Dilation of Newton Polytope and p -adic Estimate

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Abstract Let $f(X)$ be a polynomial in n variables over the finite field $\mathbb{F}_q$. Its Newton polytope $\Delta(f)$ is the convex closure in $\mathbb{R}^n$ of the origin and the exponent vectors (viewed as points in $\mathbb{R}^n$) of monomials in $f(X)$. The minimal dilation of $\Delta(f)$ such that it contains at least one lattice point of $\mathbb{Z}_{\geq 0}^n$ plays a vital role in the p -adic estimate of the number of zeros of $f(X)$ in $\mathbb{F}_q$. Using this fact, we obtain several tight and computational bounds for the dilation which unify and improve a number of previous results in this direction.

Keywords Finite field · Newton polytope · Lattice point · p -adic estimate

1 Introduction

The theory of convex polytopes has many applications across mathematics and computer science (e.g., see [6, 9–11]). Given a multivariate polynomial one may associate with it an integral polytope called its Newton polytope. Let K be an arbitrary field and $K^* = K - \{0\}$. Suppose that $f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ is a nonconstant polynomial that is written in sparse representation as a sum of m nonzero monomials

$$f(x_1, \dots, x_n) = \sum_{j=1}^m a_j x^{D_j}, \quad a_j \in K^*, \quad (1)$$

where

$$D_j = (d_{1j}, \dots, d_{nj}) \in \mathbb{Z}_{\geq 0}^n, \quad x^{D_j} = x_1^{d_{1j}} \cdots x_n^{d_{nj}}.$$

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The set $\text{Supp}(f) := \{D_1, \dots, D_m\}$ is called the *support of f* . The *Newton polytope* $\Delta(f)$ is defined to be the convex closure in $\mathbb{R}^n$ of the set $\text{Supp}(f) \cup O$ where O denotes the origin of $\mathbb{R}^n$; in our notation, $\Delta(f) = \text{conv}(\text{Supp}(f) \cup O)$.

Remark 1 The Newton polytope $\Delta(f)$ has also been defined to be the convex closure of the set $\text{Supp}(f)$ by some authors in the literature. However, for the case $O \in \text{conv}(\text{Supp}(f))$ (e.g., f has a nonzero constant), these two definitions coincide with each other.

The Newton polytope provides a geometric approach to studying polynomials. For example, according to the BKK bound (named after Bernshtein, Khovanskii and Kouchnirenko; see [3, 14–16]), the number of common zeros in $(\mathbb{C}^*)^n$ of n generic polynomials is bounded by the volume of their Newton polytopes. The decomposition of the Newton polytope, in the sense of the Minkowski sum, has a close relation to the reducibility of the associated polynomial, which was observed by Ostrowski [19] and developed by Gao and Lauder [7, 8] and others. For the polynomial defined over finite fields, Adolphson and Sperber [1] discovered that the dilation of its Newton polytope can be used to estimate the p -divisibility of the number of zeros of the polynomial. This paper will be devoted to this topic.

Let $\mathbb{F}_q$ be the finite field of q elements with characteristic p . Suppose that $f(x_1, \dots, x_n)$ is a polynomial in n variables of degree d over $\mathbb{F}_q$. Let $N(f)$ denote the number of zeros of f in $\mathbb{F}_q$ and ord_p (resp. ord_q) the p -adic additive valuation normalized such that $\text{ord}_p p = 1$ (resp. $\text{ord}_q q = 1$). One of the problems of interest is the estimate of its p -divisibility. The classical Chevalley–Warning theorem asserts $\text{ord}_p N(f) \geq 1$ if $n > d$. Let $\lceil x \rceil$ denote the least integer $\geq x$ for $x \in \mathbb{R}$, Ax showed that

Theorem 1 ([2])

$$\text{ord}_q N(f) \geq \left\lceil \frac{n}{d} \right\rceil - 1. \quad (2)$$

Ax's result was extended by Katz [13] to a system of equations by using Dwork's p -adic theory. Wan [21] provided an elementary proof to the Ax–Katz theorem. The simplest proof of the Ax–Katz theorem and its extension to character sums were given in [22]. A reduction of the Ax–Katz theorem for a system of equations to Ax's theorem for a single equation has been found by Hou [12]. O. Moreno and C.J. Moreno [18] improved the Ax–Katz theorem in terms of p -weight degree in certain special cases. Recently, Cao and Sun [4, 5] obtained some new results improving Chevalley–Warning–Ax–Katz-type estimate for the general diagonal polynomials over finite fields.

In this paper, we will extend and improve Ax's theorem by using the dilation of Newton polytope. Instead of considering all the variables, we will focus on those variables with low degree. Put $\Omega = \{1, 2, \dots, n\}$. For any nonempty subset $A \subset \Omega$, let f_A be the polynomial obtained from f by letting $x_i = 1$ for $i \in \Omega - A$. As usual, let $|A|$ denote the cardinality of A . Clearly, $f_\Omega = f$. Our result can be stated as

Theorem 2 (Main Theorem) *For any nonempty subset $A \subset \Omega$, we have*

$$\text{ord}_q N(f) \geq \left\lceil \frac{|A|}{\deg f_A} \right\rceil - 1.$$

Corollary 1 *If $|A| > \deg f_A$ for some nonempty subset $A \subset \Omega$, then $N(f) > 0$ and $q|N(f)$.*

Example 1 Let $f(x_1, x_2, x_3, x_4) = x_1x_3^4 + x_2x_4^5$. Since $\deg f = 6$ and $n = 4$, one gets the trivial estimate $q^0|N(f)$ by Ax's theorem. Now take $A = \{1, 2\}$, then $f_A = x_1 + x_2$, and one gets $q|N(f)$ by our main theorem.

As one of the reviewers pointed out, the Newton polytope of a multivariate polynomial has an internal structure which is only visible when one considers the complete family of polytopes associated to all the possible polynomials that arise by deleting one or more terms in the polynomial. Theorem 2 is an inductive argument that justifies this claim. The remainder of the paper is organized in the following way. The proof of Theorem 2 will be given in Sect. 2, before which the result of Adolphson and Sperber will be briefly introduced. In Sect. 3, we will give several bounds in special cases which unify and improve a number of previous results in this direction.

2 Adolphson–Sperber's Result and Proof of Main Theorem

By using the tool of Newton polytope and Dwork's p -adic theory, Adolphson and Sperber [1] generalized Ax's theorem to the exponential sums (for the knowledge of exponential sums, refer to Lidl and Niederreiter [17]). Let ζ_p be a fixed primitive p -th root of unity in the complex numbers. For a given polynomial $f(x_1, \dots, x_n) \in \mathbb{F}_q[x_1, \dots, x_n]$ of the form as in (1), we form the exponential sum

$$S(f) = \sum_{x_1, \dots, x_n \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(f(x_1, \dots, x_n))},$$

where $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ denotes the absolute trace map from $\mathbb{F}_q$ to the prime field $\mathbb{F}_p$.

Let $\Delta(f)$ be the Newton polytope of f as defined at the beginning of Sect. 1. Let $\omega(f)$ be the smallest positive rational number such that the dilation $\omega(f)\Delta(f)$ contains a lattice point of $\mathbb{Z}_{>0}^n$. Adolphson and Sperber proved

Theorem 3 ([1, Theorem 1.2]) $\text{ord}_q S(f) \geq \omega(f)$.

Let x_0f be the new polynomial by adding a further variable x_0 to the original polynomial f . It is easy to verify that the relation between $N(f)$ and $S(x_0f)$ is

$$qN(f) = S(x_0f) \implies \text{ord}_q N(f) = \text{ord}_q S(x_0f) - 1. \quad (3)$$

By Theorem 3, (3) and the fact that $N(f) \in \mathbb{Z}_{\geq 0}$, we have

Corollary 2 $\text{ord}_q N(f) \geq \lceil \omega(x_0 f) \rceil - 1$.

For a concrete polynomial whose Newton polytope is known in advance, the exact value of $\omega(f)$ (or $\omega(x_0 f)$) can be calculated by standard arguments in linear programming. However, we still want to find a tight and computational estimate for $\omega(f)$ (or $\omega(x_0 f)$) of the simple expression like (2) in the general case. This is just what Theorem 2 tells.

Recall a basic fact from convex geometry: all points in $\Delta(f)$ can be written in the form

$$P = \sum_{i=0}^m \beta_i D_i \quad \text{where } \beta_i \geq 0, \quad \text{and} \quad \sum_{i=0}^m \beta_i = 1.$$

Here we set $D_0 := O = (0, \dots, 0)$ for convenience.

Lemma 1

$$\omega(f) = \min \left\{ \sum_{i=0}^m \alpha_i \left| \sum_{i=0}^m \alpha_i D_i \in \mathbb{Z}_{>0}^n, \alpha_i \in \mathbb{Q}_{\geq 0} \right. \right\}.$$

Proof Assume that the point $P = \sum_{i=0}^m \alpha_i D_i$ with $\alpha_i \in \mathbb{Q}_{\geq 0}$ is a lattice point of $\mathbb{Z}_{>0}^n$. Let $\alpha = \sum_{i=1}^m \alpha_i$. We claim $\alpha > 0$; Otherwise, $P = D_0 \notin \mathbb{Z}_{>0}^n$, a contradiction. Now consider the new point $P' = \sum_{i=0}^m \frac{\alpha_i}{\alpha} D_i$. Since $\sum_{i=0}^m \frac{\alpha_i}{\alpha} = 1$, we have $P' \in \Delta(f)$. Clearly, $\alpha P' = P \in \mathbb{Z}_{>0}^n$ by our assumption. So we get $\omega(f) \leq \alpha$. The result follows from the minimality of $\omega(f)$. $\square$

We are now in a position to prove the main theorem of this paper.

Proof of Theorem 2 Without loss of generality, assume $A = \{1, 2, \dots, k\}$ with $k \leq n$. Thus

$$f_A = a_1 x_1^{d_{11}} \cdots x_k^{d_{k1}} + \cdots + a_m x_1^{d_{1m}} \cdots x_k^{d_{km}}.$$

Put $d_j := \sum_{i=1}^k d_{ij}$ for $j = 1, \dots, m$. Then $\deg f_A = \max\{d_j | j = 1, \dots, m\}$. Assume that the point $P = \sum_{i=0}^m \alpha_i D_i$ with $\alpha_i \in \mathbb{Q}_{\geq 0}$ is a lattice point of $\mathbb{Z}_{>0}^n$, then we obtain the following system:

$$\begin{cases} \alpha_1 d_{11} + \cdots + \alpha_m d_{1m} = \lambda_1, \\ \vdots \\ \alpha_1 d_{k1} + \cdots + \alpha_m d_{km} = \lambda_k, \end{cases}$$

with $\lambda_i \in \mathbb{Z}_{>0}$ for $i = 1, \dots, k$. Summing up the above equalities yields

$$\alpha_1 d_1 + \cdots + \alpha_m d_m = \lambda_1 + \cdots + \lambda_k. \quad (4)$$

Dividing the both sides of (4) by $\deg f_A$, we get

$$\sum_{i=1}^m \alpha_i \geq \alpha_1 \frac{d_1}{\deg f_A} + \cdots + \alpha_m \frac{d_m}{\deg f_A} = \frac{\lambda_1 + \cdots + \lambda_k}{\deg f_A} \geq \frac{k}{\deg f_A}. \quad (5)$$

Here we use the fact that $\deg f_A \geq d_i$ in the first inequality and the fact that $\lambda_i \geq 1$ in the last inequality. By Lemma 1 and (5), we have

$$\omega(x_0 f) \geq \frac{|A|}{\deg f_A}. \quad (6)$$

The result follows from Corollary 2 and (6). $\square$

Remark 2 Clearly, there are $2^n - 1$ choices for A . The key problem is how to find a suitable set A which corresponds to the maximal value of $|A|/\deg f_A$. We have given an example in Example 1. In fact, it is just a particular case of Theorem 6 below.

3 Further Improvement

Wan [20], Cao and Sun [4, 5] have respectively investigated the polynomials of special form over finite fields and obtained the p -adic estimate for their numbers of zeros. It can be easily seen that their estimates are tighter than Ax's estimate. In this section, we will show that their results can be deduced or improved by using our technique.

Theorem 4 ([20]) *Let $f(X)$ be a diagonal polynomial of the form*

$$f(X) = a_1 x_1^{d_1} + a_2 x_2^{d_2} + \cdots + a_n x_n^{d_n} + c, \quad (7)$$

where $a_i \in \mathbb{F}_q^*$, $c \in \mathbb{F}_q$ and $d_i \in \mathbb{Z}_{>0}$. Then we have

$$\text{ord}_q N(f) \geq \left\lceil \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} \right\rceil - 1.$$

Note that in (7) one has $d := \deg f = \max\{d_1, \dots, d_n\}$ which implies that $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} \geq \frac{n}{d}$. This shows that Wan's theorem is stronger than Ax's theorem for diagonal cases.

Cao and Sun [4, 5] have studied the so-called *general diagonal polynomial* of the form

$$f = \sum_{i=1}^r a_i x_{i1}^{d_{i1}} x_{i2}^{d_{i2}} \cdots x_{in_i}^{d_{in_i}} + c, \quad (8)$$

where $a_i \in \mathbb{F}_q^*$, $c \in \mathbb{F}_q$, $d_{ij} \in \mathbb{Z}_{\geq 0}$ and x_{ij} 's are $n (= \sum_{i=1}^r n_i)$ distinct variables. Set $d_i = \gcd(d_{i1}, \dots, d_{in_i}, q-1)$ for $i = 1, \dots, r$, and define the new polynomial

$$\widehat{f} = \sum_{i=1}^r a_i (x_{i1} x_{i2} \cdots x_{in_i})^{d_i} + c. \quad (9)$$

Theorem 5 ([5]) *With the notations as above, we have*

$$N(f) = N(\widehat{f}) \implies \text{ord}_q N(f) \geq \left\lceil \frac{n}{\deg \widehat{f}} \right\rceil - 1.$$

Since $\deg \widehat{f} \leq \deg f$, Cao–Sun’s result may also improve Ax’s theorem. We first show that Theorem 4 can be deduced by the following theorem.

Theorem 6 *Let $f(X, Y)$ be a polynomial of the form*

$$f(X, Y) = x_1^{d_1} g_1(Y) + \cdots + x_n^{d_n} g_n(Y) + c,$$

where $g_i(Y) \in \mathbb{F}_q[y_1, \dots, y_m]$ are polynomials, $c \in \mathbb{F}_q$ and $d_i \in \mathbb{Z}_{>0}$. Then

$$\text{ord}_q N(f) \geq \left\lceil \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} \right\rceil - 1.$$

Proof Consider the polynomial

$$f(X) = x_1^{d_1} + x_2^{d_2} + \cdots + x_n^{d_n}. \quad (10)$$

Applying Lemma 1 to (10), one has

$$\alpha_i d_i = \lambda_i \geq 1 \implies \alpha_i = \frac{\lambda_i}{d_i} \geq \frac{1}{d_i} \implies \sum_{i=1}^n \alpha_i \geq \sum_{i=1}^n \frac{1}{d_i}. \quad (11)$$

The result follows from (11) and Corollary 2. $\square$

As one of the reviewers pointed out, Theorem 6 is also an immediate consequence of Theorem 4. For a given $y = (y_1, \dots, y_m) \in \mathbb{F}_q^m$, let $N(f(X, y))$ denote the number of zeros of $f(X, y)$ in $\mathbb{F}_q$ and set $h := \lceil \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} \rceil - 1$. Then $q^h | N(f(X, y))$ for all $y = (y_1, \dots, y_m) \in \mathbb{F}_q^m$ by Theorem 4. Therefore one has $q^h | N(f)$.

Now we show that Theorem 5 can be generalized and improved by Theorem 6.

Corollary 3 *Let f be a general diagonal polynomial of the form as in (8) and $\widehat{f}$ defined as in (9). Then we have*

$$\text{ord}_q N(f) \geq \left\lceil \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_r} \right\rceil - 1 \geq \left\lceil \frac{n}{\deg \widehat{f}} \right\rceil - 1. \quad (12)$$

The second inequality shows that it improves Theorem 5 and hence Ax’s theorem.

Proof Consider the polynomial

$$\widehat{f} = \sum_{i=1}^r a_i x_{i1}^{d_i}.$$

The first inequality in (12) follows from the fact that $N(f) = N(\widehat{f})$ (see Theorem 5) and Theorem 6. Since $\deg \widehat{f} = \max\{n_i d_i \mid i = 1, \dots, r\}$ and $n = \sum_{i=1}^r n_i$, one gets

$$\frac{n}{\deg \widehat{f}} \leq \frac{n_1}{n_1 d_1} + \cdots + \frac{n_r}{n_r d_r} \leq \frac{1}{d_1} + \cdots + \frac{1}{d_r}.$$

This yields the second inequality in (12). $\square$

Remark 3 One sees that the Newton polytope in Theorem 6 is of special shape. The choice of A in Theorem 2 as well as the tightness of estimate for $\omega(f)$ (or $\omega(x_0 f)$) depends upon the shape of the associated Newton polytope. It may be an interesting problem to find a set A such that the value of $|A|/\deg f_A$ is the greatest among all the choices of A .

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