Even- vs. Odd-dimensional Charney–Davis Conjecture

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Abstract More than once we have heard that the Charney–Davis Conjecture makes sense only for odd-dimensional spheres. This is to point out that in fact it is also a statement about even-dimensional spheres.

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A conjecture of Heinz Hopf asserts that the sign of the Euler characteristic of a smooth Riemannian $2d$ -dimensional manifold of nonpositive sectional curvature is the same for all such manifolds, that is, the same as that of product of nonpositively curved surfaces:

$$(-1)^d \chi(M^{2d}) \geq 0.$$

For Riemannian manifolds, the condition of nonpositive sectional curvature is equivalent to being locally CAT(0). The Hopf Conjecture subsequently has been general-

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ized to include closed, piecewise Euclidean, locally CAT(0) (generalized homology) manifolds.

By work of Davis [2], Coxeter groups provide a rich source of piecewise Euclidean, locally CAT(0) spaces. Given a *flag* triangulation of a (generalized homology) sphere L^{n-1} , a construction of Davis gives a reflection (generalized homology) orbifold $\mathcal{O}^n$, with many (generalized homology) manifold covers.

Recall that the *f*-polynomial f_L of a simplicial complex L is defined by the formula

$$f_L(t) := \sum_{\sigma \in L} t^{\#\sigma},$$

and the *h*-polynomial is given by a substitution of the variable

$$(1+t)^n h_L\left(\frac{1}{1+t}\right) = t^n f_L\left(\frac{1}{t}\right),$$

where $(n-1)$ is the dimension of L . The Euler characteristic of $\mathcal{O}$ is given by the *f*- and *h*-polynomials of L as follows:

$$\chi(\mathcal{O}) = f_L(-1/2) = 2^{-n} h_L(-1).$$

Charney and Davis emphasized the combinatorial implications of the Hopf Conjecture in this context.

Conjecture [1, Conjecture D, p. 135] *If L is a flag triangulation of the (generalized homology) sphere of dimension $2d-1$, then $(-1)^d h(-1) \geq 0$.*

The Hopf Conjecture does not say anything about odd-dimensional manifolds. So, on the face of it, the Charney–Davis Conjecture should not say anything about even-dimensional (generalized homology) spheres. The point we want to make in this note is that in fact it does.

Theorem *The Charney–Davis Conjecture is equivalent to the following statement. Let L be a generalized homology sphere of dimension $2d$. Let $h_L(t)$ be its *h*-polynomial, and let $\tilde{h}_L(t)$ be defined by $h_L(t) = (1+t)\tilde{h}_L(t)$. Then*

$$(-1)^d \tilde{h}_L(-1) \geq 0.$$

Proof Let L be a suspension of $\underline{L}$. Since the *h*-polynomial is multiplicative for joins $(1+t)\tilde{h}_L(t) = h_L(t) = (1+t)h_{\underline{L}}(t)$. The statement $(-1)^d h_L(-1) = (-1)^d \tilde{h}_L(-1) \geq 0$ is just the Charney–Davis Conjecture for $\underline{L}$.

To prove the other implication recall that $\tilde{h}$ -, *h*-, and *f*-polynomials are related by the formula

$$(2+t)(1+t)^{2d-1} \tilde{h}_L\left(\frac{1}{1+t}\right) = (1+t)^{2d} h_L\left(\frac{1}{1+t}\right) = t^{2d} f_L\left(\frac{1}{t}\right).$$

Differentiating both sides, we get

$$\begin{aligned} (2+t) \left[(1+t)^{2d-1} \tilde{h}_L \left(\frac{1}{1+t} \right) \right]' + (1+t)^{2d-1} \tilde{h}_L \left(\frac{1}{1+t} \right) \\ = 2d t^{2d-1} f_L \left(\frac{1}{t} \right) - t^{2d-2} f' \left(\frac{1}{t} \right), \end{aligned}$$

where $[(1+t)^{2d-1} \tilde{h}_L(1/(1+t))]'$ is a polynomial. Substitute $t = -2$ and use the fact that, by the Dehn–Sommerville relations, $f_L(-1/2) = 0$ to get

$$(-1)^{2d-1} \tilde{h}_L(-1) = -(-2)^{2d-2} f'(-1/2).$$

To finish the proof, we observe that the sum of f -polynomials of links of vertices of L is equal to the derivative of the f -polynomial of L . We omit the proof of this straightforward claim.

Applying the above claim to the preceding equality gives

$$(-1)^d \tilde{h}_L(-1) = 4^{d-1} \sum_v (-1)^d h_{L_{k_v}}(-1).$$

The right-hand side is a sum of nonnegative terms by the Charney–Davis Conjecture. Hence the proof. $\square$

Remark The quantity $(-1)^d \tilde{h}_L(-1)$ is equal to $\gamma_d(L)$, the top coefficient of the γ -polynomial introduced in [3, Definition 2.1.4]. The calculation proving that $\gamma_d(L) > 0$, provided that the Charney–Davis Conjecture holds for links of all vertices in L , was mentioned in [3, Corollary 2.2.2] without relating it to the h -polynomial of L .

In view of the above, the Charney–Davis Conjecture for even-dimensional spheres is essentially included in (though perhaps a dramatic restatement of) Conjecture 2.1.7 in [3], which treats equally even- and odd-dimensional spheres and provides further strengthenings of the Charney–Davis Conjecture.

One may speculate about the geometric interpretation of $\tilde{h}_L(-1)$. Note that presumably $\tilde{h}_L$ is an h -polynomial of a $(2d-1)$ -dimensional sphere. For example, if L is an icosahedron, $\tilde{h}_L$ is an h -polynomial of the decagon. On the other hand, the geometry of the Davis orbifolds for the icosahedron and a suspension of the decagon are very different. The former is hyperbolic, and the latter is a product. Thus there is no hope of relating $\tilde{h}_L(-1)$ to ℓ^2 -torsion as the ℓ^2 -torsion of a product vanishes, while it is nonzero for hyperbolic one.

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