The List-Chromatic Number of Infinite Graphs Defined on Euclidean Spaces

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Abstract If D is a countable set of positive reals, $2 \leq n < \omega$, let $X_n(D)$ be the graph with the points of $\mathbf{R}^n$ as vertices where two vertices are joined iff their distance is in D . We determine the list-chromatic number of $X_n(D)$ as much as possible.

Keywords List-chromatic number · Coloring number · Graphs defined by distances · Euclidean space · Independence proofs

If D is a countable set of positive numbers, then let $X_n(D)$ denote the n -dimensional D -distance graph. That is, the vertex set is $\mathbf{R}^n$, and two points are joined if their distance is in D .

In this paper we make some remarks on $\text{List}(X_n(D))$ for certain values of D and n , where for a graph X , $\text{Chr}(X)$ denotes the chromatic number, $\text{Col}(X)$ the coloring number, and $\text{List}(X)$ the list-chromatic number of X . The inequality $\text{Chr}(X) \leq \text{List}(X) \leq \text{Col}(X)$ immediately connects these notions. In general, $\text{Chr}(X)$ and $\text{Col}(X)$ can be far away; there are bipartite graphs with large coloring number. Namely, $\text{Col}(K(\kappa, \kappa)) = \kappa$ and $\text{Col}(K(\kappa, \lambda)) = \kappa^+$ when $\kappa < \lambda$ are infinite.

The results of [8] show that the list-chromatic number can be close to the chromatic number and also to the coloring number, depending on the model of set theory. We determine $\text{List}(X_n(D))$ as much as it is possible. Namely, $\text{List}(X_2(D)) = \aleph_0$ always holds, $\text{List}(X_3(D))$ is countable if D is finite or is a sequence converging to 0 and is $\aleph_1$ otherwise, $\aleph_1 \leq \text{List}(X_n(D)) \leq c$ ($4 \leq n < \omega$) and, depending on the model of set theory, can be either of the two extrema.

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History A seminal result of Erdős and Hajnal on infinite graphs states that if (V, X) is an infinite graph that does not contain $K(n, \aleph_1)$ as subgraph for some finite n , then $\text{Col}(V, X) \leq \aleph_0$ (Corollary 5.6, [4], p. 72).

The first theorem in the topic we are interested in was given by Erdős [3], who noticed that as $\mathbf{Q}$ is countable, $X_2(\mathbf{Q})$ is easily seen not to contain the bipartite graph $K(2, \aleph_1)$, and so $\text{Col}(X_2(\mathbf{Q})) \leq \aleph_0$ by the above result.

The question whether $\text{Chr}(X_n(\mathbf{Q})) \leq \aleph_0$ for every n became one of the problems Erdős liked to mention. We notice that the above Erdős–Hajnal proof worked for any countable D ; in fact, the results in this topic were formulated for specific countable sets but worked for the general case, see our reformulated versions below.

Notice also that if D is finite, then each $\text{Chr}(X_n(D))$ is finite. Finding, however, the correct value is a notoriously hard problem even for $D = \{1\}$.

In [5] we proved that $\text{Chr}(X_3(\mathbf{Q})) \leq \aleph_0$ and noticed that $\text{Col}(X_3(\mathbf{Q})) > \aleph_0$. The argument for the latter inequality actually works for any countably infinite D unless it is a sequence converging to zero, see Lemma 2 below. The full conjecture of Erdős was eventually proved in [7]. Later Schmerl gave a very short proof of a stronger statement (Theorem 4 below, for a proof, see [9]).

Jensen and Toft [6] remarked that a result of Alon [1] implies that $\text{List}(X_2(1))$ is infinite and asked whether $\text{List}(X_2(1)) = \aleph_0$. As we saw above, this had been answered, although not explicitly, much earlier. Schmerl [10] proved that $\text{List}(X_2(1)) = \aleph_0$ and even $\text{List}(X_3(1)) = \aleph_0$, and remarked that $\text{List}(X_4(1)) > \aleph_0$ and $\text{List}(X_2(\mathbf{Q})) = \aleph_0 < \text{List}(X_3(\mathbf{Q}))$.

The recent paper [2] rediscovered some of the above results, namely, $\text{List}(X_2(\text{ODD})) = \aleph_0 < \text{List}(X_3(\text{ODD}))$, where ODD denotes the set of odd natural numbers. The authors of [2] were primarily interested in the fascinating, and apparently hard, conjecture that $\text{Chr}(X_2(\text{ODD}))$ is finite.

Definitions $\mathbf{Q}$ is the set of rationals, and $\mathbf{R}$ is the set of reals. $c = |\mathbf{R}|$ is the cardinal continuum. If S is a set, κ a cardinal, then we define $[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$ and $[S]^{<\kappa} = \{X \subseteq S : |X| < \kappa\}$. A *graph* (V, X) is any pair where V is a set (the vertices) and $X \subseteq [V]^2$ (the edges). $K(\kappa, \lambda)$ denotes the complete bipartite graph with bipartition classes of cardinalities κ, λ , respectively. A *good coloring* of a graph (V, X) is any function on V such that $f(x) \neq f(y)$ whenever $\{x, y\} \in X$. The *chromatic number* $\text{Chr}(V, X)$ or $\text{Chr}(X)$ of a graph (V, X) is the least cardinal κ such that there is a good coloring $f : V \rightarrow \kappa$. If κ is a cardinal, the graph (V, X) is κ -*choosable* if for every assignment F with $|F(x)| = \kappa$, there is a choice function $f(x) \in F(x)$ which is a good coloring of (V, X) . The *list-chromatic number* $\text{List}(V, X)$ or $\text{List}(X)$ of a graph (V, X) is the least cardinal κ such that (V, X) is κ -choosable. The *coloring number* $\text{Col}(V, X)$ or $\text{Col}(X)$ of a graph (V, X) is the smallest cardinal μ such that there is a well-ordering $<$ of the vertex set V such that $|\{y < x : \{y, x\} \in X\}| < \mu$ for every $x \in V$.

Lemma 1 $\text{Chr}(V, X) \leq \text{List}(V, X) \leq \text{Col}(V, X) \leq |V|$ for every graph (V, X) .

Proof For the first inequality, assume that (V, X) is κ -choosable. Then specifically, there is a good coloring $f(x) \in F(x)$, where $F(x) = \kappa$ for every $x \in V$, and this establishes $\text{Chr}(V, X) \leq \kappa$.

For the second inequality, assume that $\text{Col}(V, X) \leq \kappa$ for some cardinal. This means that there is a well-ordering $<$ of the vertex set V such that $|\{y < x : \{y, x\} \in X\}| < \kappa$ for every $x \in V$. If F is a function with $|F(x)| = \kappa$ for every $x \in V$, then we select a good coloring $f(x) \in F(x)$ by transfinite recursion on $<$: let $f(x)$ be an arbitrary element of the set $F(x) - \{f(y) : y < x, \{y, x\} \in X\}$. The latter set is nonempty as $|F(x)| = \kappa$, and we subtract a smaller set. It is clear that f is, indeed, a good coloring.

For proving the last inequality, it suffices to consider any well-ordering into the order type $|V|$. Recall that in axiomatic set theory, $|V|$, the cardinality of V , is identified with the smallest ordinal which is equicardinal to V , so a well-ordering into order type $|V|$ is one where each element is preceded by less than $|V|$ elements. $\square$

Lemma 2 ([2, 5, 10]) *If D is infinite and is not a sequence converging to 0, then for $n \geq 3$, $X_n(D)$ contains a $K(\aleph_0, c)$. If $D \neq \emptyset$, then for $n \geq 4$, $X_n(D)$ contains a $K(c, c)$.*

Proof In the first statement we can assume that $n = 3$. Choose distinct elements $d_0, d_1, \dots \in D$ such that $d_i > k$ ($i < \omega$) for some $k > 0$. Set $A = \{p_0, p_1, \dots\}$ where $p_i = (0, 0, \sqrt{d_i^2 - k^2})$, and let the elements of B be the points of the form $(x, y, 0)$ with $x^2 + y^2 = k^2$, i.e., B is a full circle. The distance of p_i from any element of B is $\sqrt{d_i^2 - k^2 + k^2} = d_i$.

For the second statement, we can assume that $D = \{d\}$ and $n = 4$. Then A and B can be chosen as orthogonal circles:

$$A = \left\{ (x, y, 0, 0) : x^2 + y^2 = \frac{d^2}{2} \right\}$$

and

$$B = \left\{ (0, 0, z, u) : z^2 + u^2 = \frac{d^2}{2} \right\}.$$

Obviously, the distance between a point of A and a point of B is d . $\square$

Lemma 3 *If κ is an infinite cardinal, then $\text{List}(K(\kappa, 2^\kappa)) > \kappa$.*

Proof Let (V, X) be the complete bipartite graph with bipartition classes A and B such that $|A| = \kappa$ and $|B| = 2^\kappa$. We are going to give an assignment F with $|F(x)| = \kappa$ for $x \in V$ such that no choice function $f \in \prod \{F(x) : x \in V\}$ is a good coloring of (V, X) . First, let $\{F(x) : x \in A\}$ be an arbitrary system of pairwise disjoint κ -size sets. Then, for every choice function $g \in \prod \{F(x) : x \in A\}$, select a $y_g \in B$ and make $F(y_g) = \{g(x) : x \in A\}$. As the sets $\{F(x) : x \in A\}$ are disjoint, each $F(y_g)$ is of cardinality κ . The number of choice functions is $|\prod \{F(x) : x \in A\}| = \kappa^\kappa = 2^\kappa = |B|$; therefore, it is possible to choose a different y_g for each choice function g .

Assume now that some $f \in \prod \{F(x) : x \in V\}$ is a good coloring of (V, X) . Set $g = f|A$. Then $f(y_g) \in F(y_g) = \{f(x) : x \in A\}$, and so $f(y_g) = f(x)$ for some $x \in A$, which shows that f is not a good coloring, as x and y_g are joined. $\square$

Theorem 4 (J. Schmerl) *If D is any countable set of distances, then there is a well-ordering $<$ of $\mathbf{R}^n$ such that $\inf\{d(y, x) : y < x, d(y, x) \in D\} > 0$ for every $x \in \mathbf{R}^n$.*

Corollary 5 (Komjáth [7]) $\text{Chr}(X_n(D)) \leq \aleph_0$ for every countable D and any $n < \omega$.

Proof Set $\delta(x) = \inf\{d(y, x) : y < x, d(y, x) \in D\}$. We color $\mathbf{R}^n$ as follows. For $x \in \mathbf{R}^n$, set $f(x) = (q, k)$, where $q \in \mathbf{Q}^n$ is a point with rational coordinates, and k is a natural number such that $d(q, x) < \frac{1}{2k}$ and $\delta(x) > \frac{1}{k}$. This is a coloring with countably many colors. In order to show the required property of the coloring, assume that $f(x) = f(y) = (q, k)$. Without loss of generality, $y < x$, but then, as $d(y, x) \leq d(y, q) + d(q, x) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} < \delta(x)$, we get that $d(y, x) \notin D$, as desired. $\square$

Theorem 6 (Schmerl [10]) *If D is infinite and not a sequence converging to zero, then $\text{List}(X_3(D)) \geq \aleph_1$.*

Proof By Lemma 2, $X_3(D)$ contains a $K(\aleph_0, c)$ whose list-chromatic number is at least $\aleph_1$, by Lemma 3. $\square$

Theorem 7 *If D is finite or a sequence converging to 0, then $\text{Col}(X_3(D)) \leq \aleph_0$. If D is an arbitrary countable set, then $\text{Col}(X_3(D)) \leq \aleph_1$.*

Proof We give a unified proof of the two statements. We start with the following claim.

Claim *For any infinite $Y \subseteq \mathbf{R}^3$, there is a unique smallest Z with $Y \subseteq Z \subseteq \mathbf{R}^3$, $|Z| = |Y|$, such that*

- (a) *If $x_0, x_1, x_2 \in Z$ are not collinear and $y \in \mathbf{R}^3$ has $d(y, x_i) \in D$ ($i < 3$), then $y \in Z$.*
- (b) *If $y_0, y_1 \in Z$, z is on the connecting line of y_0, y_1 , and $d(z, y_0) \in D$, then $z \in Z$.*

Proof Given x_0, x_1, x_2 as in (a), let $f(x_0, x_1, x_2)$ be the set of possible y , and given y_0, y_1 as in (b), let $g(y_0, y_1)$ be the set of possible z . Notice that $f(x_0, x_1, x_2)$ and $g(y_0, y_1)$ are always countable.

Set $Z = Z_0 \cup Z_1 \cup \dots$, where $Z_0 = Y$, and

$$Z_{n+1} = Z_n \cup \bigcup \{f(x_0, x_1, x_2) : x_0, x_1, x_2 \in Z_n\} \cup \bigcup \{g(y_0, y_1) : y_0, y_1 \in Z_n\}.$$

Then Z is as required. $\square$

Call the above set the *closure* of Y and denote it by $\text{cl}(Y)$. We prove by transfinite induction on $\kappa = |Y|$ that every subset $Y \subseteq \mathbf{R}^3$ has a well-ordering as specified in the theorem. This is obvious if κ is finite or $\aleph_0$. Assume that $\kappa > \aleph_0$ and we have the result for sets of smaller cardinality. If $|Y| = \kappa$, then we can decompose Y as $Y = \bigcup \{Y_\alpha : \alpha < \kappa\}$ into an increasing union of subsets with $Y_0 = \emptyset$, $Y_\alpha = \text{cl}(Y_\alpha)$, $|Y_\alpha| < \kappa$ for $\alpha < \kappa$, and $Y_\alpha = \bigcup \{Y_\beta : \beta < \alpha\}$ whenever $\alpha < \kappa$ is a limit ordinal. For example,

if $Y = \{y_\alpha : \alpha < \kappa\}$ is an enumeration of Y , we can have $Y_\alpha = \text{cl}(\{y_\beta : \beta < \alpha\})$ for $\alpha < \kappa$. As $|Y_{\alpha+1} - Y_\alpha| \leq |Y_{\alpha+1}| < \kappa$, we can apply the inductive hypothesis and obtain a well-ordering $<_\alpha$ of $Y_{\alpha+1} - Y_\alpha$ witnessing the (appropriate) statement of the theorem, i.e., the number of edges going down from any point is finite/countable. We define $<$ on Y as follows. Set $x < y$ if either $x, y \in Y_{\alpha+1} - Y_\alpha$ for some α and $x <_\alpha y$ or else $x \in Y_{\alpha+1} - Y_\alpha$, $y \in Y_{\beta+1} - Y_\beta$, and $\alpha < \beta$. Clearly, $<$ is a well-ordering of Y . To show the required property we have to prove that if $x \in Y_{\alpha+1} - Y_\alpha$, then the number of edges from x into Y_α is finite/countable. Let $Z \subseteq Y_\alpha$ be the set of the other endpoints, i.e., $z \in Z$ if $z \in Y_\alpha$ and $d(z, x) \in D$. Z cannot contain three noncollinear points, as that would force $x \in Y_\alpha$, a contradiction. The elements of Z therefore are all collinear, that is, $Z = Y_\alpha \cap L$ for some line L . Further, $x \notin L$, as otherwise we would get $x \in Y_\alpha$, a contradiction.

Now $Z = \{z \in L : d(x, z) \in D\}$ is countable, and it is finite if D is a zero-sequence as $x \notin L$, so it is in positive distance from L . $\square$

Theorem 8 *It is consistent that $c = \aleph_2$ and $\text{List}(X_n(D)) \leq \aleph_1$ for every $n < \omega$ and countable D .*

Proof We refer to the model of [8], where the following combinatorial principle holds.

(*) $c = \aleph_2$, and if $\{A_\alpha : \alpha < \omega_2\} \subseteq [\omega_2]^{\aleph_1}$, then there is a coloring $g : \omega_2 \rightarrow \omega$ such that g assumes every value on every A_α .

Assume that $V = \mathbf{R}^n$ and $X = X_n(D)$. We already know from Corollary 5 that $\text{Chr}(V, X) \leq \aleph_0$. It suffices therefore to deduce the following from (*): if (V, X) is a countably chromatic graph of cardinality at most $c = \aleph_2$, then $\text{List}(V, X) \leq \aleph_1$. For this, let $F(x)$ ($x \in V$) be an assignment with sets of cardinality $\aleph_1$. Without loss of generality, $V = \omega_2$ and $F(x) \in [\omega_2]^{\aleph_1}$. Let $g : \omega_2 \rightarrow \omega$ be a coloring as in (*) for the system $\{F(\alpha) : \alpha < \omega_2\}$, and let $\omega_2 = V_0 \cup V_1 \cup \dots$ be a good coloring of (ω_2, X) , i.e., all edges go between different V_i . Now, if $\alpha \in V_i$, then let $f(\alpha)$ be an element of $F(\alpha)$ such that $g(f(\alpha)) = i$. This is possible, as g assumes every value in $F(\alpha)$. f is a good coloring: if $\{\alpha, \beta\} \in X$, then $\alpha \in V_i$ and $\beta \in V_j$ for some $i \neq j$, thus $g(f(\alpha)) = i \neq j = g(f(\beta))$, and therefore $f(\alpha) \neq f(\beta)$. $\square$

Notice that similar arguments give that it is consistent that $c = \aleph_3$, or any greater cardinal, and the claim that $\text{List}(X_n(D)) \leq \aleph_1$ for every $n < \omega$ and countable D still holds.

Theorem 9 *If $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, then $\text{List}(X_n(D)) = c$ for every $D \neq \emptyset$ and $n \geq 4$.*

Proof By Lemma 1, the list chromatic number of $X_n(D)$ is at most c . By Lemma 2, $K(\aleph_1, \aleph_2)$ embeds into $X_n(D)$. By Lemma 3, the list chromatic number of $K(\aleph_1, \aleph_2)$ is greater than $\aleph_1$. $\square$

Of course, the same argument gives that if $2^\kappa = \kappa^+ = c$ for some cardinal κ or if c is weakly inaccessible and $2^\mu = c$ for every cardinal $\mu < c$, then $\text{List}(X_n(D)) = c$ when $D \neq \emptyset$, $n \geq 4$, i.e., c can be arbitrarily large, and $\text{List}(X_n(D)) = c$ may still hold.

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