 15, 16] and more general surfaces [10]. It is worth pointing out that on the torus there is precisely one admissible sequence (namely $p_5 = p_7 = 1$ and $p_i = 0$ for $i \notin \{5, 7\}$), for which an Eberhard-type result with added hexagons does not hold [12].

In this paper we consider a similar problem that is also motivated by (1). Let us suppose that we are given face sizes as before, but we are only allowed to change p_5 and p_7 (or p_{6-t} and p_{6+t} for some t , $1 \leq t \leq 3$). In this case, we think of p_k (for $k \geq 3$, $k \neq 5, 7$) as being fixed and p_5, p_7 as being free to choose. Equation (1) determines the difference $s = p_7 - p_5$, and we are asking if there exist p_5 and $p_7 = p_5 + s$ with a polyhedral realization. We give an affirmative answer to this question and derive an extension solving the corresponding problem on an arbitrary closed surface. Our construction gives simple (i.e., 3-regular) polyhedral maps on a surface, and one can impose the additional conditions that these maps have large face-width and that their graphs are 3-connected. More precisely, we prove the following:

Theorem 1.2 *Let (p_k) , $3 \leq k \leq r$, $k \neq 5, 7$, be a sequence of nonnegative integers, let S be a closed surface, and let w be a positive integer. Then there exist infinitely many pairs of integers p_5 and p_7 such that there is a 3-connected cubic map realizing S , with face-width at least w , having precisely p_k faces of size k for every $k \in \{3, \dots, r\}$.*

It is worth observing that we also fix the number p_6 of hexagonal faces. Secondly, observe that the extension of Eberhard's Theorem to a surface S other than the sphere needs an adjustment in (1); the right-hand side has to be replaced by $6\chi(S)$ where $\chi(S)$ is the Euler characteristic of S . However, in our setting the formula adjusts itself by using an appropriate number of pentagons and heptagons.

Finally, as we point out in Sect. 4, our proof suggests a general method for obtaining results of this kind.

2 Definitions

A finite sequence $p = (p_3, p_4, \dots, p_r)$ is *plausible* for a closed surface S if

$$\sum_{3 \leq k \leq r} (6 - k)p_k = 6\chi(S), \quad (2)$$

where $\chi(S)$ is the Euler characteristic of S . By Euler's formula, (2) is a necessary condition for the existence of a cubic graph embeddable in S with precisely p_k k -gons for $3 \leq k \leq r$ and no other faces. If there exists a cubic graph which is 2-cell embeddable in S with precisely p_k faces of size k for $3 \leq k \leq r$ and no other faces, then we say that p is *realizable* in S . If $\sum_{3 \leq k \leq r} (6 - k)p_k = 0$, then we call p a *neutral* sequence. For any two such sequences, one can consider their sum which is defined in the obvious way. Let us observe that the sum of a neutral sequence and a plausible sequence is a plausible sequence. We would like to understand in this context which plausible sequences are realizable and try to do so by asking when a sum of a

plausible sequence with an appropriate neutral sequence is realizable. For the neutral sequence $(0, 0, 0, 1)$, this is Eberhard's theorem.

The most important building block in both Eberhard's and our proofs is a construction called a triarc. A *triarc* is a plane graph T such that the boundary C of the outer face of T is a cycle, and moreover the following conditions are satisfied (examples are the graphs in Fig. 4 with the half-edges in the outer face removed):

- every vertex of $T - C$ has degree 3 in T ;
- C contains distinct vertices x, y, z of degree 2 (called the *corners* of the triarc) such that the degrees (in T) of the vertices on each of the three paths in $C - \{x, y, z\}$ alternate between 2 and 3, starting and ending with a vertex of degree 2.

A *side* of a triarc T as above is a subpath of C that starts and ends at distinct corners of T and does not contain the third corner. The *length* of a side P of T is the number of inner vertices of degree 2 on P ; note that although the corners of a triarc have degree 2, they are not counted when calculating the lengths of its sides. A triarc with sides of lengths a, b, c is called an (a, b, c) -triarc. Of course, we can flip or rotate such a triarc and consider it, for example, as a (b, a, c) -triarc.

Triarcs are very versatile tools. Firstly, if the length of some side of a triarc T equals the length of some side of another triarc R , then T and R can be glued together along those sides to yield a new plane graph with all inner vertices having degree 3; see, for example, Fig. 1. Secondly, every triarc T has zero total curvature; to see this, take two copies of T , turn one of them upside down, glue them along a common side to obtain a “parallelogram” (see Fig. 1 again), and identify opposite sides of this parallelogram to obtain a graph embeddable in the torus. But perhaps the most important property of triarcs is the possibility to “glue” them together to obtain larger triarcs; we describe this operation below.

Suppose we have an (a_1, b_1, c_1) -triarc and an (a_2, b_2, c_2) -triarc such that $b_1 = 2m$ and $c_2 = 2l$ are even. Then, we may combine these triarcs (and several pentagons and heptagons) to construct an $(a_1 + a_2, b_1 + b_2, c_1 + c_2)$ -triarc. To do this, we identify a corner (and an incident edge) of the first triarc with a corner (and an edge) of the second triarc—see Fig. 2—so that the two identified corners yield a vertex of degree 3 on a side of length $a_1 + a_2$ in a new triarc. Then, we can add a “parallelogram” consisting of hexagons to obtain an $(a_1 + a_2, b_1 + b_2, c_1 + c_2)$ -triarc. However, we do not want to add hexagons. Instead, we decompose the parallelogram into tiles, each consisting of four hexagons as depicted in Fig. 2, and replace each of these tiles by two pentagons and two heptagons as indicated in Fig. 3. The “tile” on the right of Fig. 3 will be used several times in the sequel, and we shall refer to it as a *H-tile*.

We are going to use this operation of gluing two triarcs into a larger one several times in the following section.

Fig. 1 Gluing two triarcs along sides of equal length. The dots represent the corners of the triarcs

Fig. 2 Glueing two triarcs with two sides of even length together using the tile of Fig. 3

Fig. 3 In a configuration of 4 hexagons we may contract the central edge and then “uncontract” it in the other direction. A “tile” consisting of two pentagons and two heptagons results; we use such tiles in Fig. 2

3 Proof of Theorem 1.2

We are ready to state and prove our main result. Let us observe that, unlike Eberhard’s theorem, we do not need to assume that the given face-sizes form a plausible sequence (although we make this assumption in the formulation of the theorem) because given a sequence (p_k) , $3 \leq k \leq r$, $k \neq 5, 7$, the sequence can always be appended by appropriate values p_5 and p_7 to become plausible.

Theorem 3.1 *Let $p = (p_3, p_4, \dots, p_r)$ be a plausible sequence for the sphere. Then there exist infinitely many integers $n \in \mathbb{N}$ such that the sequence $p + n \cdot (0, 0, 1, 0, 1)$ is realizable in the sphere.*

Proof We will give an explicit construction of a cubic graph embeddable in the sphere whose face sequence is of the form $p + n \cdot (0, 0, 1, 0, 1)$. The rough plan for this is as follows. For each face imposed by the sequence p , we create a *basic triarc* containing this face as well as some pentagons and heptagons. Then, we glue all these triarcs together and extend to a triarc with sides of suitable lengths. Finally, we construct a new triarc having the same side lengths and glue these two triarcs together (as explained later) to obtain the desired graph embedded in the sphere.

To construct a basic triarc for a k -gon (we will make p_k copies of it), we surround the k -gon by three heptagons and $k - 3$ pentagons as shown in the right half of Fig. 4

Fig. 4 A $(4, 4, 3)$ -triarc and a $(2, 2, 4)$ -triarc

(where the k -gon we are surrounding happens to be a pentagon). Note that we can always make the basic triarc isosceles with the equal sides having even length. We call the k -gon we started with the *nucleus* of this triarc.

Having constructed all basic triarcs, our next step is to glue them all together to obtain a single triarc T containing them all. We do so recursively, attaching one basic triarc at a time as shown in Fig. 2, where we use many copies of the H-tile in order to build the parallelogram needed. Each time we use this gluing operation, we are assuming that both triarcs in Fig. 2 are isosceles, with the equal sides having even length, and align them so that the two equal even sides are the upper left and upper right sides. Note that the resulting triarc is also isosceles with two equal sides of even length. Thus, we can continue recursively to glue all basic triarcs into one isosceles triarc T .

Our next aim is to enlarge T into an equilateral triarc T' with sides of length n , where n is a multiple of 8 and satisfies $n \equiv 2 \pmod{3}$, using only pentagons and heptagons. To this end, we will use the gluing operation of Fig. 2 and many copies of a $(4, 4, 3)$ -triarc and a $(2, 2, 4)$ -triarc. Figure 4 shows how to construct those triarcs with pentagons and heptagons only.

Note that gluing T' with a $(4, 4, 3)$ -triarc (as in Fig. 2) keeps it isosceles and decreases the difference of lengths between the “base” and the other two sides by 1, while gluing with a $(2, 2, 4)$ -triarc increases that difference by 2. Thus, recursively gluing with such triarcs we can enlarge T into an equilateral triarc S with sides of even length.

Moreover, using the gluing operation of Fig. 2 three times, once with a $(2, 2, 4)$ -triarc and twice with a $(4, 4, 3)$ -triarc, we can increase the side-lengths by $(2, 2, 4) + (4, 4, 3) + (4, 4, 3) = (10, 10, 10)$. Thus we can increase the length of each side of S by 10. Since $10 \equiv 1 \pmod{3}$, we can use this operation to enlarge S into an equilateral triarc S' with even sides of length $2 \pmod{3}$. Moreover, since performing this operation three times increases the length of each side by 30, and $30 \equiv 6 \pmod{8}$, we can enlarge S' into an equilateral triarc T' with the length of each side being a multiple of 8 and congruent to 2 modulo 3.

Fig. 5 Gluing R and T' together along a “ring” consisting of pentagons and heptagons. This operation is possible because we made sure that every side of T' , and thus also of R , has length congruent to 2 (mod 3)

Next, we are going to construct a triarc R that has the same side lengths as T' but consists of pentagons and heptagons only. By gluing together a $(2, 2, 4)$ -triarc, a $(2, 4, 2)$ -triarc, and a $(4, 2, 2)$ -triarc (that is, the same triarc in three different rotations), we get an $(8, 8, 8)$ -triarc, which we will call D . Since the sides of T' have length a multiple of 8, by gluing copies of D together recursively as in Fig. 2 we can indeed construct a triarc R that has the same dimensions as T' .

We can now combine R and T' together to produce a cubic graph tiling the sphere as shown in Fig. 5. By construction, this graph has for every $k \in \mathbb{N} \setminus \{0, 1, 2, 5, 7\}$, precisely p_k faces of size k , and moreover it has at least p_5 pentagons and at least p_7 heptagons. Thus its face sequence is of the form $p + (0, 0, n, 0, m)$ for some $n, m \in \mathbb{N}_+$. Since both p and $p + (0, 0, n, 0, m)$ satisfy Euler's formula (the former by assumption, the latter because the plane graph we just constructed implements it), we have $n = m$.

This completes the construction and shows the existence of one particular value of n as desired. However, observe that the construction of T' and R allows us to make the side lengths of these triarcs arbitrarily large. This shows that we can get examples for infinitely many values of n and thus completes the proof. $\square$

We now turn from planar graphs to maps on arbitrary (compact) surfaces. A *map* on a surface S is a graph together with a 2-cell embedding in S . A map is *polyhedral* if all faces are closed disks in the surface and the intersection of any two faces is either empty, a common vertex, or a common edge. If the graph of the map is cubic, then we say that the map is *simple*.

A cycle contained in the graph of a map is *contractible* if it bounds a disk on the surface. The *edge-width* of a map M is the length of a shortest noncontractible cycle in M . The *face-width* of M is the minimum number of faces, the union of whose boundaries contains a noncontractible cycle. We refer the reader to [14] for more about the basic properties and the importance of these parameters of maps. At this point we only note that a map is polyhedral if and only if its graph is 3-connected and its face-width is at least three, see [14, Proposition 5.5.12]. We also note that if r is the largest size of a face of M , then the edge-width of M cannot exceed $\frac{r}{2}$ times the face-width of M .

We now restate and prove our main result, Theorem 1.2.

Corollary 3.2 *Let (p_k) , $3 \leq k \leq r$, $k \neq 5, 7$, be a sequence of nonnegative integers, let S be a closed surface, and let w be a positive integer. Then there exist infinitely many pairs of integers p_5 and p_7 such that the sequence $(p_3, p_4, p_5, \dots, p_r)$ is realizable in S and there is a 3-connected realizing cubic map of face-width at least w .*

Proof Let us first describe a construction that does not necessarily achieve the desired face-width; we will later explain how to modify this construction in order to get large face-width.

The rough sketch of this construction is as follows. Firstly, we increase the number of hexagons in the sequence (p_k) to $p'_6 := p_6 + 2h + c$, where h is the number of handles of S and c the number of its crosscaps (by the surface classification theorem we may assume that one of h, c is zero, but we do not have to). It follows from Theorem 3.1 that we can increase the numbers p_5 and p_7 of this sequence to some appropriate values so that the resulting sequence p' is realized by a map on the sphere. We will then use the $2h + c$ auxiliary hexagons of this map we added above to introduce some handles and/or crosscaps. After doing so, all auxiliary hexagons will have disappeared, and we obtain a map on S whose sequence of faces differs from (p_k) by some pentagons and heptagons only.

More precisely, similarly to the proof of Theorem 3.1, we construct a basic triarc for each face in p' , but with one modification: for each hexagon, we construct a triarc like the one in Fig. 6 (on the left) rather than one with two even sides of equal lengths (in fact, we need this modification for the auxiliary hexagons only, but we might as well use it for the original hexagons in p as well).

Next, we proceed as in Theorem 3.1 to glue all basic triarcs together into one triarc T . However, since we now have basic triarcs with all sides odd (the ones of Fig. 6), the gluing operation of Fig. 2 will not work for these triarcs. For this reason, we first extend each such triarc into an equilateral triarc with even sides using three copies of the $(2, 2, 4)$ -triarc of Fig. 4 as shown in Fig. 6 (right).

We continue imitating the proof of Theorem 3.1 to obtain a cubic graph G embedded in a homeomorphic copy S' of the sphere that contains all basic triarcs. We will

Fig. 6 *On the left: the new basic triarc for a hexagon. On the right: extending the triarc from the left into an equilateral triarc with even sides*

Fig. 7 The situation arising after introducing a handle. The 12-cycle C consists of the dashed and the thick edges

now perform some cutting and gluing operations on both S' and G to obtain a new surface, homeomorphic to S , with a cubic graph G' embedded in it.

Suppose that $h > 0$. Then, pick h pairs $(F_1, F'_1), \dots, (F_h, F'_h)$ of hexagonal faces of G such that all the faces F_i and F'_i are distinct (there are enough hexagonal faces by our choice of the sequence p'). Now, for each pair (F_i, F'_i) , perform the following operations. Cut out the two discs of S' corresponding to F_i, F'_i and glue their boundaries together with a half-twist; that is, each vertex of the boundary of F_i is identified with the midpoint of an edge of F'_i and vice-versa. This operation creates a handle in S' , and the embedded graph remains cubic; however, it also gives rise to some unwanted faces: the size of each face that was incident to F_i or F'_i has now been increased by 1. We thus have the situation depicted in Fig. 7, where C is the cycle of length 12 resulting from the boundaries of F_i and F'_i . Recall that since every hexagon is put in a basic triarc like the one in Fig. 6, the sizes of the faces on each side of C alternate between 6 and 8 as shown in Fig. 7. But now, contracting and uncontracting each of the three thick edges (in the way explained in Fig. 3) turns each of the faces incident with C into a heptagon.

On the other hand, if $c > 0$, then pick c distinct hexagonal faces $F_1, \dots, F_c$, and for every i , cut out the disc corresponding to F_i and glue in its place the outside of the hexagon of Fig. 8 with a half twist. Each such operation gives rise to a new crosscap and also to unwanted faces just like in Fig. 7. But again, contracting and uncontracting each of the three thick edges, we can turn all these unwanted hexagons and octagons into heptagons.

Thus, after all these operations have been completed, we obtain a surface with h handles and c crosscaps with a cubic graph embedded in it whose face sequence is $p + (0, 0, n, 0, m)$ for some $n, m \in \mathbb{N}_+$. Note that all auxiliary hexagons in $p' - p$ have disappeared after the above operations.

It is easy to check that our maps are 3-connected. Indeed, our “building blocks”—the basic triarcs and the triarcs of Fig. 4—are 3-connected after suppressing the de-

Fig. 8 The gadget used to create a crosscap *inside* a hexagon

gree 2 vertices. This property is also true for triarcs in Fig. 6, and it is maintained by the gluing operation of Fig. 2. By gluing two triarcs along a “circumference,” using the ring in Fig. 5, we get a 3-connected graph. The gadget we used for introducing cross-caps (Fig. 8) is 3-connected, and it is not hard to see that we maintain 3-connectivity when adding this gadget or when creating a handle as depicted in Fig. 7.

It remains to discuss how to modify this construction to obtain maps with arbitrarily large face-width. By the remark preceding Corollary 3.2, it suffices to construct maps with arbitrarily large edge-width z since the face sizes are bounded from above by r . This is achieved as follows.

First of all, we make every basic triarc used in the construction large enough that the distance from its nucleus to the boundary of the triarc is at least z , and the length of each side of each triarc is at least $3z$. To achieve this, we first glue the triarc with several $(2, 2, 4)$ -triarcs (or any other triarcs) both on the left and on the right, to obtain a large triarc with the nucleus in the middle of the bottom side. Then we possibly glue it with a $(4, 4, 3)$ -triarcs to create a triarc with all sides even. Finally, we rotate the triarc by 120° and perform more glueing with $(2, 2, 4)$ -triarcs to get the nucleus away from the boundary. (The notions “left”, “right”, and “ 120° ” in this paragraph refer to the glueing operation of Fig. 2.)

Next, we replace the auxiliary hexagons used in order to add handles and crosscaps with $6N$ -gons, where N is odd and greater than $z/2$. Of course, this will force us to add some more pentagons to our sequence p_k to make it plausible. Note that we can generalize the triarc on the left of Fig. 6 so that the inner 6-gon is replaced by a $6N$ -gon surrounded by three heptagons and $6N - 3$ pentagons, arranged in a symmetric way so that any two heptagons separate $2N - 1$ pentagons from the rest. We will make use of the fact that $2N - 1$ is odd. We need to adapt the right half of Fig. 6 as well, since the inner triarc has now grown larger. For this, note that each side of the inner triarc has now length $2N + 1$, and so in order to use the method of the right half of Fig. 6, the three peripheral triarcs must have a base of length $2N + 2$ (in addition to having their other two sides of equal length). Since we chose N to

be odd, it turns out that $2N + 2$ is a multiple of four, and so we can construct the required peripheral triarcs by gluing several $(2, 2, 4)$ -triarcs together, using Fig. 2, into a $(N + 1, N + 1, 2N + 2)$ -triarc.

Moreover, the crosscap gadget shown in Fig. 8 can be generalized so that the inner 6-gon is replaced by a $6N$ -gon that is surrounded by $3N$ heptagons and $3N$ pentagons, arranged alternately around the $6N$ -gon (here it is also important that we chose N to be odd).

When the time comes to insert crosscaps or glue pairs of such $6N$ -gons together (after a half-twist), we obtain a similar configuration as in Fig. 7, but with $3N$ thick edges. Some of these thick edges are surrounded by faces of sizes 8, 6, 8, 6 (as in Fig. 7), while others are surrounded by four hexagons or by one octagon and three hexagons. Note, however, that for parity reasons, we can make sure that every octagon is incident with a thick edge, and still every fourth edge on the dashed cycle is thick. Finally, the contract–uncontract operation of Fig. 3 turns these faces into pentagons and heptagons only.

It is easy to see that these changes did not hurt 3-connectivity. Let us now argue that the resulting map G has edge-width at least z . Recall that the surface S is obtained from a plane graph G' , embedded in the sphere, that is composed of large basic triarcs $T_1, \dots, T_m$, some large parallelograms used to glue the basic triarcs together into a large triarc T , and a remainder X comprising the material we used to enlarge T into T' , the ring of Fig. 5, and the triarc R . Let L_i be the nucleus of T_i . Then S is obtained from G' by gluing the crosscap gadget into some of the $6N$ -gons L_i and/or by identifying some pairs L_i, L_j of the $6N$ -gons to create handles.

We claim that for every basic triarc T_i such that the nucleus L_i of T_i is a $6N$ -gon and

for every side P of T_i , there is a set of z pairwise disjoint L_i – P paths. (3)

Indeed, recall that in order to construct T_i , we first surrounded L_i by several pentagons and heptagons, $6N$ in total, to obtain a triarc T_i^1 , then we performed the operation of the right half of Fig. 6 to obtain a triarc T_i^2 , and finally we enlarged this into a larger triarc $T_i^3 = T_i$ using the operation of Fig. 2 several times (this final step was described later, in the part of the current proof concerning large face-width). Now given any side P' of T_i^2 , it is possible to find, within T_i^2 , a set of z pairwise disjoint L_i – P' paths, see Fig. 9. Then, every time we use the operation of Fig. 2 while enlarging T_i^2 into T_i^3 , it is possible to recursively propagate those paths to reach the side of T_i^3 corresponding to P' ; if P' is included within a side of T_i^3 , then nothing needs to be done, and if not, then we can propagate our paths through the parallelogram of Fig. 2 while keeping them disjoint (this is true even after performing the contract–uncontract operations of Fig. 3). This proves our claim (3).

Next, we claim that any two nuclei L_i, L_j can be joined by z pairwise disjoint paths in G' . Indeed, this follows easily from (3) and the fact that whenever we glue two triarcs T, T' together as in Fig. 2 by a parallelogram R with side-lengths m, n , then we can find a set of m pairwise disjoint paths within R joining its two opposite sides of length m , as well as a set of $\min(m, n)$ pairwise disjoint paths within R joining the sides of T and T' incident with R .

Fig. 9 Constructing z disjoint L_i – P' paths, in the case that the auxiliary hexagons are replaced with 42-gons ($6N$ for $N = 7$). In light gray are the $(2, 2, 4)$ -triarc, in dark gray the H-tiles resulting from the glueing operation of Figs. 2 and 3. The empty triangle at the bottom is a part of the triarc, isomorphic to the top-left and top-right ones. As our paths do not use the bottom part, we don't show the details in the figure. The white triangular shape in the middle represents the nucleus. The 16 thick paths are the ones we need in order to prove that our graphs have large face-width

We distinguish two cases.

Case 1: the surface S is orientable There are three types of noncontractible cycles in G . The first one comes from a path P in G' that connects two nuclei L_i, L_j such that these nuclei are glued to create a handle. As the distance from each nucleus to the boundary of the corresponding triarc T_i is at least z , the length of P is at least z as well (even at least $2z$).

The second type of noncontractible cycle C comes from a cycle C' in G' such that $|C'| \leq |C|$ and C' separates some nucleus L_i from some other nucleus L_j in G' . We use the above construction of z pairwise disjoint paths from L_i to L_j to conclude that $|C| \geq z$ as desired (in fact, we have $|C| \geq 2z$ because the graph is cubic and so any two paths that have a common inner vertex must have a common edge).

The last type is similar to the second one: it is a cycle C that crosses some cycle L of G obtained by gluing two nuclei L_i, L_j to introduce a handle. Such a cycle C comes from a k -tuple of paths in G' , where k is the number of times that C crosses L , half of which paths have ends on L_i , and the other half of them on L_j . We may assume that none of these paths P leaves the triarc containing the endpoints of P , for otherwise $|P| \geq z$ holds. We will consider again the z pairwise disjoint paths connecting L_i to L_j . In fact, we only need to consider their parts that are contained in the triarcs T_i, T_j : Let these parts be $P_{i,1}, \dots, P_{i,z}$ (connecting L_i to the boundary of T_i) and $P_{j,1}, \dots, P_{j,z}$ (connecting L_j to the boundary of T_j). As the paths $P_{i,t}$ start regularly along two thirds of the nucleus L_i and the same holds for L_j , we can use them to create $z/2$ pairwise disjoint paths $Q_1, \dots, Q_{z/2}$ in G connecting the boundary of T_i to the boundary of T_j . Note that the part of the surface S containing T_i and T_j is a cylinder, and the cycle C goes around this cylinder. Thus C must intersect all of the paths Q_t . As the graph is cubic, each intersection with a path has to use at least two vertices, proving again that $|C| \geq z$.

Case 2: S is nonorientable In this case a noncontractible cycle C in G will either yield a cycle C' as above, in which case the same argument applies, or it will yield

a path P' in G' whose endpoints were identified when introducing crosscaps. Recall that we made every basic triarc used in the construction large enough that the distance from its nucleus to the boundary of the triarc is at least z ; thus P' is, without loss of generality, contained within one of the triarcs in which a crosscap was introduced. With the help of Fig. 9 and Fig. 8 (modified with a $6N$ -gon replacing the hexagon as described above), it is now not hard to see that $|P'| \geq z$ as desired. $\square$

4 Other Neutral Sequences

In this paper we concentrated on the neutral sequence $(0, 0, 1, 0, 1)$, but we believe that our methods and results apply in a much more general setting—see also Sect. 5—and it is the purpose of this section to explain this.

In Sect. 3 we showed that every plausible sequence can be extended into a realizable one by adding pentagons and heptagons only. In what follows we are going to give a rough sketch of a proof that an arbitrary neutral sequence s can be used to extend any plausible sequence into a realizable one under the assumption that a couple of basic building blocks can be constructed using precisely the faces that appear in some multiple of s . We expect that our construction will help yield more general results in the future, by showing that these building blocks can indeed be constructed.

So let $p = (p_3, p_4, \dots, p_r)$ be a plausible sequence for the sphere or the torus, and let $s = (p'_3, p'_4, \dots, p'_r)$ be a neutral sequence. In order to prove that there is some n such that $p + ns$ is realizable, it suffices to find some $k \in \mathbb{N}$ for which it is possible to construct the following building blocks using precisely the faces that appear in some multiple of s :

- (i) a (k, k, k) -triarc
- (ii) a $(k, k, k - 1)$ -triarc
- (iii) for every nonzero entry p_l in p , a triarc containing a face of size l , such that the length of two of the sides of this triarc is a multiple of k
- (iv) a “ring” like the one in Fig. 5 (using the faces from s in the right proportion rather than pentagons and heptagons) for combining two equilateral triarcs

Indeed, to begin with, construct a parallelogram with all sides of length k out of two $(k, k, k - 1)$ -triarcs (supplied by (ii)) as shown in Fig. 10. (In figures explaining

Fig. 10 Constructing a parallelogram out of two $(k, k, k - 1)$ -triarcs

Fig. 11 Gluing R and T' together. The black dots depict the faces imposed by the sequence p

our construction, we shall use triarcs made of hexagonal faces, but this is for illustration purposes only; in fact they have to be made of multiples of s .) This also allows us to construct any parallelogram with dimensions mk, lk for every $m, l \in \mathbb{N}$.

Next, similarly to the construction in Theorem 3.1, construct a “basic” triarc as in (iii) for each face-size l for which $p_l \neq 0$; in fact, we construct p_l copies of this basic triarc for every l . Then, using the parallelograms, we constructed earlier, we recursively glue all those triarcs together into a single triarc T , in a manner very similar to the operation of Fig. 2.

By recursively gluing the resulting triarc with a $(k, k, k - 1)$ -triarc provided by (ii) using the gluing operation of Fig. 2, we can transform T into an equilateral (mk, mk, mk) -triarc T' for some (large) $m \in \mathbb{N}$.

Using the glueing operation of Fig. 2 it is possible to construct a triarc R with the same side-lengths as T' , using only (k, k, k) -triarcs (provided by (i)) and the above parallelograms.

In the case of the sphere, we can combine R and T' by using the “ring” provided by (iv) to complete the construction.

If the underlying surface S is the torus, we glue R and T' together along one of their sides to obtain a parallelogram and glue two opposite sides of this parallelogram together to obtain a cylinder C both of whose bounding cycles are in–out alternating cycles of length mk , see Fig. 11. We then glue the two bounding cycles of C together to obtain a realization of a torus.

If p is plausible for some other surface S , then we would need additional gadgets like those used in the proof of Corollary 3.2.

5 Outlook

Trying to achieve a better understanding of the implications of Euler’s formula, we studied the question of whether, given a plausible sequence p and a neutral sequence q , it is possible to combine p and q into a realizable sequence $p + nq$, but we did so in very restricted cases. The general problem remains wide open; in particular, we would be interested to see an answer to the following problem.

Problem 5.1 *Given a closed surface S , is it true that for every plausible sequence p for S and every neutral sequence q , there is an $n \in \mathbb{N}$ such that $p + nq$ is realizable in S with the exception of only finitely many pairs (p, q) ?*

(As mentioned in the introduction, if S is the torus, then the list of exceptional pairs (p, q) is not empty.)

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