Symmetric Versions of Laman’s Theorem

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Abstract Recent work has shown that if an isostatic bar-and-joint framework possesses nontrivial symmetries, then it must satisfy some very simply stated restrictions on the number of joints and bars that are “fixed” by various symmetry operations of the framework.

For the group C_3 which describes 3-fold rotational symmetry in the plane, we verify the conjecture proposed by Connelly et al. (Int. J. Solids Struct. 46:762–773, 2009) that these restrictions on the number of fixed structural components, together with the Laman conditions, are also sufficient for a framework with C_3 symmetry to be isostatic, provided that its joints are positioned as generically as possible subject to the given symmetry constraints.

In addition, we establish symmetric versions of Henneberg’s theorem and Crapo’s theorem for C_3 which provide alternate characterizations of “generically” isostatic graphs with C_3 symmetry.

As shown in (Schulze, Combinatorial and geometric rigidity with symmetry constraints, Ph.D. thesis, York University, Toronto, Canada, 2009; Schulze, Symmetrized Laman theorems for the groups C_2 and C_s , in preparation, 2009), our techniques can be extended to establish analogous results for the symmetry groups C_2 and C_s which are generated by a half-turn and a reflection in the plane, respectively.

Keywords Generic rigidity · Infinitesimal rigidity · Bar-and-joint frameworks · Symmetric frameworks · Laman graphs · Henneberg construction · 3Tree2 partition

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1 Introduction

A bar-and-joint framework is said to be *isostatic* if it is minimal infinitesimally rigid, in the sense that it is infinitesimally rigid and the removal of any bar results in an infinitesimally flexible framework (see [8, 9, 31, 33], for example).

In 1970, G. Laman provided an elegant characterization of *generically 2-isostatic* graphs, that is, graphs whose generic two-dimensional realizations as bar-and-joint frameworks are isostatic [14]. There are well-known difficulties in extending this result to higher dimensions (see [8, 9, 33], for example).

Using techniques from group representation theory, it was recently shown in [4] that if a two-dimensional isostatic bar-and-joint framework possesses non-trivial symmetries, then it must not only satisfy the Laman conditions but also some very simply stated extra conditions concerning the number of joints and bars that are fixed by various symmetry operations of the framework (see also [17–19]). In particular, these restrictions imply that a two-dimensional isostatic framework must belong to one of only six possible point groups. In the Schoenflies notation [3], these groups are denoted by C_1 , C_2 , C_3 , C_s , C_{2v} , and C_{3v} .

It was conjectured in [4] that the Laman conditions, together with the corresponding additional conditions concerning the number of fixed structural components, are not only necessary but also sufficient for a symmetric framework to be isostatic, provided that its joints are positioned as generically as possible subject to the given symmetry constraints.

In this paper, we use the definition of “generic” for symmetry groups established in [20] to verify this conjecture for the symmetry group C_3 which describes 3-fold rotational symmetry in the plane ($\mathbb{Z}_3$ as an abstract group). The result is striking in its simplicity: to test a “generic” framework with C_3 symmetry for isostaticity, we just need to check the number of joints that are “fixed” by the 3-fold rotation, as well as the standard conditions for generic rigidity without symmetry.

By defining appropriate symmetrized inductive construction techniques and appropriate symmetrized tree partitions of graphs, we also establish symmetric versions of Henneberg’s theorem [9, 13] and Crapo’s theorem [5, 9, 26] for the group C_3 . These results provide us with some alternate techniques to give a “certificate” that a graph is “generically” isostatic modulo C_3 symmetry. Furthermore, they enable us to generate all such graphs by means of an inductive construction sequence.

With each of the main results presented in this paper, we also lay the foundation to design algorithms that decide whether a given graph is generically isostatic modulo C_3 symmetry.

It is shown in [19, 21] that our techniques can be extended to establish symmetric versions of Laman’s theorem, Henneberg’s theorem, and Crapo’s theorem for the symmetry groups C_2 and C_s which are generated by a half-turn and a reflection, respectively, as well. However, it turns out that these proofs, in particular the ones for C_s , are considerably more complex than the ones for C_3 . For simplicity, we therefore restrict our attention to the group C_3 in this paper.

The Laman-type conjectures for the dihedral groups C_{2v} and C_{3v} are still open. For a discussion on the difficulties that arise in proving these conjectures (as well as a variety of related conjectures), we refer the interested reader to [19].

2 Rigidity Theoretic Definitions and Preliminaries

2.1 Graph Theory Terminology

All graphs considered in this paper are finite graphs without loops or multiple edges. The *vertex set* of a graph G is denoted by $V(G)$, and the *edge set* of G is denoted by $E(G)$. Two vertices $u \neq v$ of G are said to be *adjacent* if $\{u, v\} \in E(G)$, and *independent* otherwise. A set S of vertices of G is *independent* if every two vertices of S are independent. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices that are adjacent to v , and the elements of $N_G(v)$ are called the *neighbors* of v .

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. The simplest type of subgraph of G is that obtained by deleting a vertex or an edge from G . Let v be a vertex, and e be an edge of G . Then we write $G - \{v\}$ for the subgraph of G that has $V(G) \setminus \{v\}$ as its vertex set and whose edges are those of G that are not incident with v . Similarly, we write $G - \{e\}$ for the subgraph of G that has $V(G)$ as its vertex set and $E(G) \setminus \{e\}$ as its edge set. The deletion of a set of vertices or a set of edges from G is defined and denoted analogously.

If u and v are independent vertices of G , then we write $G + \{\{u, v\}\}$ for the graph that has $V(G)$ as its vertex set and $E(G) \cup \{\{u, v\}\}$ as its edge set. The addition of a set of edges is again defined and denoted analogously.

For a nonempty subset U of $V(G)$, the subgraph $\langle U \rangle$ of G *induced* by U is the graph having vertex set U and whose edges are those of G that are incident with two elements of U .

The *intersection* $G = G_1 \cap G_2$ of two graphs G_1 and G_2 is the graph with $V(G) = V(G_1) \cap V(G_2)$ and $E(G) = E(G_1) \cap E(G_2)$. Similarly, the *union* $G = G_1 \cup G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

An *automorphism* of a graph G is a permutation α of $V(G)$ such that $\{u, v\} \in E(G)$ if and only if $\{\alpha(u), \alpha(v)\} \in E(G)$. The automorphisms of a graph G form a group under composition, which is denoted by $\text{Aut}(G)$.

Let H be a subgraph of G and $\alpha \in \text{Aut}(G)$. We define $\alpha(H)$ to be the subgraph of G that has $\alpha(V(H))$ as its vertex set and $\alpha(E(H))$ as its edge set, where $\{u, v\} \in \alpha(E(H))$ if and only if $\alpha^{-1}(\{u, v\}) = \{\alpha^{-1}(u), \alpha^{-1}(v)\} \in E(H)$.

We say that H is *invariant under* α if $\alpha(V(H)) = V(H)$ and $\alpha(E(H)) = E(H)$, in which case we write $\alpha(H) = H$.

The graph G in Fig. 1(a), for example, has the automorphism $\alpha = (v_1 v_2 v_3) \times (v_4 v_5 v_6)$. The subgraph H_1 of G is invariant under α , but the subgraph H_2 of G is not, because $\alpha(E(H_2)) \neq E(H_2)$.

2.2 Infinitesimal Rigidity

Definition 2.1 [8, 9, 31, 33] A *framework* in $\mathbb{R}^d$ is a pair (G, p) , where G is a graph, and $p : V(G) \rightarrow \mathbb{R}^d$ is a map with the property that $p(u) \neq p(v)$ for all $\{u, v\} \in E(G)$. We also say that (G, p) is a d -dimensional *realization* of the *underlying graph* G .

An ordered pair $(v, p(v))$, where $v \in V(G)$, is a *joint* of (G, p) , and an unordered pair $\{(u, p(u)), (v, p(v))\}$ of joints, where $\{u, v\} \in E(G)$, is a *bar* of (G, p) .

Fig. 1 An invariant (b) and a noninvariant subgraph (c) of the graph G under $\alpha = (v_1 v_2 v_3) \times (v_4 v_5 v_6) \in \text{Aut}(G)$

Fig. 2 The arrows indicate the nonzero displacement vectors of an infinitesimal rigid motion (a) and infinitesimal flexes (b), (c) of frameworks in $\mathbb{R}^2$

Definition 2.2 [9, 31, 33] Let (G, p) be a framework in $\mathbb{R}^d$ with $V(G) = \{v_1, v_2, \dots, v_n\}$. An *infinitesimal motion* of (G, p) is a function $u : V(G) \rightarrow \mathbb{R}^d$ such that

$$(p(v_i) - p(v_j)) \cdot (u(v_i) - u(v_j)) = 0 \quad \text{for all } \{v_i, v_j\} \in E(G).$$

An infinitesimal motion u of (G, p) is an *infinitesimal rigid motion* if there exists a skew-symmetric matrix S (a rotation) and a vector t (a translation) such that $u(v) = Sp(v) + t$ for all $v \in V(G)$. Otherwise u is an *infinitesimal flex* of (G, p) .

(G, p) is *infinitesimally rigid* if every infinitesimal motion of (G, p) is an infinitesimal rigid motion. Otherwise (G, p) is said to be *infinitesimally flexible*.

Physically, an infinitesimal motion of (G, p) is a set of displacement vectors, one at each joint, which preserve the lengths of all bars of (G, p) at first order (see also Fig. 2).

For a framework (G, p) whose underlying graph G has a vertex set that is indexed from 1 to n , say $V(G) = \{v_1, v_2, \dots, v_n\}$, we will frequently denote $p(v_i)$ by p_i for $i = 1, 2, \dots, n$. The k th component of a vector x is denoted by $(x)_k$.

Definition 2.3 [8, 9, 31, 33] Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_n\}$, and let $p : V(G) \rightarrow \mathbb{R}^d$. The *rigidity matrix* of (G, p) is the $|E(G)| \times dn$ matrix

$$\mathbf{R}(G, p) = \begin{pmatrix} 0 & \dots & 0 & p_i - p_j & 0 & \dots & 0 & p_j - p_i & 0 & \dots & 0 \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \end{pmatrix},$$

that is, for each edge $\{v_i, v_j\} \in E(G)$, $\mathbf{R}(G, p)$ has the row with $(p_i - p_j)_1, \dots, (p_i - p_j)_d$ in the columns $d(i-1) + 1, \dots, di$, $(p_j - p_i)_1, \dots, (p_j - p_i)_d$ in the columns $d(j-1) + 1, \dots, dj$, and 0 elsewhere.

Note that if we identify an infinitesimal motion of (G, p) with a column vector in $\mathbb{R}^{dn}$ (by using the order on $V(G)$), then the kernel of the rigidity matrix $\mathbf{R}(G, p)$ is the space of all infinitesimal motions of (G, p) .

Theorem 2.1 [2, 7] A framework (G, p) in $\mathbb{R}^d$ is infinitesimally rigid if and only if either $\text{rank}(\mathbf{R}(G, p)) = d|V(G)| - \binom{d+1}{2}$ or G is a complete graph K_n and the points $p(v)$, $v \in V(G)$, are affinely independent.

Definition 2.4 [9, 31, 33] A framework (G, p) is *independent* if the row vectors of the rigidity matrix $\mathbf{R}(G, p)$ are linearly independent. A framework which is both independent and infinitesimally rigid is called *isostatic*.

An isostatic framework is minimal infinitesimally rigid, in the sense that the removal of any bar results in an infinitesimally flexible framework (see also [8, 9, 31, 33]).

Theorem 2.2 [9, 33] For a d -dimensional realization (G, p) of a graph G with $|V(G)| \geq d$, the following are equivalent:

- (i) (G, p) is isostatic
- (ii) (G, p) is infinitesimally rigid, and $|E(G)| = d|V(G)| - \binom{d+1}{2}$
- (iii) (G, p) is independent, and $|E(G)| = d|V(G)| - \binom{d+1}{2}$

2.3 Generic Rigidity

Generic rigidity is concerned with the infinitesimal rigidity of “almost all” geometric realizations of a given graph. One of the “standard” definitions of “generic” that is frequently used in rigidity theory is the following.

Definition 2.5 [8, 9] Let G be a graph with $V(G) = \{v_1, \dots, v_n\}$, and K_n be the complete graph on $V(G)$. A framework (G, p) is *generic* if the determinant of any submatrix of $\mathbf{R}(K_n, p)$ is zero only if it is (identically) zero in the variables p'_i .

There are two fundamental facts regarding this definition of generic. First, it follows immediately from Definition 2.5 that the set of all generic realizations of a given graph G in $\mathbb{R}^d$ forms a dense open subset of all possible realizations of G in $\mathbb{R}^d$ [9].

Secondly, (G, p) is infinitesimally rigid (independent, isostatic) for some map $p : V(G) \rightarrow \mathbb{R}^d$ if and only if every d -dimensional generic realization of G is infinitesimally rigid (independent, isostatic) [8, 9, 33]. Thus, for generic frameworks, infinitesimal rigidity is purely combinatorial and hence is a property of the underlying graph. This gives rise to the following definition of infinitesimal rigidity for graphs:

Definition 2.6 A graph G is *generically d -rigid* (*d -independent*, *d -isostatic*) if d -dimensional generic realizations of G are infinitesimally rigid (independent, isostatic).

In 1970, G. Laman proved the following combinatorial characterization of generically 2-isostatic graphs.

Theorem 2.3 (Laman, 1970, [14]) A graph G with $|V(G)| \geq 2$ is generically 2-isostatic if and only if

- (i) $|E(G)| = 2|V(G)| - 3$
- (ii) $|E(H)| \leq 2|V(H)| - 3$ for all $H \subseteq G$ with $|V(H)| \geq 2$

Various proofs of Laman's theorem can be found in [8, 9, 16, 26, 30], for example.

Throughout this paper, we will refer to conditions (i) and (ii) in Theorem 2.3 as the *Laman conditions*.

A combinatorial characterization of generically isostatic graphs in dimension 3 or higher is not yet known. See [8, 9, 27], for example, for a detailed discussion of this problem.

In 1911, L. Henneberg showed that generically 2-isostatic graphs can also be characterized using an inductive construction sequence.

Definition 2.7 [27, 31] Let G be a graph, $U \subseteq V(G)$ with $|U| = 2$, and $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = E(G) \cup \{\{v, u\} | u \in U\}$ is called a *vertex 2-addition* (by v) of G . (See also Fig. 3(a).)

Definition 2.8 [27, 31] Let G be a graph, $U \subseteq V(G)$ with $|U| = 3$, and $\{u_1, u_2\} \in E(G)$ for some $u_1, u_2 \in U$. Further, let $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = (E(G) \setminus \{\{u_1, u_2\}\}) \cup \{\{v, u\} | u \in U\}$ is called an *edge 2-split* (on $u_1, u_2; v$) of G . (See also Fig. 3(b).)

Fig. 3 Illustrations of a vertex 2-addition (a) and an edge 2-split (b)

Fig. 4 A proper (a) and a nonproper (b) 3Tree2 partition

Theorem 2.4 (Henneberg, 1911, [13]) *A graph is generically 2-isostatic if and only if it may be constructed from a single edge by a sequence of vertex 2-additions and edge 2-splits.*

For a proof of Henneberg’s theorem, see [9] or [27], for example.

Another way of characterizing generically 2-isostatic graphs is due to H. Crapo and uses partitions of a graph into edge disjoint trees.

Definition 2.9 [5, 15, 26] A *3Tree2 partition* of a graph G is a partition of $E(G)$ into the edge sets of three edge disjoint trees T_0, T_1, T_2 such that each vertex of G belongs to exactly two of the trees. (See also Fig. 4.)

A 3Tree2 partition is called *proper* if no nontrivial subtrees of distinct trees T_i have the same span (i.e., the same vertex sets).

Remark 2.1 If a graph G has a 3Tree2 partition, then it satisfies $|E(G)| = 2|V(G)| - 3$. This follows from the presence of exactly two trees at each vertex of G and the fact that for every tree T , we have $|E(T)| = |V(T)| - 1$. Moreover, note that a 3Tree2 partition of a graph G is proper if and only if every nontrivial subgraph H of G satisfies the count $|E(H)| \leq 2|V(H)| - 3$ [15].

Theorem 2.5 (Crapo, 1989, [5]) *A graph G is generically 2-isostatic if and only if G has a proper 3Tree2 partition.*

3 Symmetric Frameworks

A *symmetry operation* of a framework (G, p) in $\mathbb{R}^d$ is an isometry x of $\mathbb{R}^d$ such that for some $\alpha \in \text{Aut}(G)$, we have $x(p(v)) = p(\alpha(v))$ for all $v \in V(G)$ [11, 18–20].

The set of all symmetry operations of a framework (G, p) forms a group under composition, called the *point group* of (G, p) [1, 3, 11, 19, 20]. Since translating a framework does not change its rigidity properties, we may assume wlog that the point group of any framework in this paper is a *symmetry group*, i.e., a subgroup of the orthogonal group $O(\mathbb{R}^d)$ [18–20].

We use the Schoenflies notation for the symmetry operations and symmetry groups considered in this paper, as this is one of the standard notations in the literature about symmetric structures (see [1, 3, 4, 6, 10, 11, 18–20], for example). In particular, we

Fig. 5 Isostatic frameworks in $\mathbb{R}^2$ with point group C_3 : (a) a realization of the triangular prism graph G in $\mathcal{R}_{(G, C_3, \Phi)}$, where $\Phi : C_3 \rightarrow \text{Aut}(G)$ is the homomorphism defined by $\Phi(C_3) = (v_1 v_2 v_3)(v_4 v_5 v_6)$; (b) a realization of the complete bipartite graph $K_{3,3}$ in $\mathcal{R}_{(K_{3,3}, C_3, \Psi)}$, where $\Psi : C_3 \rightarrow \text{Aut}(K_{3,3})$ is the homomorphism defined by $\Psi(C_3) = (v_1 v_2 v_3)(v_4 v_5 v_6)$

denote a rotation about the origin by an angle of $\frac{2\pi}{3}$ by C_3 and a cyclic group of order three generated by a rotation C_3 by C_3 .

Given a symmetry group S in dimension d and a graph G , we let $\mathcal{R}_{(G, S)}$ denote the set of all d -dimensional realizations of G whose point group is either equal to S or contains S as a subgroup [18–20]. In other words, the set $\mathcal{R}_{(G, S)}$ consists of all realizations (G, p) of G for which there exists a map $\Phi : S \rightarrow \text{Aut}(G)$ so that

$$x(p(v)) = p(\Phi(x)(v)) \quad \text{for all } v \in V(G) \text{ and all } x \in S. \quad (1)$$

A framework $(G, p) \in \mathcal{R}_{(G, S)}$ satisfying the equations in (1) for the map $\Phi : S \rightarrow \text{Aut}(G)$ is said to be of type Φ , and the set of all realizations in $\mathcal{R}_{(G, S)}$ which are of type Φ is denoted by $\mathcal{R}_{(G, S, \Phi)}$ (see again [18–20] and Fig. 5).

Remark 3.1 Note that a set $\mathcal{R}_{(G, S)}$ can possibly be empty and that for a nonempty set $\mathcal{R}_{(G, S)}$, it is also possible that $\mathcal{R}_{(G, S, \Phi)} = \emptyset$ for some map $\Phi : S \rightarrow \text{Aut}(G)$. For examples and further details, see [19, 20].

The following symmetry-adapted notion of generic for the set $\mathcal{R}_{(G, S, \Phi)}$ was introduced in [20] (see also [19]).

Definition 3.1 Let G be a graph with $V(G) = \{v_1, \dots, v_n\}$, and let K_n be the complete graph on $V(G)$. Further, let S be a symmetry group, and Φ be a map from S to $\text{Aut}(G)$. A framework $(G, p) \in \mathcal{R}_{(G, S, \Phi)}$ is (S, Φ) -generic if the determinant of any submatrix of $\mathbf{R}(K_n, p)$ is zero only if it is zero for all p' satisfying the symmetry equations in (1).

Intuitively, an (S, Φ) -generic realization of a graph G is obtained by placing the vertices of a set of representatives for the symmetry orbits $Sv = \{\Phi(x)(v) \mid x \in S\}$ into “generic” positions. The positions for the remaining vertices of G are then uniquely determined by the symmetry constraints imposed by S and Φ (see [19, 20] for further details).

It is shown in [20] that the set of (S, Φ) -generic realizations of a graph G is an open dense subset of the set $\mathcal{R}_{(G,S,\Phi)}$. Moreover, the infinitesimal rigidity properties are the same for all (S, Φ) -generic realizations of G , as the following theorem shows.

Theorem 3.1 [19, 20] *Let G be a graph, S be a symmetry group, and Φ be a map from S to $\text{Aut}(G)$ such that $\mathcal{R}_{(G,S,\Phi)} \neq \emptyset$. The following are equivalent.*

- (i) *There exists a framework $(G, p) \in \mathcal{R}_{(G,S,\Phi)}$ that is infinitesimally rigid (independent, isostatic).*
- (ii) *Every (S, Φ) -generic realization of G is infinitesimally rigid (independent, isostatic).*

So, being infinitesimally rigid (independent, isostatic) is an (S, Φ) -generic property. This gives rise to the following:

Definition 3.2 Let G be a graph, S be a symmetry group, and Φ be a map from S to $\text{Aut}(G)$. Then G is said to be (S, Φ) -generically infinitesimally rigid (independent, isostatic) if all realizations of G which are (S, Φ) -generic are infinitesimally rigid (independent, isostatic).

Using techniques from group representation theory, it is shown in [4] that if a symmetric isostatic framework (G, p) belongs to a set $\mathcal{R}_{(G,S,\Phi)}$, where S is a nontrivial symmetry group, and $\Phi : S \rightarrow \text{Aut}(G)$ is a homomorphism, then (G, p) needs to satisfy certain restrictions on the number of joints and bars that are “fixed” by various symmetry operations of (G, p) (see also [6, 19, 20]). An alternate way of deriving these restrictions is given by Owen and Power [17].

To state the relevant result for the symmetry group C_3 in dimension 2, we need the following definition.

Definition 3.3 [18, 19] Let G be a graph, Φ be a map from C_3 to $\text{Aut}(G)$, and $(G, p) \in \mathcal{R}_{(G,C_3,\Phi)}$. Then a joint $(v, p(v))$ of (G, p) is fixed by C_3 with respect to Φ if $\Phi(C_3)(v) = v$. The number of joints of (G, p) that are fixed by C_3 with respect to Φ is denoted by $j_{\Phi(C_3)}$.

Remark 3.2 If a joint $(v, p(v))$ of a framework $(G, p) \in \mathcal{R}_{(G,C_3,\Phi)}$ is fixed by C_3 with respect to Φ , then we have $C_3(p(v)) = p(\Phi(C_3)(v)) = p(v)$. Thus, any joint of (G, p) that is fixed by C_3 with respect to Φ must lie at the center of the rotation C_3 , i.e., at the origin in $\mathbb{R}^2$.

Theorem 3.2 [4, 19] *Let G be a graph, $\Phi : C_3 \rightarrow \text{Aut}(G)$ be a homomorphism, and (G, p) be an isostatic framework in $\mathcal{R}_{(G,C_3,\Phi)}$ with the property that the points $p(v)$, $v \in V(G)$, span all of $\mathbb{R}^2$. Then G satisfies the Laman conditions, and we have $j_{\Phi(C_3)} = 0$.*

In Sect. 5, we verify the conjecture proposed in [4] that the necessary conditions in Theorem 3.2 are also sufficient for a (C_3, Φ) -generic realization of G to be isostatic.

In addition, we provide Henneberg-type and Crapo-type characterizations of $(\mathcal{C}_3, \Phi)$ -generically isostatic graphs.

The techniques used in these proofs are extended in [19, 21] to prove the corresponding Laman-type conjectures for the groups $\mathcal{C}_2$ and $\mathcal{C}_s$ generated by a half-turn and a reflection, respectively, and analogous Henneberg-type and Crapo-type results for these groups. Characterizations of $(\mathcal{C}_{2v}, \Phi)$ - or $(\mathcal{C}_{3v}, \Phi)$ -generically isostatic graphs, however, have not yet been established (see again [19, 21]).

While, initially, the fact that $\mathcal{C}_3$ allows the easiest and most natural proof for the Laman-type conjecture stated in [4] (as well as for a symmetric version of Crapo's theorem) came as somewhat of a surprise, we can now identify some clear indications for this.

For example, Crapo's theorem uses partitions of the edges of G into *three* edge-disjoint trees, so that it is most natural to extend this result to the cyclic group $\mathcal{C}_3$ of order three. Moreover, the condition $j_{\Phi(\mathcal{C}_3)} = 0$ implies that for any subgraph H of G with full $\mathcal{C}_3$ symmetry, we must have that both $|V(H)|$ and $|E(H)|$ are multiples of three, so that H cannot satisfy the count $|E(H)| = 2|V(H)| - 4$ or $|E(H)| = 2|V(H)| - 5$. As we will see, this turns out to be extremely useful in the proof of the Laman-type result for $\mathcal{C}_3$.

4 Symmetric Henneberg Moves and 3Tree2 Partitions for $\mathcal{C}_3$

We need the following inductive construction techniques to obtain a symmetrized Henneberg theorem for $\mathcal{C}_3$.

Definition 4.1 Let G be a graph, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. Let v_1, v_2 be two distinct vertices of G , and $v, w, z \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v, w, z\}$ and $E(\widehat{G}) = E(G) \cup \{\{v, v_1\}, \{v, v_2\}, \{w, \Phi(C_3)(v_1)\}, \{w, \Phi(C_3)(v_2)\}, \{z, \Phi(C_3^2)(v_1)\}, \{z, \Phi(C_3^2)(v_2)\}\}$ is called a $(\mathcal{C}_3, \Phi)$ vertex addition (by (v, w, z)) of G . (See also Fig. 6.)

Definition 4.2 Let G be a graph, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. Let v_1, v_2, v_3 be three distinct

Fig. 6 A $(\mathcal{C}_3, \Phi)$ vertex addition of a graph G , where $\Phi(C_3) = \gamma$ and $\Phi(C_3^2) = \gamma^2$

Fig. 7 A (C_3, Φ) edge split of a graph G , where $\Phi(C_3) = \gamma$ and $\Phi(C_3^2) = \gamma^2$

Fig. 8 A (C_3, Φ) Δ extension of a graph G , where $\Phi(C_3) = \gamma$ and $\Phi(C_3^2) = \gamma^2$

vertices of G such that $\{v_1, v_2\} \in E(G)$ and not both of v_1 and v_2 are fixed by $\Phi(C_3)$, and let $v, w, z \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v, w, z\}$ and $E(\widehat{G}) = (E(G) \setminus \{\{v_1, v_2\}, \Phi(C_3)(\{v_1, v_2\}), \Phi(C_3^2)(\{v_1, v_2\})\}) \cup \{\{v, v_i\} \mid i = 1, 2, 3\} \cup \{\{w, \Phi(C_3)(v_i)\} \mid i = 1, 2, 3\} \cup \{\{z, \Phi(C_3^2)(v_i)\} \mid i = 1, 2, 3\}$ is called a (C_3, Φ) edge split (on $(\{v_1, v_2\}, \Phi(C_3)(\{v_1, v_2\}), \Phi(C_3^2)(\{v_1, v_2\}))$; (v, w, z)) of G . (See also Fig. 7.)

Definition 4.3 Let G be a graph, $C_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \rightarrow \text{Aut}(G)$ be a homomorphism. Let v_0 be a vertex of G that is not fixed by $\Phi(C_3)$ and let $v, w, z \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v, w, z\}$ and $E(\widehat{G}) = E(G) \cup \{\{v, w\}, \{w, z\}, \{z, v\}, \{v, v_0\}, \{w, \Phi(C_3)(v_0)\}, \{z, \Phi(C_3^2)(v_0)\}\}$ is called a (C_3, Φ) Δ extension (by (v, w, z)) of G . (See also Fig. 8.)

Remark 4.1 Each of the constructions in Definitions 4.1, 4.2, and 4.3 has the property that if the graph G satisfies the Laman conditions, then so does $\widehat{G}$. This follows from Theorems 2.3 and 2.4 and the fact that we can obtain a (C_3, Φ) vertex addition of G by a sequence of three vertex 2-additions, a (C_3, Φ) edge split of G by a sequence of

Fig. 9 (C_3, Φ) 3Tree2 partitions of graphs, where $\Phi(C_3) = \gamma$ and $\Phi(C_3^2) = \gamma^2$

three edge 2-splits, and a (C_3, Φ) Δ extension of G by a vertex 2-addition followed by two edge 2-splits.

In order to extend Crapo's theorem to C_3 , we need the following symmetrized definition of a 3Tree2 partition.

Definition 4.4 Let G be a graph, $C_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \rightarrow \text{Aut}(G)$ be a homomorphism. A (C_3, Φ) 3Tree2 partition of G is a 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$ of G such that $\Phi(C_3)(T_i) = T_{i+1}$ for $i = 0, 1, 2$, where the indices are added modulo 3. (See also Fig. 9.)

5 The Main Result

Theorem 5.1 Let G be a graph with $|V(G)| \geq 3$, $C_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : C_3 \rightarrow \text{Aut}(G)$ be a homomorphism. The following are equivalent:

- (i) $\mathcal{R}_{(G, C_3, \Phi)} \neq \emptyset$, and G is (C_3, Φ) -generically isostatic
- (ii) $|E(G)| = 2|V(G)| - 3$, $|E(H)| \leq 2|V(H)| - 3$ for all $H \subseteq G$ with $|V(H)| \geq 2$ (Laman conditions), and $j_{\Phi(C_3)} = 0$
- (iii) there exists a (C_3, Φ) construction sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G_k, \Phi_k) = (G, \Phi)$$

such that

- (a) G_{i+1} is a (C_3, Φ_i) vertex addition, a (C_3, Φ_i) edge split, or a (C_3, Φ_i) Δ extension of G_i with $V(G_{i+1}) = V(G_i) \cup \{v_{i+1}, w_{i+1}, z_{i+1}\}$ for all $i = 0, 1, \dots, k-1$
- (b) $\Phi_0 : C_3 \rightarrow \text{Aut}(K_3)$ is a nontrivial homomorphism, and for all $i = 0, 1, \dots, k-1$, $\Phi_{i+1} : C_3 \rightarrow \text{Aut}(G_{i+1})$ is the homomorphism defined by $\Phi_{i+1}(x)|_{V(G_i)} = \Phi_i(x)$ for all $x \in C_3$ and $\Phi_{i+1}(C_3)|_{\{v_{i+1}, w_{i+1}, z_{i+1}\}} = (v_{i+1} w_{i+1} z_{i+1})$
- (iv) G has a proper (C_3, Φ) 3Tree2 partition

We break the proof of this result up into four lemmas.

Lemma 5.2 *Let G be a graph with $|V(G)| \geq 3$, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. If $\mathcal{R}_{(G, \mathcal{C}_3, \Phi)} \neq \emptyset$ and G is $(\mathcal{C}_3, \Phi)$ -generically isostatic, then G satisfies the Laman conditions, and we have $j_{\Phi(C_3)} = 0$.*

Proof The result follows immediately from Laman's theorem (Theorem 2.3) and Theorem 3.2. $\square$

Lemma 5.3 *Let G be a graph with $|V(G)| \geq 3$, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. If G satisfies the Laman conditions and we also have $j_{\Phi(C_3)} = 0$, then there exists a $(\mathcal{C}_3, \Phi)$ construction sequence for G .*

Proof We employ induction on $|V(G)|$. Note first that if for a graph G , there exists a homomorphism $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ such that $j_{\Phi(C_3)} = 0$, then $|V(G)| \equiv 0 \pmod{3}$. The only graph with three vertices that satisfies the Laman conditions is the graph K_3 , and if $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(K_3)$ is a homomorphism such that $j_{\Phi(C_3)} = 0$, then Φ is clearly a nontrivial homomorphism. This proves the base case.

So we let $n \geq 3$ and assume that the result holds for all graphs with n or fewer than n vertices.

Let G be a graph with $|V(G)| = n + 3$ that satisfies the Laman conditions and suppose that $j_{\Phi(C_3)} = 0$ for a homomorphism $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$. In the following, we denote $\Phi(C_3)$ by γ and $\Phi(C_3^2)$ by γ^2 .

Since G satisfies the Laman conditions, it is easy to verify that G has a vertex of valence 2 or 3 (see [8, 9, 19], for example).

We assume first that G has a vertex v of valence 2, say $N_G(v) = \{v_1, v_2\}$. Note that v , $\gamma(v)$, and $\gamma^2(v)$ are three distinct vertices of G , because $j_\gamma = 0$. Suppose that two of these vertices are adjacent, wlog $\{v, \gamma(v)\} \in E(G)$. Then $\{\gamma(v), \gamma^2(v)\}, \{\gamma^2(v), v\} \in E(G)$, because $\gamma \in \text{Aut}(G)$. Let $G' = G - \{v, \gamma(v), \gamma^2(v)\}$. Then

$$|E(G')| = |E(G)| - 3 = 2|V(G)| - 6 = 2|V(G')|.$$

Since $|V(G)| \geq 6$, we have $|V(G')| \geq 3$, and hence G' violates the Laman conditions, a contradiction.

Therefore, $\{v, \gamma(v), \gamma^2(v)\}$ is an independent subset of $V(G)$, which says that the six edges $\{v, v_i\}, \{\gamma(v), \gamma(v_i)\}, \{\gamma^2(v), \gamma^2(v_i)\}$, $i = 1, 2$, are all pairwise distinct. Thus,

$$|E(G')| = |E(G)| - 6 = 2|V(G)| - 9 = 2|V(G')| - 3.$$

Also, for $H \subseteq G'$ with $|V(H)| \geq 2$, we have $H \subseteq G$, and hence

$$|E(H)| \leq 2|V(H)| - 3.$$

Therefore, G' satisfies the Laman conditions.

Let $\Phi' : \mathcal{C}_3 \rightarrow \text{Aut}(G')$ be the homomorphism with $\Phi'(x) = \Phi(x)|_{V(G')}$ for all $x \in \mathcal{C}_3$. Then we have $j_{\Phi'(C_3)} = 0$, and hence, by the induction hypothesis, there

Fig. 10 If a graph G satisfies the conditions in Theorem 5.1(ii) and has a vertex v of valence 3, then G is a graph of one of the types depicted above

exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G_k, \Phi_k) = (G', \Phi')$$

satisfying the conditions in Theorem 5.1(iii). Since G is a $(\mathcal{C}_3, \Phi')$ vertex addition of G' with $V(G) = V(G') \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G', \Phi'), (G, \Phi)$$

is a sequence with the desired properties.

Suppose now that G has a vertex v of valence 3, say $N_G(v) = \{v_1, v_2, v_3\}$, and no vertex of valence 2. Note that v , $\gamma(v)$, and $\gamma^2(v)$ are again three distinct vertices of G , as are $v_i, \gamma(v_i)$ and $\gamma^2(v_i)$ for each $i = 1, 2, 3$, because $j_\gamma = 0$. We need to consider the following three cases (see also Fig. 10):

- Case 1:* $\{v, \gamma(v), \gamma^2(v)\}$ is an independent subset of $V(G)$, and all three of these vertices share a common neighbor, say wlog v_1 . Since $\gamma \in \text{Aut}(G)$, this says that each of v , $\gamma(v)$, and $\gamma^2(v)$ has the same neighbors, namely $v_1, \gamma(v_1)$, and $\gamma^2(v_1)$.
- Case 2:* $\{v, \gamma(v), \gamma^2(v)\}$ is an independent subset of $V(G)$, and there is no vertex in G that is adjacent to all three of these vertices.
- Case 3:* $\{v, \gamma(v), \gamma^2(v)\}$ is not independent in G , and hence $\{v, \gamma(v)\}, \{\gamma(v), \gamma^2(v)\}, \{\gamma^2(v), v\} \in E(G)$.

Case 1: By the Edge 2-Split theorem (see Proposition 3.3 in [27]) and Laman's theorem (Theorem 2.3), there exists a pair $\{a, b\}$ of vertices in $\{v_1, \gamma(v_1), \gamma^2(v_1)\}$ such that $G - \{v\} + \{\{a, b\}\}$ satisfies the Laman conditions. By the same argument, applied two more times, it follows that the graph $\tilde{G} = G - \{\{v, \gamma(v), \gamma^2(v)\}\} + \{\{v_1, \gamma(v_1)\}, \{\gamma(v_1), \gamma^2(v_1)\}, \{\gamma^2(v_1), v_1\}\}$ also satisfies the Laman conditions.

Further, if we define $\tilde{\Phi}$ by $\tilde{\Phi}(x) = \Phi(x)|_{V(\tilde{G})}$ for all $x \in \mathcal{C}_3$, then $\tilde{\Phi}(x) \in \text{Aut}(\tilde{G})$ for all $x \in \mathcal{C}_3$, and $\tilde{\Phi} : \mathcal{C}_3 \rightarrow \text{Aut}(\tilde{G})$ is a homomorphism. Since we clearly also have $j_{\tilde{\Phi}(\mathcal{C}_3)} = 0$, it follows from the induction hypothesis that there exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G_k, \Phi_k) = (\tilde{G}, \tilde{\Phi})$$

Fig. 11 The graphs G , G_0 , and G' in Case 2 of the proof of Lemma 5.3

satisfying the conditions in Theorem 5.1(iii). Since G is a $(\mathcal{C}_3, \tilde{\Phi})$ edge split of $\tilde{G}$ with $V(G) = V(\tilde{G}) \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (\tilde{G}, \tilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case 2: By the Edge 2-Split theorem (Proposition 3.3 in [27]) and Laman's theorem (Theorem 2.3), there exists $\{i_2, j_2\} \subseteq \{1, 2, 3\}$ such that $G_2 = G - \{\gamma^2(v)\} + \{\{\gamma^2(v_{i_2}), \gamma^2(v_{j_2})\}\}$ satisfies the Laman conditions. By the same argument, there exist $\{i_1, j_1\} \subseteq \{1, 2, 3\}$ and $\{i_0, j_0\} \subseteq \{1, 2, 3\}$ such that both $G_1 = G_2 - \{\gamma(v)\} + \{\{\gamma(v_{i_1}), \gamma(v_{j_1})\}\}$ and $G_0 = G_1 - \{v\} + \{\{v_{i_0}, v_{j_0}\}\}$ also satisfy the Laman conditions. We assume wlog that $\{i_0, j_0\} = \{1, 2\}$. Then for every subgraph H of $G_0 - \{\{v_1, v_2\}\}$ with $v_1, v_2 \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 4$. Moreover, for every subgraph H of $G' = G - \{v, \gamma(v), \gamma^2(v)\}$ with $v_1, v_2 \in V(H)$, we also have $|E(H)| \leq 2|V(H)| - 4$, because G' is obtained from $G_0 - \{\{v_1, v_2\}\}$ by deleting the edges $\{\gamma(v_{i_1}), \gamma(v_{j_1})\}$ and $\{\gamma^2(v_{i_2}), \gamma^2(v_{j_2})\}$ (see also Fig. 11).

Since G' is invariant under γ , every subgraph H of G' with $\gamma(v_1), \gamma(v_2) \in V(H)$ or $\gamma^2(v_1), \gamma^2(v_2) \in V(H)$ also satisfies $|E(H)| \leq 2|V(H)| - 4$. Note that $\{v_1, v_2\}$, $\{\gamma(v_1), \gamma(v_2)\}$, and $\{\gamma^2(v_1), \gamma^2(v_2)\}$ are three distinct pairs of vertices (though not edges, by the above counts) of G , as the following argument shows.

Suppose $\{v_1, v_2\} = \{\gamma(v_1), \gamma(v_2)\}$. Then $v_1 = \gamma(v_2)$ and $v_2 = \gamma(v_1)$, because G satisfies $j_\gamma = 0$. Therefore, $\gamma(v_1) = \gamma^2(v_2)$, and hence $v_2 = \gamma^2(v_2)$, contradicting $j_\gamma = 0$. Similarly, $\{\gamma(v_1), \gamma(v_2)\} \neq \{\gamma^2(v_1), \gamma^2(v_2)\}$ and $\{v_1, v_2\} \neq \{\gamma^2(v_1), \gamma^2(v_2)\}$.

We claim that $\tilde{G} = G' + \{\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}, \{\gamma^2(v_1), \gamma^2(v_2)\}\}$ satisfies the Laman conditions. We clearly have

$$|E(\tilde{G})| = |E(G')| + 3 = |E(G)| - 6 = 2|V(G)| - 9 = 2|V(\tilde{G})| - 3.$$

Suppose that there exists a subgraph H of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H)$ and $|E(H)| = 2|V(H)| - 4$. Then there also exists $\gamma(H) \subseteq G'$ with $\gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(\gamma(H))$ and $|E(\gamma(H))| = 2|V(\gamma(H))| - 4$, and $\gamma^2(H) \subseteq G'$ with $\gamma^2(v_1), \gamma^2(v_2), v_1, v_2 \in V(\gamma^2(H))$ and $|E(\gamma^2(H))| = 2|V(\gamma^2(H))| - 4$, because G' is invariant under γ . Let $H' = H \cup \gamma(H)$. Then

$$\begin{aligned} |E(H')| &= |E(H)| + |E(\gamma(H))| - |E(H \cap \gamma(H))| \\ &\geq 2|V(H)| - 4 + 2|V(\gamma(H))| - 4 - (2|V(H \cap \gamma(H))| - 4) \\ &= 2|V(H')| - 4, \end{aligned}$$

because $H \cap \gamma(H)$ is a subgraph of G' with $\gamma(v_1), \gamma(v_2) \in V(H \cap \gamma(H))$. Since H' is also a subgraph of G' with $\gamma(v_1), \gamma(v_2) \in V(H')$, it follows that

$$|E(H')| = 2|V(H')| - 4.$$

Similarly, it can be shown that $H'' = H' \cup \gamma^2(H)$ satisfies

$$|E(H'')| = 2|V(H'')| - 4,$$

because $H' \cap \gamma^2(H)$ is a subgraph of G' with $v_1, v_2 \in V(H' \cap \gamma^2(H))$. However, H'' is invariant under γ and satisfies $j_{\gamma|_{V(H'')}} = 0$, so that $|V(H'')| \equiv 0 \pmod{3}$ and $|E(H'')| \equiv 0 \pmod{3}$, contradicting the count $|E(H'')| = 2|V(H'')| - 4$.

Therefore, every subgraph H of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H)$, $\gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(H)$, or $\gamma^2(v_1), \gamma^2(v_2), v_1, v_2 \in V(H)$ satisfies $|E(H)| \leq 2|V(H)| - 5$.

It is now only left to show that for every subgraph H of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(H)$, we have $|E(H)| \leq 2|V(H)| - 6$. Suppose to the contrary that there exists a subgraph H of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2), \gamma^2(v_1), \gamma^2(v_2) \in V(H)$ and $|E(H)| = 2|V(H)| - 5$. Then there also exist $\gamma(H) \subseteq G'$ and $\gamma^2(H) \subseteq G'$ with the same properties, because G' is invariant under γ . Let $H' = H \cup \gamma(H)$. Then

$$\begin{aligned} |E(H')| &= |E(H)| + |E(\gamma(H))| - |E(H \cap \gamma(H))| \\ &\geq 2|V(H)| - 5 + 2|V(\gamma(H))| - 5 - (2|V(H \cap \gamma(H))| - 5) \\ &= 2|V(H')| - 5, \end{aligned}$$

because $H \cap \gamma(H)$ is a subgraph of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H \cap \gamma(H))$. Since H' is also a subgraph of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H')$, it follows that

$$|E(H')| = 2|V(H')| - 5.$$

Similarly, it can be shown that $H'' = H' \cup \gamma^2(H)$ satisfies

$$|E(H'')| = 2|V(H'')| - 5,$$

because $H' \cap \gamma^2(H)$ is a subgraph of G' with $v_1, v_2, \gamma(v_1), \gamma(v_2) \in V(H' \cap \gamma^2(H))$. However, H'' is invariant under γ , and we have $j_{\gamma|_{V(H'')}} = 0$, so that $|V(H'')| \equiv 0 \pmod{3}$ and $|E(H'')| \equiv 0 \pmod{3}$, contradicting the count $|E(H'')| = 2|V(H'')| - 5$.

Thus, $\tilde{G} = G' + \{\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}, \{\gamma^2(v_1), \gamma^2(v_2)\}\}$ indeed satisfies the Laman conditions.

Further, if we define $\tilde{\Phi}$ by $\tilde{\Phi}(x) = \Phi(x)|_{V(\tilde{G})}$ for all $x \in \mathcal{C}_3$, then $\tilde{\Phi}(x) \in \text{Aut}(\tilde{G})$ for all $x \in \mathcal{C}_3$ and $\tilde{\Phi} : \mathcal{C}_3 \rightarrow \text{Aut}(\tilde{G})$ is a homomorphism. Since we also have $j_{\tilde{\Phi}(\mathcal{C}_3)} = 0$, it follows from the induction hypothesis that there exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G_k, \Phi_k) = (\tilde{G}, \tilde{\Phi})$$

satisfying the conditions in Theorem 5.1(iii). Since G is a $(\mathcal{C}_3, \tilde{\Phi})$ edge split of $\tilde{G}$ with $V(G) = V(\tilde{G}) \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (\tilde{G}, \tilde{\Phi}), (G, \Phi)$$

is a sequence with the desired properties.

Case 3: Note that $G' = G - \{v, \gamma(v), \gamma^2(v)\}$ satisfies

$$|E(G')| = |E(G)| - 6 = 2|V(G)| - 9 = 2|V(G')| - 3.$$

Also, for $H \subseteq G'$ with $|V(H)| \geq 2$, we have $H \subseteq G$, and hence

$$|E(H)| \leq 2|V(H)| - 3,$$

so that G' satisfies the Laman conditions.

If we define Φ' by $\Phi'(x) = \Phi(x)|_{V(G')}$ for all $x \in \mathcal{C}_3$, then $\Phi'(x) \in \text{Aut}(G')$ for all $x \in \mathcal{C}_3$, and $\Phi' : \mathcal{C}_3 \rightarrow \text{Aut}(G')$ is a homomorphism. Since we also have $j_{\Phi'(C_3)} = 0$, it follows from the induction hypothesis that there exists a sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G_k, \Phi_k) = (G', \Phi')$$

satisfying the conditions in Theorem 5.1(iii). Since G is a $(\mathcal{C}_3, \Phi')$ Δ extension of G' with $V(G) = V(G') \cup \{v, \gamma(v), \gamma^2(v)\}$,

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G', \Phi'), (G, \Phi)$$

is a sequence with the desired properties. $\square$

Lemma 5.4 *Let G be a graph with $|V(G)| \geq 3$, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. If there exists a $(\mathcal{C}_3, \Phi)$ construction sequence for G , then G has a proper $(\mathcal{C}_3, \Phi)$ 3Tree2 partition.*

Proof We proceed by induction on $|V(G)|$. Let $V(K_3) = \{v_1, v_2, v_3\}$ and wlog let $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(K_3)$ be the homomorphism defined by $\Phi(C_3) = (v_1 v_2 v_3)$. Then K_3 has the proper $(\mathcal{C}_3, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$, where $T_0 = \langle \{v_1, v_2\} \rangle$, $T_1 = \langle \{v_2, v_3\} \rangle$, and $T_2 = \langle \{v_3, v_1\} \rangle$.

Assume, then, that the result holds for all graphs with n or fewer than n vertices, where $n \geq 3$.

Let G be a graph with $|V(G)| = n + 3$, and let $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism such that there exists a $(\mathcal{C}_3, \Phi)$ construction sequence

$$(K_3, \Phi_0) = (G_0, \Phi_0), (G_1, \Phi_1), \dots, (G_k, \Phi_k) = (G, \Phi)$$

satisfying the conditions in Theorem 5.1(iii). By Remark 4.1, G satisfies the Laman conditions, and hence, by Remark 2.1, any 3Tree2 partition of G must be proper. Therefore, it suffices to show that G has some $(\mathcal{C}_3, \Phi)$ 3Tree2 partition. We let $\Phi(C_3) = \gamma$ and $\Phi(C_3^2) = \gamma^2$.

Fig. 12 Construction of a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_3, \Phi_{k-1})$ vertex addition of G_{k-1}

By the induction hypothesis, G_{k-1} has a $(\mathcal{C}_3, \Phi_{k-1})$ 3Tree2 partition $\{E(T_0^{(k-1)}), E(T_1^{(k-1)}), E(T_2^{(k-1)})\}$. In the following, we compute the indices i of the trees $T_i^{(k-1)}$ modulo 3.

Suppose first that G is a $(\mathcal{C}_3, \Phi_{k-1})$ vertex addition by $(v w z)$ of G_{k-1} with $N_G(v) = \{v_1, v_2\}$, where $w = \gamma(v)$ and $z = \gamma^2(v)$. Since $\Phi_{k-1}(\mathcal{C}_3) = \gamma|_{V(G_{k-1})}$, we have $N_G(w) = \{\gamma(v_1), \gamma(v_2)\}$ and $N_G(z) = \{\gamma^2(v_1), \gamma^2(v_2)\}$. Note that both v_1 and v_2 belong to exactly two of the trees $T_i^{(k-1)}$. Therefore, there exists $l \in \{0, 1, 2\}$ such that $v_1 \in V(T_l^{(k-1)})$ and $v_2 \in V(T_{l+1}^{(k-1)})$. It follows that $\gamma(v_1) \in V(T_{l+1}^{(k-1)})$, $\gamma^2(v_1), \gamma(v_2) \in V(T_{l+2}^{(k-1)})$, and $\gamma^2(v_2) \in V(T_l^{(k-1)})$. So, if we define $T_l^{(k)}$ to be the tree with

$$\begin{aligned} V(T_l^{(k)}) &= V(T_l^{(k-1)}) \cup \{v, z\}, \\ E(T_l^{(k)}) &= E(T_l^{(k-1)}) \cup \{\{v, v_1\}, \{z, \gamma^2(v_2)\}\}, \end{aligned}$$

$T_{l+1}^{(k)}$ to be the tree with

$$\begin{aligned} V(T_{l+1}^{(k)}) &= V(T_{l+1}^{(k-1)}) \cup \{v, w\}, \\ E(T_{l+1}^{(k)}) &= E(T_{l+1}^{(k-1)}) \cup \{\{v, v_2\}, \{w, \gamma(v_1)\}\}, \end{aligned}$$

and $T_{l+2}^{(k)}$ to be the tree with

$$\begin{aligned} V(T_{l+2}^{(k)}) &= V(T_{l+2}^{(k-1)}) \cup \{w, z\}, \\ E(T_{l+2}^{(k)}) &= E(T_{l+2}^{(k-1)}) \cup \{\{w, \gamma(v_2)\}, \{z, \gamma^2(v_1)\}\}, \end{aligned}$$

then $\{E(T_0^{(k)}), E(T_1^{(k)}), E(T_2^{(k)})\}$ is a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G . (See also Fig. 12.)

Suppose next that G is a $(\mathcal{C}_3, \Phi_{k-1})$ edge split on $(\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}, \{\gamma^2(v_1), \gamma^2(v_2)\})$; (v, w, z) of G_{k-1} with $E(G_k) = (E(G_{k-1}) \setminus \{\{v_1, v_2\}, \{\gamma(v_1), \gamma(v_2)\}, \{\gamma^2(v_1), \gamma^2(v_2)\}\}) \cup \{\{v, v_i\} | i = 1, 2, 3\} \cup \{\{w, \gamma(v_i)\} | i = 1, 2, 3\} \cup$

Fig. 13 Construction of a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_3, \Phi_{k-1})$ edge split of G_{k-1}

$\{\{z, \gamma^2(v_i)\} \mid i = 1, 2, 3\}$, where $w = \gamma(v)$ and $z = \gamma^2(v)$. Wlog we may assume that $\{v_1, v_2\} \in E(T_0^{(k-1)})$. Then $\{\gamma(v_1), \gamma(v_2)\} \in E(T_1^{(k-1)})$ and $\{\gamma^2(v_1), \gamma^2(v_2)\} \in E(T_2^{(k-1)})$. Note that v_3 belongs to a tree $T_l^{(k-1)}$, where $l \neq 0$. Suppose $v_3 \in T_1^{(k-1)}$. Then $\gamma(v_3) \in T_2^{(k-1)}$ and $\gamma^2(v_3) \in T_0^{(k-1)}$. So, if we define $T_0^{(k)}$ to be the tree with

$$\begin{aligned} V(T_0^{(k)}) &= V(T_0^{(k-1)}) \cup \{v, z\}, \\ E(T_0^{(k)}) &= (E(T_0^{(k-1)}) \setminus \{v_1, v_2\}) \cup \{\{v, v_1\}, \{v, v_2\}, \{z, \gamma^2(v_3)\}\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V(T_1^{(k)}) &= V(T_1^{(k-1)}) \cup \{w, v\}, \\ E(T_1^{(k)}) &= (E(T_1^{(k-1)}) \setminus \{\gamma(v_1), \gamma(v_2)\}) \\ &\quad \cup \{\{w, \gamma(v_1)\}, \{w, \gamma(v_2)\}, \{v, v_3\}\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V(T_2^{(k)}) &= V(T_2^{(k-1)}) \cup \{z, w\}, \\ E(T_2^{(k)}) &= (E(T_2^{(k-1)}) \setminus \{\gamma^2(v_1), \gamma^2(v_2)\}) \\ &\quad \cup \{\{z, \gamma^2(v_1)\}, \{z, \gamma^2(v_2)\}, \{w, \gamma(v_3)\}\}, \end{aligned}$$

then $\{E(T_0^{(k)}), E(T_1^{(k)}), E(T_2^{(k)})\}$ is a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G . (See also Fig. 13.) If $v_3 \in T_2^{(k-1)}$, then we obtain a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G in an analogous manner.

Finally, suppose that G is a $(\mathcal{C}_3, \Phi_{k-1})$ Δ extension by $(v w z)$ of G_{k-1} with $E(G) = E(G_{k-1}) \cup \{\{v, w\}, \{w, z\}, \{z, v\}, \{v, v_0\}, \{w, \gamma(v_0)\}, \{z, \gamma^2(v_0)\}\}$, where $w = \gamma(v)$ and $z = \gamma^2(v)$. Wlog we may assume that $v_0 \in V(T_0^{(k-1)})$. Then $\gamma(v_0) \in V(T_1^{(k-1)})$ and $\gamma^2(v_0) \in V(T_2^{(k-1)})$. So, if we define $T_0^{(k)}$ to be the tree with

Fig. 14 Construction of a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G in the case where G is a $(\mathcal{C}_3, \Phi_{k-1})$ Δ extension of G_{k-1}

$$\begin{aligned} V(T_0^{(k)}) &= V(T_0^{(k-1)}) \cup \{v, w\}, \\ E(T_0^{(k)}) &= E(T_0^{(k-1)}) \cup \{\{v, v_0\}, \{v, w\}\}, \end{aligned}$$

$T_1^{(k)}$ to be the tree with

$$\begin{aligned} V(T_1^{(k)}) &= V(T_1^{(k-1)}) \cup \{w, z\}, \\ E(T_1^{(k)}) &= E(T_1^{(k-1)}) \cup \{\{w, \gamma(v_0)\}, \{w, z\}\}, \end{aligned}$$

and $T_2^{(k)}$ to be the tree with

$$\begin{aligned} V(T_2^{(k)}) &= V(T_2^{(k-1)}) \cup \{v, z\}, \\ E(T_2^{(k)}) &= E(T_2^{(k-1)}) \cup \{\{z, \gamma^2(v_0)\}, \{z, v\}\}, \end{aligned}$$

then $\{E(T_0^{(k)}), E(T_1^{(k)}), E(T_2^{(k)})\}$ is a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G . (See also Fig. 14.) $\square$

In order to show that (iv) implies (i) in Theorem 5.1, we use an approach that is in the style of Tay's proof (see [26]) of Crapo's original result. This requires the notion of a "frame," i.e., a generalized notion of a framework that allows joints to be located at the same point in space, even if their corresponding vertices are adjacent.

Definition 5.1 [26] Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. A *frame* in $\mathbb{R}^2$ is a triple (G, p, q) , where $p : V(G) \rightarrow \mathbb{R}^2$ and $q : E(G) \rightarrow \mathbb{R}^2 \setminus \{0\}$ are maps with the property that for all $\{v_i, v_j\} \in E(G)$, there exists a scalar $\lambda_{ij} \in \mathbb{R}$ (possibly zero) such that $p(v_i) - p(v_j) = \lambda_{ij}q(\{v_i, v_j\})$.

Fig. 15 The frame (G, p, q)

Definition 5.2 The *generalized rigidity matrix* of a frame (G, p, q) in $\mathbb{R}^2$ is the matrix

$$\mathbf{R}(G, p, q) = \begin{pmatrix} 0 & \dots & 0 & q(\{v_i, v_j\}) & 0 & \dots & 0 & -q(\{v_i, v_j\}) & 0 & \dots & 0 \\ \vdots & & & & & & & & & & \\ 0 & \dots & 0 & -q(\{v_i, v_j\}) & 0 & \dots & 0 & q(\{v_i, v_j\}) & 0 & \dots & 0 \\ \vdots & & & & & & & & & & \end{pmatrix},$$

i.e., for each edge $\{v_i, v_j\} \in E(G)$, $\mathbf{R}(G, p, q)$ has the row with $(q(\{v_i, v_j\}))_1$ and $(q(\{v_i, v_j\}))_2$ in the columns $2i - 1$ and $2i$, $-(q(\{v_i, v_j\}))_1$ and $-(q(\{v_i, v_j\}))_2$ in the columns $2(j - 1)$ and $2j$, and 0 elsewhere.

We say that (G, p, q) is *independent* if $\mathbf{R}(G, p, q)$ has linearly independent rows.

Remark 5.1 If (G, p, q) is a frame with the property that $p(v_i) \neq p(v_j)$ whenever $\{v_i, v_j\} \in E(G)$, then we obtain the rigidity matrix of the framework (G, p) by multiplying each row of $\mathbf{R}(G, p, q)$ by its corresponding scalar λ_{ij} . Therefore, if (G, p, q) is independent, so is (G, p) .

Lemma 5.5 Let G be a graph with $|V(G)| \geq 3$, $\mathcal{C}_3 = \{Id, C_3, C_3^2\}$ be a symmetry group in dimension 2, and $\Phi : \mathcal{C}_3 \rightarrow \text{Aut}(G)$ be a homomorphism. If G has a proper $(\mathcal{C}_3, \Phi)$ 3Tree2 partition, then $\mathcal{R}_{(G, \mathcal{C}_3, \Phi)} \neq \emptyset$, and G is $(\mathcal{C}_3, \Phi)$ -generically isostatic.

Proof Suppose that G has a proper $(\mathcal{C}_3, \Phi)$ 3Tree2 partition $\{E(T_0), E(T_1), E(T_2)\}$. By Theorem 3.1, it suffices to find some framework $(G, p) \in \mathcal{R}_{(G, \mathcal{C}_3, \Phi)}$ that is isostatic. Since G has a 3Tree2 partition, G satisfies the count $|E(G)| = 2|V(G)| - 3$ (see Remark 2.1), and hence, by Theorem 2.2, it suffices to find a map $p : V(G) \rightarrow \mathbb{R}^2$ such that $(G, p) \in \mathcal{R}_{(G, \mathcal{C}_3, \Phi)}$ is independent. In the following, we again denote $\Phi(C_3)$ by γ and $\Phi(C_3^2)$ by γ^2 .

Let $e_0 = (0, 0)$, $e_1 = (1, 0)$, and $e_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Also, for $i = 0, 1, 2$, let V_i be the set of vertices of G that are not in $V(T_i)$, and let (G, p, q) be the frame depicted in Fig. 15 with $p : V(G) \rightarrow \mathbb{R}^2$ and $q : E(G) \rightarrow \mathbb{R}^2$ defined by

$$p(v) = e_i \quad \text{if } v \in V_i,$$

$$q(b) = \begin{cases} e_2 - e_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) & \text{if } b \in E(T_0) \\ e_0 - e_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) & \text{if } b \in E(T_1) \\ e_1 - e_0 = (1, 0) & \text{if } b \in E(T_2). \end{cases}$$

We claim that the generalized rigidity matrix $\mathbf{R}(G, p, q)$ has linearly independent rows. To see this, we first rearrange the columns of $\mathbf{R}(G, p, q)$ in such a way that we obtain the matrix $\mathbf{R}'(G, p, q)$ which has the $(2i - 1)$ st column of $\mathbf{R}(G, p, q)$ in its i th column and the $(2i)$ th column of $\mathbf{R}(G, p, q)$ in its $(|V(G)| + i)$ th column for $i = 1, 2, \dots, |V(G)|$. Let F_b denote the row vector of $\mathbf{R}'(G, p, q)$ that corresponds to the edge $b \in E(G)$. We then rearrange the rows of $\mathbf{R}'(G, p, q)$ in such a way that we obtain the matrix $\mathbf{R}''(G, p, q)$ which has the vectors F_b with $b \in E(T_0)$ in the rows $1, 2, \dots, |E(T_0)|$, the vectors F_b with $b \in E(T_1)$ in the following $|E(T_1)|$ rows, and the vectors F_b with $b \in E(T_2)$ in the last $|E(T_2)|$ rows. So $\mathbf{R}''(G, p, q)$ is a matrix of the form

$$\left(\begin{array}{ccc|ccc} -\frac{1}{2} & & \frac{1}{2} & \frac{\sqrt{3}}{2} & & -\frac{\sqrt{3}}{2} \\ & & \vdots & & & \vdots \\ & -\frac{1}{2} & & \frac{\sqrt{3}}{2} & & -\frac{\sqrt{3}}{2} \\ \hline & -\frac{1}{2} & & \frac{1}{2} & & \\ & & \vdots & & & \frac{\sqrt{3}}{2} \\ & & -\frac{1}{2} & & \frac{\sqrt{3}}{2} & \\ \hline 1 & & & -1 & & \\ & & \vdots & & & \\ & 1 & & -1 & & \mathbf{0} \end{array} \right).$$

Clearly, $\mathbf{R}(G, p, q)$ has a row dependency if and only if $\mathbf{R}''(G, p, q)$ does. Suppose that $\mathbf{R}''(G, p, q)$ has a row dependency of the form

$$\sum_{b \in E(G)} \alpha_b F_b = 0,$$

where $\alpha_b \neq 0$ for some $b \in E(T_2)$. Then, since T_2 is a tree, we have

$$\sum_{b \in E(T_2)} \alpha_b F_b \neq 0.$$

Thus, there exists a vertex $v_s \in V(T_2)$, $s \in \{1, 2, \dots, |V(G)|\}$, such that

$$\sum_{b \in E(T_2)} \alpha_b (F_b)_s = C \neq 0.$$

Since $v_s \in V(T_2)$, v_s belongs to either T_0 or T_1 , say wlog $v_s \in V(T_1)$ and $v_s \notin V(T_0)$. Therefore, $(F_b)_s = 0$ and $(F_b)_{|V(G)|+s} = 0$ for all $b \in E(T_0)$, and

$$\sum_{b \in E(T_1)} \alpha_b (F_b)_s = -C.$$

This says that

$$\sum_{b \in E(T_1)} \alpha_b(F_b)_{|V(G)|+s} = \sum_{b \in E(G)} \alpha_b(F_b)_{|V(G)|+s} = -\sqrt{3}C \neq 0,$$

a contradiction. Therefore, if $\sum_{b \in E(G)} \alpha_b F_b = 0$ is a row dependency of $\mathbf{R}''(G, p, q)$, then $\alpha_b = 0$ for all $b \in E(T_2)$.

So, it is now only left to show that the matrix $\tilde{\mathbf{R}}(G, p, q)$ which is obtained from $\mathbf{R}''(G, p, q)$ by deleting those rows of $\mathbf{R}''(G, p, q)$ that correspond to the edges of T_2 has linearly independent rows. This can be done by multiplying $\tilde{\mathbf{R}}(G, p, q)$ by appropriate matrices of basis transformations and then using arguments analogous to those above. So, as claimed, the frame (G, p, q) is independent.

Now, if (G, p) is not a framework, then we need to symmetrically pull apart those joints of (G, p, q) that have the same location e_i in $\mathbb{R}^2$ and whose vertices are adjacent. So, wlog suppose that $|V_0| \geq 2$. Then we also have $|V_0| = |V_1| = |V_2| \geq 2$, because $\{E(T_0), E(T_1), E(T_2)\}$ is a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition of G . Since $\{E(T_0), E(T_1), E(T_2)\}$ is proper, one of $\langle V_0 \rangle \cap T_i$, $i = 1, 2$, say wlog $\langle V_0 \rangle \cap T_2$, is not connected, and hence $\langle V_1 \rangle \cap T_0$ and $\langle V_2 \rangle \cap T_1$ are also not connected. Let A be the set of vertices in one of the components of $\langle V_0 \rangle \cap T_2$, and let $\gamma(A)$ and $\gamma^2(A)$ be the vertex sets of the corresponding components of $\langle V_1 \rangle \cap T_0$ and $\langle V_2 \rangle \cap T_1$, respectively. For $t \in \mathbb{R}$, we define $p_t: V(G) \rightarrow \mathbb{R}^2$ and $q_t: E(G) \rightarrow \mathbb{R}^2$ by

$$p_t(v) = \begin{cases} \left(-\frac{1}{2}t, -\frac{\sqrt{3}}{2}t\right) & \text{if } v \in A \\ (1+t, 0) & \text{if } v \in \gamma(A) \\ \left(\frac{1}{2}(1-t), \frac{\sqrt{3}}{2}(1+t)\right) & \text{if } v \in \gamma^2(A) \\ p(v) & \text{otherwise,} \end{cases}$$

$$q_t(b) = \begin{cases} \left(1 + \frac{1}{2}t, \frac{\sqrt{3}}{2}t\right) & \text{if } b \in E_{A, V_1 \setminus \gamma(A)} \\ \left(1 + \frac{3}{2}t, \frac{\sqrt{3}}{2}t\right) & \text{if } b \in E_{A, \gamma(A)} \\ \left(-\frac{1}{2} - t, \frac{\sqrt{3}}{2}\right) & \text{if } b \in E_{\gamma(A), V_2 \setminus \gamma^2(A)} \\ \left(-\frac{1}{2} - \frac{3}{2}t, \frac{\sqrt{3}}{2}(1+t)\right) & \text{if } b \in E_{\gamma(A), \gamma^2(A)} \\ \left(-\frac{1}{2}(1-t), -\frac{\sqrt{3}}{2}(1+t)\right) & \text{if } b \in E_{\gamma^2(A), V_0 \setminus A} \\ \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} - \sqrt{3}t\right) & \text{if } b \in E_{\gamma^2(A), A} \\ q(b) & \text{otherwise,} \end{cases}$$

where for disjoint sets $X, Y \in V(G)$, $E_{X,Y}$ denotes the set of edges of G incident with a vertex in X and a vertex in Y . (See also Fig. 16.) Then $(G, p_t, q_t) = (G, p, q)$ if $t = 0$. Now, if we let t' be an indeterminate, then the rows of $(G, p_{t'}, q_{t'})$ are linearly dependent (over the quotient field of $\mathbb{R}[t']$) if and only if the determinants of all $|E(G)| \times |E(G)|$ submatrices of $(G, p_{t'}, q_{t'})$ are identically zero. These determinants are polynomials in t' . Thus, the set of all $t \in \mathbb{R}$ with the property that $\mathbf{R}(G, p_t, q_t)$ has a nontrivial row dependency is a variety F whose complement, if nonempty, is a dense open set. Since $t = 0$ is in the complement of F , we can conclude that

Fig. 16 The frame (G, p_t, q_t)

for almost all t , (G, p_t, q_t) is independent. Therefore, there exists a $t_0 \in \mathbb{R}$, $t_0 \neq 0$, such that the frame (G, p_{t_0}, q_{t_0}) is independent. This process can be continued until we obtain an independent frame $(G, \hat{p}, \hat{q})$ with $\hat{p}(u) \neq \hat{p}(v)$ for all $\{u, v\} \in E(G)$. Then, by Remark 5.1, $(G, \hat{p})$ is an independent framework, and the right translation of $(G, \hat{p})$ yields an independent framework in the set $\mathcal{R}_{(G, \mathcal{C}_3, \Phi)}$. $\square$

Lemmas 5.2, 5.3, 5.4, and 5.5 provide a complete proof for Theorem 5.1.

As shown in [19], there also exists a direct *geometric* proof for the fact that condition (iii) implies condition (i) in Theorem 5.1, i.e., that the existence of a $(\mathcal{C}_3, \Phi)$ construction sequence for G implies that $\mathcal{R}_{(G, \mathcal{C}_3, \Phi)} \neq \emptyset$ and that G is $(\mathcal{C}_3, \Phi)$ -generically isostatic. By generalizing the basic geometric techniques used in this proof, we can construct classes of (S, Φ) -generically isostatic graphs for a variety of symmetry groups S . These techniques also allow us to prove (or at least conjecture) characterizations of (S, Φ) -generically isostatic graphs in situations where symmetric tree partitions are too complex. Moreover, they provide significant results for (S, Φ) -generically independent graphs.

An immediate consequence of the symmetrized Laman theorems for $\mathcal{C}_3$, $\mathcal{C}_2$, and $\mathcal{C}_s$ (and the analogous conjectures for $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$) is that there is (would be) a polynomial-time algorithm to determine whether a given graph G is (S, Φ) -generically isostatic. In fact, although the Laman conditions involve an exponential number of subgraphs of G , there are several algorithms that determine whether they hold in $c|V(G)||E(G)|$ steps, where c is a constant. The pebble game [12] is an example for such an algorithm. The additional symmetry conditions for the number of fixed structural components can trivially be checked in constant time, from the graph automorphisms.

Another algorithm to determine whether G is generically 2-isostatic is given in [5]. This is an adaptation of Edmond's algorithm (for minimal partitions of a matroid into independent sets) which checks whether Crapo's 3Tree2 condition is satisfied. Since the algorithm in [5] always produces a 3Tree2 partition in which one of the trees is spanning, one has to run through the corresponding check $|V(G)| - 1$ times. In a modified algorithm that determines whether G is $(\mathcal{C}_3, \Phi)$ -generically isostatic, this can be cut down by a factor of 3, because all the trees in a $(\mathcal{C}_3, \Phi)$ 3Tree2 partition are of the same size.

6 Further Work

6.1 Pinned Frameworks and Other Framework Systems

In mechanical and structural engineering, one is often interested in the rigidity and flexibility properties of *pinned* frameworks, i.e., frameworks that have some of their joints firmly anchored (“pinned”) to the ground (see, for example, [6, 23, 24]). Using the techniques presented in [4, 6, 19] or, alternatively, the ones presented in [17], it is straightforward to show that an isostatic symmetric *pinned* framework (G, p) must again satisfy some very simply stated restrictions on the number of (unpinned) joints and bars of (G, p) that are fixed by various symmetry operations of (G, p) . While there are only six possible point groups that allow isostatic frameworks in the plane, it turns out that an isostatic *pinned* framework can be constructed for *any* point group in dimension 2.

We conjecture that the standard Laman-type conditions for a pinned graph G (see [23], for example), together with the additional necessary conditions concerning the number of fixed joints and bars, are also sufficient for pinned two-dimensional realizations of G that are as generic as possible subject to the given symmetry constraints to be isostatic.

For C_3 , C_2 , and C_s , a proof of this conjecture follows in fact directly from Theorem 5.1 (and the corresponding results in [19, 21]), since we can always “replace” the pinned vertex set by an (S, Φ) -generically isostatic (unpinned) graph on the same vertex set (using a symmetric Henneberg construction).

Note that in [17] necessary conditions for various other symmetric framework systems (such as point-line systems and other geometric constraint systems arising in CAD, for example) to be isostatic have also been established. For each of these systems, one may again ask whether the standard (nonsymmetric) necessary conditions, together with the additional symmetry conditions, are also sufficient for symmetric generic isostaticity.

6.2 Frameworks in Dimension $d > 2$

A combinatorial characterization of generically d -isostatic graphs in dimension $d > 2$ has not yet been found [8, 9, 33]. However, there are a number of inductive construction techniques which are known to preserve the d -dimensional generic rigidity properties of a graph (see [15, 31], for example).

It is shown in [19] that, unlike in dimension 2, symmetry in dimension $d > 2$ induces extra conditions for a graph G to be (S, Φ) -generically isostatic beyond those of

- (a) G being generically d -isostatic and
- (b) the symmetry conditions derived in [4] concerning the number of fixed structural components of G (and of all symmetric subgraphs H of G with the full count $|E(H)| = d|V(H)| - \binom{d+1}{2}$).

We conjecture that “flatness” caused by symmetry is the only additional concern and that it can be made into a finite set of added combinatorial conditions, for each symmetry group. See [4, 19, 22] for further details.

6.3 Body-Bar and Body-Hinge Structures

Faced with the difficulties of characterizing generically rigid graphs in dimension $d > 2$, in contrast with the well-developed theory in the plane, there has recently been a careful study of a special class of frameworks, the class of *body-bar frameworks*. These structures have a basically complete combinatorial theory which exhibits all the key theorems and algorithms of the well-understood plane frameworks (see, for example, [25, 29, 31]).

For a body-bar framework (G, p) that possesses nontrivial symmetries, joint work with S. Guest and W. Whiteley shows that in addition to the conditions in Tay's theorem (see [25]), there exist further necessary conditions for (G, p) to be isostatic [10]. These can be formulated as restrictions on the number of bars and bodies that are fixed by various symmetry operations of (G, p) . While these extra conditions are analogous to the ones derived for bar and joint frameworks, the modified context holds the promise of converting them into necessary and sufficient conditions for an *arbitrary-dimensional* body-bar realization of G to be isostatic, provided that this realization is as generic as possible subject to the given symmetry constraints. These conjectures, as well as various additional conjectures concerning combinatorial characterizations of d -dimensional symmetric body-bar frameworks, are stated in [10].

For the groups C_3 , C_2 , and C_s in dimension 2, these conjectures can readily be proven by modeling a symmetric body-bar framework as a framework (in the sense of Definition 2.1) with isostatic bar and joint bodies of required symmetry, and then applying the results of this paper.

We suspect that, analogously to the nonsymmetric situation, the results and conjectures in [10] concerning symmetric generic body-bar frameworks also translate directly to symmetric generic body-hinge frameworks [28, 31–33]. Moreover, we conjecture that a symmetric version of the Molecular Conjecture (see [28, 32], for example) holds, i.e., that under symmetric generic conditions, body-bar frameworks and molecular frameworks also possess the same rigidity properties (see [19] for details).

We note that a number of biomolecules possess rotational symmetry, including a number of virus shells which exhibit the symmetry of the rotational icosahedral group. The potential for such applications and for understanding human-built structures which are designed to have symmetry is a further motivation for giving explicit results for symmetric body-bar, body-hinge, and molecular structures.

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