A Jordan–Brouwer Separation Theorem for Polyhedral Pseudomanifolds

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Abstract The Jordan Curve Theorem referring to a simple closed curve in the plane has a particularly simple proof in the case that the curve is polygonal, called the “raindrop proof”. We generalize the notion of a simple closed polygon to that of a polyhedral $(d - 1)$ -pseudomanifold ($d \geq 2$) and prove a Jordan–Brouwer Separation Theorem for such a manifold embedded in $\mathbb{R}^d$. As a by-product, we get bounds on the polygonal diameter of the interior and exterior of such a manifold which are almost tight. This puts the result within the frame of computational geometry.

Keywords Bing’s house · Dual graph · Euler’s formula · Jordan–Brouwer theorem · Jordan’s curve theorem · Jordan exterior (interior) · Polygonal diameter · Polyhedral manifold · Triangulation

1 Introduction

Jordan’s Curve Theorem asserts that if γ is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus \gamma$ is the disjoint union of two non-empty connected open sets (“domains” in the old terminology), and that γ is the *common boundary* of these two domains. Moreover, the bounded component of $\mathbb{R}^2 \setminus \gamma$ is simply connected (this follows from the Jordan–Schönflies Theorem, see [32] or [23, p. 68]), and a one-point compactification of $\mathbb{R}^2$

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transforming it into an S^2 results in a one-point extension of the unbounded component of $\mathbb{R}^2 \setminus \gamma$, which again is simply connected, see [23, p. 71]. The separation property of γ assured by the Jordan Curve Theorem does generalize to $(d - 1)$ -spheres in $\mathbb{R}^d$, $d \geq 3$ (or in S^d) (Jordan–Brouwer Separation Theorem, see [31, p. 198, Theorem 15]), but for $d \geq 3$ the two domains separated by S^{d-1} embedded in $\mathbb{R}^d$ (or S^d) are not necessarily simply connected (“Alexander’s horned sphere”, first described in [2]). For various proofs of Jordan’s basic (and deep) Curve Theorem, we refer to [8, 20, 32], [23, p. 31 ff], [3, Vol. I, pp. 39–64], [19, pp. 285 ff], and the survey [11]. In the case when γ is a simple closed polygon (in the plane), this theorem specializes to the Jordan theorem for simple closed polygons which has a particularly simple proof, known as the “raindrop proof”. (See [10, pp. 267–269], [15, pp. 281–285], [7, pp. 27–29], or [23, pp. 16–18]. An interesting historical article is [14].) The purpose of this paper is to extend the theorem and its “raindrop proof” (for the case of a simple closed polygon) to d -dimensional space $\mathbb{R}^d$. In Definition 1.3 below, we shall define a polyhedral $(d - 1)$ -pseudomanifold (abbreviated $(d - 1)$ -PPM) in $\mathbb{R}^d$, and we shall prove, for these pseudomanifolds, the following

Theorem 1.1 (Jordan–Brouwer Separation Theorem for polyhedral $(d - 1)$ -pseudomanifolds) *If $\mathcal{C}$ is a $(d - 1)$ -PPM in $\mathbb{R}^d$ ($d \geq 1$) and $|\mathcal{C}|$ denotes the body of $\mathcal{C}$ (= the union of all cells of $\mathcal{C}$), then $\mathbb{R}^d \setminus |\mathcal{C}|$ is the disjoint union of two non-empty connected open sets (= “domains”), one bounded and one unbounded, and $|\mathcal{C}|$ is the common boundary of these two components.*

For related results in $\mathbb{R}^3$ see [9] and [17], and for the Jordan–Brouwer Theorem in $\mathbb{R}^d$, $d \geq 2$, see [12].

Definition 1.1 The bounded component of $\mathbb{R}^d \setminus |\mathcal{C}|$ is the *interior*, denoted by $\text{int } \mathcal{C}$ of $\mathcal{C}$, and the unbounded component, denoted by $\text{ext } \mathcal{C}$, is the *exterior* of $\mathcal{C}$.

Historical notes Theorem 1.1 is a special case of the Jordan–Brouwer Separation Theorem for $(d - 1)$ -pseudomanifolds in $\mathbb{R}^d$ formulated in the mid 1940s, perhaps earlier, and proved by homology methods (see below). The main novelty of Theorem 1.1 over the general Jordan–Brouwer Theorem is its pure polyhedral formulation and its elementary “raindrop” proof, which makes it accessible to every geometer who does not feel at home with the machinery of algebraic topology. The relative easiness and transparency of this proof over the complications that arise in the general case may be well illustrated by the planar case, where the “raindrop” proof is much easier than the proof of the Jordan theorem for general curves. Moreover, this raindrop proof serves to compute the maximal polygonal diameter of the interior (resp., exterior) of a $(d - 1)$ -PPM, which puts it in the frame of computational geometry not accessible by the methods of algebraic topology. The question of the maximal polygonal diameter of $\text{int } \mathcal{C}$ (resp., $\text{ext } \mathcal{C}$) for a general $(d - 1)$ -pseudomanifold $\mathcal{C}$ in $\mathbb{R}^d$ is, in fact, meaningless (it is unbounded even for C^∞ -smooth manifolds).

For the sake of historical accuracy, we give now one of the earliest formulations of the Jordan–Brouwer Separation Theorem for $(d - 1)$ -pseudomanifolds which we could find in the literature (quote):

The Jordan–Brouwer Theorem “Every $(d - 1)$ -dimensional closed pseudomanifold in S^d is orientable (and) separates S^d into precisely two domains (“domain” is the old terminology for “open connected set”) and is the common boundary of these two domains” (see [3, Vol. III, p. 54, Theorem 3.44]). (The proof uses homology theory.)

In fact, our “polyhedral $(d - 1)$ -pseudomanifold” (= $(d - 1)$ -PPM) is essentially the piecewise linear (PL) structure underlying Aleksandrov’s “ d -dimensional pseudomanifold” defined in *ibid*, Vol. II, p. 72, Definition 3.11. Aleksandrov’s “strongly connected d -complex” there is essentially the condition on the connectedness of the dual graph (our strong connectivity condition (D) below; see *ibid*, Vol. I, p. 200, Definition 5.24), and what makes it there into a “pseudomanifold” is the condition that every subfacet is common to *exactly* two facets, i.e., our no branching condition (C) below. (In *ibid*, p. 72, Definition 3.12 “exactly” is replaced by “either one or two” (hence, *not* zero), a generalization which makes the pseudomanifold into a “pseudomanifold with boundary”). Since Aleksandrov’s book [3] is relatively old, we tried to find out whether the Jordan–Brouwer Separation Theorem for $(d - 1)$ -pseudomanifolds appears in more recent books, and we could not find anything. For example, [31, p. 198, Theorem 15] has simply a $(d - 1)$ -sphere embedded in S^d , whereas [24, p. 446, Corollary 74.2] has a “compact connected triangulable $(d - 1)$ -manifold” (without singularities) embedded in S^d , and on p. 425, Corollary 71.2, it has a homeomorph A of S^{d-1} embedded in S^d such that “ (S^d, A) is triangulable”, which is a certain condition needed for the variant of the homology theory developed there.

The reader may justifiably ask now (as indeed one of our referees did) whether this “raindrop proof” is not merely a specialization of the homology proof to our piecewise linear setting. The answer is definitely “no”. The “raindrop proof” does not mimic any part of the homology approach. It uses a sort of height function, thus resembling a Morse’s theory approach rather than a homology approach, where the $(d - 1)$ -manifold is gradually sliced by parallel hyperplanes so that the singularities encountered in this process are scrutinized. Such a process was used already in [1] to show that a 2-manifold of genus 1 in $\mathbb{R}^3$ separates $\mathbb{R}^3$ into two components one of which is homeomorphic to the interior of a 2-torus (“tubular region” in Alexander’s terminology) as conjectured by H. Tietze. Alexander’s proof is, however, very sketchy.

Definition 1.2 A polyhedral cell complex $\mathcal{C}$ in $\mathbb{R}^d$ is a finite non-empty collection of convex polytopes (= cells) in $\mathbb{R}^d$ that satisfies the following two conditions:

- (A) If $P \in \mathcal{C}$ and if F is a *face* of P (i.e., the intersection of P with a supporting hyperplane), then $F \in \mathcal{C}$ as well.
- (B) If $P, Q \in \mathcal{C}$, then the intersection $P \cap Q$ is a face of both P and Q .

If $P \in \mathcal{C}$ has dimension k , then we refer to P as a k -cell of $\mathcal{C}$. The *dimension* of $\mathcal{C}$ is defined as $\max\{\dim P : P \in \mathcal{C}\}$, and $|\mathcal{C}|$, the *body* of $\mathcal{C}$, is defined as the union of all cells of $\mathcal{C}$. A cell $Q \in \mathcal{C}$ is *maximal* if Q is not included in any other cell of $\mathcal{C}$. The cell complex $\mathcal{C}$ is *homogeneously k -dimensional* if all maximal cells of $\mathcal{C}$ are k -dimensional. The *dual graph* of a homogeneously k -dimensional complex $\mathcal{C}$ is defined as follows: its vertices are the k -cells of $\mathcal{C}$. Two distinct vertices are joined by an edge if the corresponding k -cells share a $(k - 1)$ -face.

Fig. 1 A 2-PPM embedded in $\mathbb{R}^3$

Definition 1.3 A $(d - 1)$ -dimensional polyhedral pseudomanifold (abbr. $(d - 1)$ -PPM) in $\mathbb{R}^d$ is a homogeneously $(d - 1)$ -dimensional polyhedral complex in $\mathbb{R}^d$ that satisfies the following conditions:

- (C) (No branching) Every $(d - 2)$ -cell is included in exactly two $(d - 1)$ -cells.
- (D) (Strong connectivity) The dual graph is connected.

If $\mathcal{C}$ is a $(d - 1)$ -PPM in $\mathbb{R}^d$, then the $(d - 1)$ -cells of $\mathcal{C}$ are the *facets* and the $(d - 2)$ -cells are the *subfacets* of $\mathcal{C}$. We denote by $\text{vert } \mathcal{C}$ the set of vertices of $\mathcal{C}$, by $\text{skel}_k(\mathcal{C})$ the k -skeleton of $\mathcal{C}$ ($0 \leq k \leq d$), and by $f_k(\mathcal{C})$ the number of k -cells of $\mathcal{C}$.

Remark 1.1

- (1) Notice that we do not assume any regularity conditions on the cells of dimension $d - 3$ or less. Thus, for $d \geq 3$ the body $|\mathcal{C}|$ of a $(d - 1)$ -PPM $\mathcal{C}$ may be homeomorphic to a pseudomanifold, such as, e.g., a “degenerate torus” consisting of a $(d - 1)$ -sphere with two diametrical points identified. An easy realization of this $(d - 1)$ -pseudomanifold as the body $|\mathcal{C}|$ of a $(d - 1)$ -PPM is the following: Let P be a d -prism with (parallel) bases Q, Q' and barycenter c . The complex $\mathcal{C}$ is obtained from the boundary complex $\mathcal{B}(P)$ of P by removing the facets Q, Q' and adding $\{c\}$ and all pyramids $\text{conv}(\{c\} \cup F)$ (with apex c and base F), where F ranges over all proper faces of Q and of Q' .
- (2) For $d = 3$, the body of a 2-PPM $\mathcal{C}$ (even without singularities) may be homeomorphic to an arbitrary closed orientable 2-manifold, i.e., a 2-sphere with a finite number of handles. Another example of a 2-PPM is the trefoil knot (= 3-leaf clover) in the logo of the Mathematics Faculty of the Chemnitz University of Technology; see Fig. 1. In [6], it is proved that the Heawood map of seven countries on the torus is not realizable as a 2-PPM in $\mathbb{R}^3$, and in [29] it is proved that the dual $(3, 7)_8$ of the Klein map $(7, 3)_8$ is realizable as a 2-PPM in $\mathbb{R}^3$. The authors note that the Klein map $(7, 3)_8$ itself is not realizable in such a way; see l.c., p. 541, lines 27–31. In fact, as follows from [13, Exercise 13.2.3 on p. 253], no simple 2-map of genus $g \geq 1$ (such as the Klein map $\{7, 3\}_8$) is realizable in any $\mathbb{R}^d$ (when we require the faces to be convex).
- (3) Theorem 1.1 trivially holds for $d = 1$. For $d = 2$, a $(d - 1)$ -PPM is just a simple closed polygon, and the theorem reduces to Jordan’s theorem for simple closed polygons in the plane.
- (4) For $d \geq 3$, a polyhedral $(d - 1)$ -complex $\mathcal{C}$ in $\mathbb{R}^d$ whose dual graph is connected (i.e., $\mathcal{C}$ satisfies condition (D) above) and in which every $(d - 1)$ -cell is included in *at least* two $(d - 1)$ -cells (instead of *exactly* two $(d - 1)$ -cells as in condition

Fig. 2 Simplified Bing's house

(C) above), the complement $\mathbb{R}^d \setminus \mathcal{C}$ may be connected, as the following example shows.

Example 1.1 (Simplified Bing's house with two rooms) Begin with the boundary of a 3-cube, to which we add in the middle a horizontal (2-dimensional square) barrier (Fig. 2(a)). So far this is a polyhedral 2-dimensional complex which separates $\mathbb{R}^3$ into three components (the exterior and two floors inside).

Next make a square hole near the “north-west” corner of the upper ceiling, just below it we make a parallel hole in the middle ceiling, and between these two holes form a “chimney” in the upper floor by adding four vertical panels as shown in Fig. 2(b).

A person standing on the roof may fall now into the lower storey through the chimney. Repeat the construction symmetrically near the opposite “south-east” corner in the lower floor. The polyhedral 2-complex $\mathcal{C}$ obtained (see Fig. 2(c)) has a connected dual graph, every edge of $\mathcal{C}$ is contained in at least two facets, and $\mathcal{C}$ does not separate $\mathbb{R}^3$, i.e., $\mathbb{R}^3 \setminus |\mathcal{C}|$ is connected. This example easily generalizes to $d \geq 3$.

- (5) It is well known that if γ is a simple closed polygon in $\mathbb{R}^2$, then $\gamma \cup \text{int } \gamma$ can be triangulated without adding vertices (see, e.g., [15], pp. 286–287). This does not generalize to $(d-1)$ -PPM's in $\mathbb{R}^d$ for $d \geq 3$, i.e., there is a 2-PPM $\mathcal{C}$ in $\mathbb{R}^3$ such that $\mathcal{C} \cup \text{int } \mathcal{C}$ cannot be tetrahedrized without additional vertices. Two such 2-PPM's in $\mathbb{R}^3$ are presented in Sect. 5 below. But other results based on the piecewise Jordan separation theorem may be generalized using Theorem 1.1, e.g., the characterization of those geometric graphs which are 1-skeletons of unstacked triangulations of simple closed polygons given in [18].

For the sake of the proof of Theorem 1.1, we split our main theorem into two statements, namely: *Let $\mathcal{C}$ be a $(d-1)$ -PPM in $\mathbb{R}^d$. Then*

- (E) $\mathbb{R}^d \setminus |\mathcal{C}|$ is the disjoint union of two open sets, $\text{int } \mathcal{C}$ and $\text{ext } \mathcal{C}$. The boundary of each of these sets is $|\mathcal{C}|$; $\text{int } \mathcal{C}$ is bounded, and $\text{ext } \mathcal{C}$ is unbounded.
 (F) The sets $\text{int } \mathcal{C}$ and $\text{ext } \mathcal{C}$ are (polygonally) connected.

We shall prove (E) in Sect. 2 by constructing a continuous function $f: \mathbb{R}^d \setminus |\mathcal{C}| \rightarrow \{0, 1\}$ which attains both values 0 and 1 in every neighborhood of every points $x \in |\mathcal{C}|$, and defining $\text{ext } \mathcal{C} = f^{-1}(0)$, $\text{int } \mathcal{C} = f^{-1}(1)$. The (polygonal) connectedness of $\text{int } \mathcal{C}$ and $\text{ext } \mathcal{C}$ is proved in Sect. 3 (see in particular Theorem 3.2).

Notation $B^k(c, r)$ denotes the closed ball of dimension k centered at c and having radius r .

2 A “Raindrop” Proof of (E)

The construction of f will be performed in four steps.

Step I: Choosing a “generic” direction.

Let $F_1, \dots, F_t$ be all hyperplanes ($= (d-1)$ -flats) spanned by subsets of $\text{vert } \mathcal{C}$. For $i = 1, \dots, t$, let $F_i^0 =_{\text{def}} F_i - F_i$ be the linear subspace parallel to F_i . Choose a unit vector $v \in \mathbb{R}^d \setminus \bigcup_{i=1}^t F_i^0$ (“ v ” for “vertical”). Vector v is our direction “up”, and $-v$ is pointing “down”.

By the choice of v , a flat F spanned by vertices of $\mathcal{C}$, other than $\mathbb{R}^d$, will meet a line parallel to v in at most one point. For each point $p \in \mathbb{R}^d \setminus |\mathcal{C}|$, denote by $R(p)$ the closed vertical “pointing down” half-line $R(p) =_{\text{def}} \{p - \lambda v : 0 \leq \lambda < \infty\}$. $R(p)$ is the path of a “raindrop” emanating from p . We divide $\mathbb{R}^d \setminus |\mathcal{C}|$ into three disjoint sets (recall that a subfacet means a $(d-2)$ -cell):

$$\begin{aligned} S_0 &= \{p \in \mathbb{R}^d \setminus |\mathcal{C}| : R(p) \text{ does not meet any subfacet of } \mathcal{C}\}, \\ S_1 &= \{p \in \mathbb{R}^d \setminus |\mathcal{C}| : R(p) \text{ meets exactly one subfacet of } \mathcal{C}\}, \\ S_2 &= \{p \in \mathbb{R}^d \setminus |\mathcal{C}| : R(p) \text{ meets more than one subfacet of } \mathcal{C}\}. \end{aligned}$$

We shall first define f on S_0 ($=$ Step II), then extend its definition (continuously) to S_1 ($=$ Step III), and finally extend it (continuously) to S_2 ($=$ Step IV).

In the sequel, we shall use the following notation: For a set $A \subset \mathbb{R}^d$, $A^+ =_{\text{def}} \{a + \lambda v : a \in A, \lambda \geq 0\}$. A^+ is the set of points of $\mathbb{R}^d$ that lie “above” A . Note that (for all $p \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$):

$$R(p) \text{ meets } A \text{ iff } p \in A^+. \quad (1)$$

Step II: Define f on S_0 .

For $p \in S_0$, denote by $r(p)$ the number of facets ($= (d-1)$ -cells of $\mathcal{C}$) that meet $R(p)$, and define $f(p) =_{\text{def}} \frac{1}{2}(1 - (-1)^{r(p)})$ to be the parity of $r(p)$ ($f(p) = 0$ if $r(p)$ is even, 1 if $r(p)$ is odd).

Next we show that S_0 is a dense open subset of $\mathbb{R}^d$, and that $f : S_0 \rightarrow \{0, 1\}$ is a continuous, hence locally constant, function. In view of (1), we can write $S_0 = \mathbb{R}^d \setminus (|\mathcal{C}| \cup |\text{skel}_{d-2}\mathcal{C}|^+)$, where $|\text{skel}_{d-2}\mathcal{C}|$ denotes the union of all subfacets of $\mathcal{C}$. The set $|\text{skel}_{d-2}\mathcal{C}|$ is compact, same as $\mathcal{C}$, and therefore $|\text{skel}_{d-2}\mathcal{C}|^+$ is closed. Thus S_0 is an open subset of $\mathbb{R}^d$. Moreover, the set $|\mathcal{C}| \cup |\text{skel}_{d-2}\mathcal{C}|^+$ can be covered by a finite number of hyperplanes in $\mathbb{R}^d$. It follows that S_0 is dense in $\mathbb{R}^d$.

Now for the continuity of f . Assume $x \in S_0$. Let ε be the (positive) distance from x to $|\mathcal{C}| \cup |\text{skel}_{d-2}\mathcal{C}|^+ (= \mathbb{R}^d \setminus S_0)$. If $x' \in \mathbb{R}^d$, $\|x - x'\| < \varepsilon$, then the segment $[x, x']$ does not meet $|\mathcal{C}| \cup |\text{skel}_{d-2}\mathcal{C}|^+$. Let F be any facet of $\mathcal{C}$. The set F^+ is a closed, convex, unbounded and full-dimensional polyhedral subset of $\mathbb{R}^d$ whose boundary consists of the base F and of the side facets G^+ , where G ranges over all $(d-2)$ -faces of F . Thus, $\text{bd } F^+ \subset |\mathcal{C}| \cup |\text{skel}_{d-2}\mathcal{C}|^+$, and therefore the segment $[x, x']$ does not meet the boundary of F^+ . It follows that $x' \in F^+$ iff $x \in F^+$, i.e., $R(x)$ meets F iff $R(x')$ meets F . This is true for all facets F of $\mathcal{C}$. Therefore, $r(x) = r(x')$,

hence $f(x') = f(x)$. Thus we have shown that the function $f : S_0 \rightarrow \{0, 1\}$ is locally constant, hence continuous.

Step III: Extend f continuously from S_0 to $S_0 \cup S_1$.

Suppose $p \in S_1$. Let G be the unique subfacet of $\mathcal{C}$ that meets $R(p)$ (or, equivalently, such that $p \in G^+$). $R(p)$ meets G in a unique point $g \in \text{relint } G$. (The uniqueness of g follows from our choice of the vertical direction. If g were not a relative interior point of G , then g would belong to a proper face J of G , which, in turn, would be included in another subfacet G' , contradicting our assumption that $R(p)$ meets only one subfacet.) Hence $p \in \text{relint } G^+$.

Let F_1, F_2 be the facets of $\mathcal{C}$ that include G . Define $H =_{\text{def}} \text{aff } G + \mathbb{R}v$. H is the vertical hyperplane through G . Denote by H^-, H^+ the two closed half-spaces of $\mathbb{R}^d$ bounded by H . The two facets F_1, F_2 may be in the same half-space, or in different half-spaces. Choose the notation in such a way that either $(\alpha) F_1 \subset H^-, F_2 \subset H^+$ or $(\beta) F_1 \cup F_2 \subset H^+$.

Now consider the closed set $\Delta =_{\text{def}} |\mathcal{C}| \cup |\text{skel}_{d-2}(\mathcal{C}) \setminus \text{relint } G|^+$. (Here $|\text{skel}_{d-2}(\mathcal{C}) \setminus \text{relint } G|$ is the union of all subfacets of $\mathcal{C}$, except $\text{relint } G$.) This set includes the boundary of F^+ , for every facet F of $\mathcal{C}$, except for F_1^+ and F_2^+ . It also includes the boundaries of F_1^+ and F_2^+ , except for $\text{relint}(G^+)$. In particular, it includes the relative boundary of G^+ . But it does not contain the point p . Put $\varepsilon =_{\text{def}} \text{dist}(p, \Delta)$, and define $U =_{\text{def}} \{x \in \mathbb{R}^d : \|x - p\| < \varepsilon\} = \text{int } B^d(p, \varepsilon)$. Note that if $x \in U$, then the closed interval $[p, x]$ does not meet Δ . Now make the following observations.

- (I) If F is any facet of $\mathcal{C}$, other than F_1 and F_2 , then the interval $[p, x]$ does not meet the boundary of F^+ , and therefore p and x are either both in F^+ , or both not in F^+ .
- (II) If, say, $F_1 \subset H^-$ and $x \in \text{int } H^-$ then, moving along the interval $[p, x]$ from p to x , we start at a point $p \in \text{relint } G^+ \subset \text{bd } F_1^+$, move into $\text{int } F_1^+$, and do not hit the boundary of F_1^+ again. Therefore, $x \in \text{int } F_1^+$. The same holds with H^- replaced by H^+ , and/or F_1 replaced by F_2 . It follows in case (α) : if $x \in U \setminus H$, then x belongs to exactly one of the sets F_1^+, F_2^+ . And it follows in case (β) : if $x \in U \cap \text{int } H^-$, then x belongs to none of the sets F_1^+, F_2^+ ; if $x \in U \cap H^+$, then x belongs to both.
- (III) If G' is any subfacet of $\mathcal{C}$ other than G , then $G'^+ \subset \Delta$, and therefore $x \notin G'^+$, for any $x \in U$.
- (IV) If $x \in U \setminus H$, then clearly $x \notin G^+$. If $x \in U \cap H$, then the interval $[p, x]$ lies in $\text{aff } G^+ (= H)$, contains a point $p \in \text{relint } G^+$, and does not meet the relative boundary of G^+ (which is part of Δ); therefore, $x \in \text{relint } G^+$.

From these observations we conclude:

- (A) $U \setminus H \subset S_0$, and f is constant on $U \setminus H$.
- (B) $U \cap H \subset S_1$.

Now define $f(p)$ to be the constant value that f takes on $U \setminus H$. Clearly, if we apply the same procedure to any point $q \in U \cap H$, we will end up with a value $f(q)$ equal to the value $f(p)$ just defined. Thus we have extended f to a continuous function $f : S_0 \cup S_1 \rightarrow \{0, 1\}$.

Step IV: Extend f continuously from $S_0 \cup S_1$ to $S_0 \cup S_1 \cup S_2 (= \mathbb{R}^d \setminus |\mathcal{C}|)$.

Assume $p \in S_2$. Then the ray $R(p)$ meets at least two subfacets of $\mathcal{C}$. (This can happen only for $d \geq 3$, by our choice of v .) Assume that $R(p)$ meets exactly t distinct subfacets $G_1, \dots, G_t$ of $\mathcal{C}$, $t \geq 2$. Denote by $\mathcal{G}$ the subcomplex of $\mathcal{C}$ that consists of all cells that *do not* meet $R(p)$. Then $p \notin |\mathcal{C}| \cup |\mathcal{G}|^+$. Define $\varepsilon =_{\text{def}} \text{dist}(p, |\mathcal{C}| \cup |\mathcal{G}|^+)$, and let $U =_{\text{def}} B^d(p, \varepsilon) \subset \mathbb{R}^d$ (the open ball of radius ε centered at p). By our choice of ε , we have $S_2 \cap U = \bigcup_{i=1}^t \bigcup_{j=i+1}^t (G_i^+ \cap G_j^+ \cap U)$. In Lemma 2.1 below, we will show that each of the sets $G_i^+ \cap G_j^+$ ($1 \leq i < j \leq t$) is included in a $(d-2)$ -flat. This will imply (see Lemma 2.2 below) that the union of these sets does not separate U . The integer valued function f , defined and continuous on the connected set $(S_0 \cup S_1) \cap U (= U \setminus \bigcup_{i=1}^t \bigcup_{j=i+1}^t (G_i^+ \cap G_j^+))$, takes a constant value c ($c = 0$ or $c = 1$) there. We define $f(p)$ to be c as well. This rule, applied to any other point $q \in S_2 \cap U$, will assign to q the same value $f(q) = c$. Thus we have extended f to a continuous (hence locally constant) function from $S_0 \cup S_1 \cup S_2 (= \mathbb{R}^d \setminus |\mathcal{C}|)$ to $\{0, 1\}$.

To complete the proof of statement (E), we define, as indicated above, the sets $\text{ext}\mathcal{C} =_{\text{def}} f^{-1}(0)$ and $\text{int}\mathcal{C} =_{\text{def}} f^{-1}(1)$. These are clearly two disjoint open sets in $\mathbb{R}^d$, whose union is $\text{dom } f = \mathbb{R}^d \setminus |\mathcal{C}|$. Note that $\mathbb{R}^d \setminus \text{conv } |\mathcal{C}| \subset \text{ext}\mathcal{C}$, and therefore $\text{int}\mathcal{C} \subset \text{conv } |\mathcal{C}|$. Thus $\text{ext}\mathcal{C}$ is unbounded, and $\text{int}\mathcal{C}$ is bounded.

We still have to show that every point of $|\mathcal{C}|$ is a common boundary point of $\text{int}\mathcal{C}$ and $\text{ext}\mathcal{C}$ (and therefore $\text{int}\mathcal{C} \neq \emptyset, \text{ext}\mathcal{C} \neq \emptyset$). Since the boundaries of $\text{int}\mathcal{C}$ and of $\text{ext}\mathcal{C}$ are closed sets, it suffices to show that the common boundary points of $\text{int}\mathcal{C}$ and $\text{ext}\mathcal{C}$ are dense in $|\mathcal{C}|$. If G is a subfacet ($= (d-2)$ -face) of $\mathcal{C}$, then the set $G + \mathbb{R}v$ is included in a vertical hyperplane, and therefore intersects a facet F of $\mathcal{C}$ in a convex set of dimension $\leq d-2$. Thus $F \setminus \bigcup \{G + \mathbb{R}v : G \in \text{skel}_{d-2}\mathcal{C}\}$ is dense in F , and $|\mathcal{C}| \setminus (\mathbb{R}v + |\text{skel}_{d-2}\mathcal{C}|)$ is dense in $|\mathcal{C}|$. If $x \in |\mathcal{C}| \setminus (\mathbb{R}v + |\text{skel}_{d-2}\mathcal{C}|)$, then x belongs to the relative interior of some facet F of $\mathcal{C}$. If $\varepsilon > 0$ is sufficiently small, then the points $x + \varepsilon v, x - \varepsilon v$ are both in S_0 , the ray $R(x + \varepsilon v)$ meets F , in addition to all facets met by $R(x - \varepsilon v)$. Thus $r(x + \varepsilon v) = 1 + r(x - \varepsilon v)$, and $f(x + \varepsilon v) \neq f(x - \varepsilon v)$, i.e., $\{f(x - \varepsilon v), f(x + \varepsilon v)\} = \{0, 1\}$. Thus x is a common boundary point of $\text{int}\mathcal{C}$ and $\text{ext}\mathcal{C}$.

Lemma 2.1 *For $1 \leq i < j \leq t$ we have $\dim(G_i^+ \cap G_j^+) \leq d-2$.*

Proof Case I. If $G_i^+ \cap G_j^+ = (G_i \cap G_j)^+$, then

$$\dim(G_i^+ \cap G_j^+) = \dim(G_i \cap G_j)^+ = 1 + \dim(G_i \cap G_j) \leq 1 + (d-3) = d-2.$$

(The intersection of two distinct $(d-2)$ -cells of $\mathcal{C}$ is a cell of $\mathcal{C}$, of dimension $\leq d-3$.)

Case II. $G_i^+ \cap G_j^+ \supsetneq (G_i \cap G_j)^+$.

This means that we can find a point q that lies above a point $q' \in G_i$, and above a point $q'' \in G_j$, $q' \neq q''$. Thus $q' = q - \lambda'v$, $q'' = q - \lambda''v$, $\lambda' > 0$, $\lambda'' > 0$, $\lambda' \neq \lambda''$. The sets $(\text{aff } G_i)^+$ and $(\text{aff } G_j)^+$ are both $(d-1)$ -flats. If they are distinct, then $G_i^+ \cap G_j^+$ is included in their intersection, which is a $(d-2)$ -flat. There remains the possibility that $(\text{aff } G_i)^+ = (\text{aff } G_j)^+ =_{\text{def}} H$. If this happens, then the set

$W =_{\text{def}} \text{vert } G_i \cup \text{vert } G_j$ is included in the $(d-1)$ -flat H , and the difference set of $\text{aff } W$ contains the vector $q' - q'' = (\lambda' - \lambda'')v$, contrary to our choice of v . $\square$

Lemma 2.2 *Let $U \subset \mathbb{R}^d$ be a convex set with non-empty interior ($\dim U = d$). For $i = 1, \dots, t$, let J_i be a $(d-2)$ -flat in $\mathbb{R}^d$, and let S_i be a subset of J_i . Then the set $T =_{\text{def}} U \setminus \bigcup_{i=1}^t S_i$ is polygonally connected. In fact, T is an L_2 -set (every two points of T can be joined by a polygonal path in T with at most two edges).*

Proof Assume $a, b \in T$. If $[a, b] \subset T$, we are done. Otherwise define, for $i = 1, \dots, t$, $A_i =_{\text{def}} \text{aff}(J_i \cup \{a\})$, $B_i =_{\text{def}} \text{aff}(J_i \cup \{b\})$. A_i and B_i are flats of dimension $d-1$ or $d-2$.

Since $\dim U = d$, we can find a point $c \in U \setminus \bigcup_{i=1}^t (A_i \cup B_i)$. In fact, we can find such a point c arbitrarily close to $\frac{1}{2}(a+b)$. If the line $\text{aff}(a, c)$ meets J_i at some point x other than a , then $c \in \text{aff}(a, x) \subset \text{aff}(\{a\} \cup J_i) = A_i$, contrary to our choice of c . Therefore, $[a, c] \cap J_i = \emptyset$ for $i = 1, \dots, t$, hence $[a, c] \subset U \setminus \bigcup_{i=1}^t J_i \subset U \setminus \bigcup_{i=1}^t S_i = T$ (so $[a, c] \subset T$). The same argument, with a and A_i replaced by b and B_i , shows that $[c, b] \subset T$. $\square$

Remark 2.1 Lemma 2.2 can be extended beyond the realm of convex sets, as follows:

If U is an open connected set in $\mathbb{R}^d$, and $S_1, \dots, S_t$ are subsets of U with $\dim \text{aff } S_i \leq d-2$ for $i = 1, \dots, t$, then the set $T = U \setminus \bigcup_{i=1}^t S_i$ is polygonally connected. If two points $a, b \in T$ can be joined by a polygonal path with n edges within S , then they can be joined by a polygonal path with at most $\max(2, n)$ edges within T . The proof is similar.

3 Polygonal Connectedness of $\text{int } \mathcal{C}$ and of $\text{ext } \mathcal{C}$

In this section we show that the open sets $\text{int } \mathcal{C}$ and $\text{ext } \mathcal{C}$ are (polygonally) connected. For any two points $a, b \in \text{int } \mathcal{C}$, we construct a polygonal path from a to b that lies entirely in $\text{int } \mathcal{C}$, stays mostly close to $|\mathcal{C}|$ and far from $|\text{skel}_{d-3} \mathcal{C}|$. We do similarly for $\text{ext } \mathcal{C}$. We shall also bound from above the number of edges of the constructed paths. The bound will be approximately $\frac{n}{d}$, where $n =_{\text{def}} f_{d-1}(\mathcal{C})$ is the number of facets of $\mathcal{C}$.

3.1 Some Preliminary Lemmata

Let the facets of $\mathcal{C}$ be $F_1, \dots, F_n$ ($n = f_{d-1}(\mathcal{C})$). For $i = 1, \dots, n$, let u_i be a unit vector perpendicular to $\text{aff } F_i$. Choose the direction of u_i in such a way that for each point $b \in \text{relint } F_i$ and for all sufficiently small positive values of ε , $b + \varepsilon u_i \in \text{ext } \mathcal{C}$ and $b - \varepsilon u_i \in \text{int } \mathcal{C}$. If a subfacet G of $\mathcal{C}$ is the intersection of F_i and F_j ($1 \leq i < j \leq n$), define $u_{ij} =_{\text{def}} u_i + u_j$.

Lemma 3.1 *If $g \in \text{relint } G$ and ε is a sufficiently small positive number, then $g + \varepsilon u_{ij} \in \text{ext } \mathcal{C}$, $g - \varepsilon u_{ij} \in \text{int } \mathcal{C}$.*

Fig. 3 Proof of Lemma 3.1

Proof The set $\text{aff } G$ is a $(d - 2)$ -flat. Denote by A the $(d - 2)$ -dimensional linear subspace of $\mathbb{R}^d$ parallel to $\text{aff } G$. ($A = \text{aff } G - \text{aff } G$ and $\text{aff } G = g + A$.) The facets F_i and F_j lie in two half-hyperplanes H_i and H_j bounded by $\text{aff } G$, say, $H_i = \text{cone}_g(F_i) = \text{aff } G + \mathbb{R}^+v_i$, $H_j = \text{cone}_g(F_j) = \text{aff } G + \mathbb{R}^+v_j$, where v_i and v_j are suitable unit vectors in the orthogonal complement $A^\perp$ of A . If ε is a sufficiently small positive number ($0 < \varepsilon < \text{dist}(g, |\mathcal{C} \setminus \{F_i, F_j, G\}|)$), then $B^d(g, \varepsilon) \setminus |\mathcal{C}| = B^d(g, \varepsilon) \setminus (H_i \cup H_j)$. The union $H_i \cup H_j$ divides $B^d(g, \varepsilon)$ into two open wedges, $B^d(g, \varepsilon) \cap \text{int } \mathcal{C}$ and $B^d(g, \varepsilon) \cap \text{ext } \mathcal{C}$. If H_i and H_j lie in the same hyperplane ($v_j = -v_i$), then each of these two wedges is an open half-ball. In this case, $u_i = u_j$ (Fig. 3(a)), $u_{ij} = 2u_i = 2u_j$, and the lemma holds trivially. If u_i and u_j lie in different hyperplanes, then one of the wedges is larger than a half-ball, and the other is smaller.

In all cases we have

$$\langle u_i, v_j \rangle = \langle u_j, v_i \rangle = \sin \alpha, \quad (2)$$

where α is the dihedral angle of the wedge $B^d(g, \varepsilon) \cap \text{ext } \mathcal{C}$ at g . If $\langle u_i, v_j \rangle < 0$, then $B(g, \varepsilon) \cap \text{ext } \mathcal{C}$ is the larger wedge (Fig. 3(b)), and if $\langle u_i, v_j \rangle > 0$, then $B(g, \varepsilon) \cap \text{int } \mathcal{C}$ is the larger wedge (Fig. 3(c)). Adding the equations $u_i = \langle u_i, u_j \rangle u_j + \langle u_i, v_j \rangle v_j$ and $u_j = \langle u_j, u_i \rangle u_i + \langle u_j, v_i \rangle v_i$ and using (2), we find that $(1 - \langle u_i, u_j \rangle)(u_i + u_j) = \sin \alpha(v_i + v_j)$.

If $u_i \neq u_j$, then $1 - \langle u_i, u_j \rangle > 0$, and therefore $u_{ij} = u_i + u_j = \frac{\sin \alpha}{1 - \langle u_i, u_j \rangle} \times (v_i + v_j)$. Thus u_{ij} is a positive (resp., negative) multiple of $v_i + v_j$ when $\sin \alpha > 0$ (resp., $\sin \alpha < 0$). In both cases, u_{ij} points towards $\text{ext } \mathcal{C}$, and $-u_{ij}$ towards $\text{int } \mathcal{C}$. $\square$

Lemma 3.2 (“Push away from $|\mathcal{C}|$ ”)

- Suppose F_i is a facet of $\mathcal{C}$, $G = F_i \cap F_j$ is a subfacet, $b \in \text{relint } F_i$, $g \in \text{relint } G$, and u is a vector that satisfies $\langle u, u_i \rangle > 0$. Define $I_0 =_{\text{def}} [b, g]$, $I_\varepsilon =_{\text{def}} [b + \varepsilon u, g + \varepsilon u_{ij}]$ (u_i, u_j and $u_{ij} = u_i + u_j$ denote the same vectors as in the preceding lemma). If ε is a sufficiently small positive number, then $I_\varepsilon \subset \text{ext } \mathcal{C}$, $I_{-\varepsilon} \subset \text{int } \mathcal{C}$. (The required smallness of ε may depend on the choice of the points b, g and of the vector u .)
- Suppose F_i, F_j, F_k are distinct facets of $\mathcal{C}$, $G = F_i \cap F_j$, $H = F_j \cap F_k$ are subfacets ($G \neq H$), $g \in \text{relint } G$, $h \in \text{relint } H$. Define $J_0 =_{\text{def}} [g, h]$, $J_\varepsilon =_{\text{def}} [g + \varepsilon u_{ij}, h + \varepsilon u_{jk}]$. If $\varepsilon > 0$ is sufficiently small, then $J_\varepsilon \subset \text{ext } \mathcal{C}$ and $J_{-\varepsilon} \subset \text{int } \mathcal{C}$.

Proof (a) First note that I_0 does not meet any facet of $\mathcal{C}$ except F_i and F_j . The same holds for I_ε , provided $|\varepsilon| < \min(\frac{1}{2}, \frac{1}{\|u\|}) \cdot \text{dist}(I_0, |\mathcal{C} \setminus \{F_i, F_j, G\}|)$. By Lemma 3.1

above, $g + \varepsilon u_{ij} \in \text{ext}\mathcal{C}$ and $g - \varepsilon u_{ij} \in \text{int}\mathcal{C}$, provided ε is positive and sufficiently small. To complete the proof, it suffices to show that $I_\varepsilon \cap F_i = \emptyset$ and $I_\varepsilon \cap F_j = \emptyset$ (for sufficiently small $|\varepsilon|$, $\varepsilon \neq 0$).

As for F_i : $\langle u_i, u \rangle > 0$ (given) and $\langle u_i, u_{ij} \rangle = 1 + \langle u_i, u_j \rangle > 0$. Therefore, for any $\varepsilon \neq 0$ both endpoints of I_ε lie (strictly) on the same side of the hyperplane $\text{aff } F_i$, hence $F_i \cap I_\varepsilon = \emptyset$.

As for F_j : If F_j and F_i lie in the same hyperplane ($u_i = u_j$), then the previous argument shows that $F_j \cap I_\varepsilon = \emptyset$ for all $\varepsilon \neq 0$ as well. If $u_i \neq u_j$, consider first the case where $\langle u_i, v_j \rangle < 0$ (Fig. 3(b)). For $\varepsilon > 0$, I_ε lies in the open half-space $\{x \in \mathbb{R}^d : \langle u_i, x \rangle > \langle u_i, g \rangle\}$, whereas F_j lies in the closed half-space $\{x \in \mathbb{R}^d : \langle u_i, x \rangle \leq \langle u_i, g \rangle\}$. Therefore, $I_\varepsilon \cap F_j = \emptyset$. For $\varepsilon < 0$, $\langle u_j, g + \varepsilon u_{ij} \rangle = \langle u_j, g \rangle + \varepsilon(1 + \langle u_i, u_j \rangle) < \langle u_j, g \rangle$. On the other hand, $\langle u_j, b \rangle < \langle u_j, g \rangle$ (for any point $b \in \text{relint } F_i$, since $\langle u_j, v_i \rangle < 0$), and therefore $\langle u_j, b + \varepsilon u \rangle < \langle u_j, g \rangle$ for sufficiently small $|\varepsilon|$, $\varepsilon \neq 0$. Thus both endpoints of I_ε lie on the same open side of the hyperplane $\text{aff } F_j$, hence $I_\varepsilon \cap F_j = \emptyset$.

In the case where $\langle u_i, v_j \rangle > 0$ (Fig. 3(c) above), just repeat the previous argument with the roles of $\varepsilon > 0$ and $\varepsilon < 0$ interchanged.

(b) The proof is similar to that of (a). First, note that J_0 does not meet any facet of $\mathcal{C}$ except F_i , F_j , and F_k . The same holds for J_ε provided $|\varepsilon| < \frac{1}{2} \cdot \text{dist}(J_0, |\mathcal{C} \setminus \{F_i, F_j, F_k, G, H\}|)$. By Lemma 3.1 above, $g + \varepsilon u_{ij}$, $h + \varepsilon u_{jk} \in \text{ext}\mathcal{C}$ and $g - \varepsilon u_{ij}$, $h - \varepsilon u_{jk} \in \text{int}\mathcal{C}$, provided ε is positive and sufficiently small. To complete the proof, it suffices to show that $J_\varepsilon \cap F_i = \emptyset$, $J_\varepsilon \cap F_j = \emptyset$, and $J_\varepsilon \cap F_k = \emptyset$ (for sufficiently small $|\varepsilon|$, $\varepsilon \neq 0$).

As for F_j : $\langle u_j, u_{ij} \rangle = 1 + \langle u_j, u_i \rangle > 0$ and $\langle u_j, u_{jk} \rangle = 1 + \langle u_j, u_k \rangle > 0$. Therefore, for any $\varepsilon \neq 0$, both endpoints of J_ε lie on the same open side of the hyperplane $\text{aff } F_j$, hence $F_j \cap J_\varepsilon = \emptyset$.

As for F_i : If F_j and F_i lie in the same hyperplane ($u_i = u_j$), then the previous argument shows that $F_i \cap J_\varepsilon = \emptyset$ for all $\varepsilon \neq 0$ as well. If $u_i \neq u_j$, consider first the case $\langle u_i, v_j \rangle < 0$ (Fig. 3(b) above). For $\varepsilon > 0$, J_ε lies in the open half-space $\{x \in \mathbb{R}^d : \langle u_j, x \rangle > \langle u_j, g \rangle\}$, whereas F_i lies in the closed half-space $\{x \in \mathbb{R}^d : \langle u_j, x \rangle \leq \langle u_j, g \rangle\}$. Therefore, $J_\varepsilon \cap F_i = \emptyset$. For $\varepsilon < 0$, we have $\langle u_i, g + \varepsilon u_{ij} \rangle = \langle u_i, g \rangle + \varepsilon(1 + \langle u_i, u_j \rangle) < \langle u_i, g \rangle$. On the other hand, $\langle u_i, h \rangle < \langle u_i, g \rangle$ (for any point $h \in F_j \setminus G$, since $\langle u_i, v_j \rangle < 0$), and therefore $\langle u_i, h + \varepsilon u_{jk} \rangle < \langle u_i, g \rangle$ for sufficiently small $|\varepsilon|$. Thus both endpoints of J_ε lie on the same open side of the hyperplane $\text{aff } F_i$, hence $J_\varepsilon \cap F_i = \emptyset$.

In the case where $\langle u_i, v_j \rangle > 0$ (Fig. 3(c) above), just repeat the previous argument with the roles of $\varepsilon > 0$ and $\varepsilon < 0$ interchanged.

As for F_k : Since the roles of F_i and F_k are interchangeable, the statement proved above for F_i applies to F_k as well. $\square$

Definition 3.1 Let p be a point in $\mathbb{R}^d \setminus |\mathcal{C}|$ ($= \text{ext}\mathcal{C} \cup \text{int}\mathcal{C}$), and F be a facet of $\mathcal{C}$. We say that p sees F if, for some point $a \in \text{relint } F$, $[p, a] \cap |\mathcal{C}| = \{a\}$.

Lemma 3.3 Assume $p \in \mathbb{R}^d \setminus |\mathcal{C}|$. Then p sees at least one facet of $\mathcal{C}$.

Proof Assume, w.l.o.g., that $p \in \text{ext}\mathcal{C}$. Let q be a point in $\text{int}\mathcal{C}$. Let U be a neighborhood of q that lies entirely in $\text{int}\mathcal{C}$. Choose a point $q' \in U$ such that the line $\text{aff}(p, q')$

does not meet any subfacet of $\mathcal{C}$. (This condition can be met by avoiding a finite number of hyperplanes through p .) Then the line segment $[p, q']$ must meet $|\mathcal{C}|$. Let a be the first point of $|\mathcal{C}|$ on $[p, q']$ (starting from p). Then a is a relative interior point of some facet F of $\mathcal{C}$; and $[p, a] \cap |\mathcal{C}| = \{a\}$. $\square$

Definition 3.2 Let p be a point in $\mathbb{R}^d \setminus |\mathcal{C}|$. We say that p has a *bounded horizon* (“does not see the sky”) with respect to $\mathcal{C}$ if every ray in $\mathbb{R}^d$ that emanates from p meets $|\mathcal{C}|$. (This is always the case when $p \in \text{int}\mathcal{C}$.) Otherwise we say that p has an *unbounded horizon*. Of course, p can be of unbounded horizon only if $p \in \text{ext}\mathcal{C}$.

Proposition 3.1 *If p has a bounded horizon, then p sees at least $d + 1$ different facets of $\mathcal{C}$.*

Proof By Lemma 3.3, p sees at least one facet of $\mathcal{C}$. Assume we know that p sees k facets $F_1, \dots, F_k$ of $\mathcal{C}$, $1 \leq k \leq d$. Choose points $a_i \in \text{relint } F_i$, $1 \leq i \leq k$, such that $[p, a_i] \cap |\mathcal{C}| = \{a_i\}$, and unit vectors u_i such that $\langle u_i, x \rangle = \langle u_i, a_i \rangle$ for all $x \in F_i$, and $\langle u_i, p \rangle < \langle u_i, a_i \rangle$. (If $p \in \text{int}\mathcal{C}$, then our vectors u_i are the same as the vectors u_i defined before Lemma 3.1 above. If $p \in \text{ext}\mathcal{C}$, then our vectors u_i are the negatives of the vectors u_i defined before Lemma 3.1.)

If $k < d$, or if $k = d$ and the vectors $u_1, \dots, u_d$ are linearly dependent, let w be a unit vector that is orthogonal to $u_1, \dots, u_k$. If $k = d$ and the vectors $u_1, \dots, u_d$ are linearly independent, let w be a unit vector that satisfies $\langle u_i, w \rangle < 0$ for $i = 1, \dots, d$. By our choice of w , the ray $R(p, w) = \{p + \lambda w : \lambda \geq 0\}$ does not meet any of the facets $F_1, \dots, F_k$. Since these facets are bounded, the same holds for the ray $R(p, w')$, provided w' is sufficiently close to w . Choose w' close to w and such that the ray $R(p, w')$ does not meet any subfacet of $\mathcal{C}$. Let a_{k+1} be the first point of $|\mathcal{C}|$ on $R(p, w')$. The point a_{k+1} belongs to the relative interior of a facet F_{k+1} of $\mathcal{C}$. Clearly, p sees F_{k+1} , and F_{k+1} is not one of the previous facets $F_1, \dots, F_k$. $\square$

3.2 d -Connectedness of the Dual Graph of $\mathcal{C}$

Theorem 3.1 (Generalized Balinski–Klee Theorem) *If $\mathcal{C}$ is a $(d - 1)$ -PPM ($d \geq 2$), then the dual graph $D(\mathcal{C})$ of $\mathcal{C}$ is d -connected.*

Remarks (1) For the definition of a $(d - 1)$ -PPM, see Definitions 1.2 and 1.3 above. Theorem 3.1 does not make use of the assumption that the complex $\mathcal{C}$ lies in $\mathbb{R}^d$; $\mathcal{C}$ may lie in $\mathbb{R}^{d'}$ for some $d' > d$. In fact, Theorem 3.1 can be stated for abstract polyhedral pseudomanifolds (that might, or might not be realizable).

(2) The theorem generalizes Balinski’s Theorem on the d -connectedness of the graph (and of the dual graph) of a d -polytope (see [5] or [13, p. 213]). In fact, the proof given below uses Balinski’s Theorem. For *simplicial* $(d - 1)$ -pseudomanifolds Theorem 3.1 was formulated and proved by V. Klee; see [16, Theorem 5].

Proof Note that if $d = 2$, then the 1-PPM $\mathcal{C}$ is just a simple closed polygon, and the dual graph of $\mathcal{C}$ is a simple circuit, which is 2-connected. Therefore, we assume in the sequel that $d \geq 3$. In the sequel we denote the dual graph of $\mathcal{C}$ by $D(\mathcal{C})$.

Given two facets A, B of $\mathcal{C}$, a *path* (of length l) that connects A to B in $D(\mathcal{C})$ is just a sequence $\mathcal{P} = (F_0, F_1, \dots, F_l)$ of facets of $\mathcal{C}$ that satisfies: $F_0 = A, F_l = B$, and $\dim(F_{i-1} \cap F_i) = d - 2$ for $i = 1, \dots, l$. The facets $F_1, \dots, F_{l-1}$ are the *interior nodes* of $\mathcal{P}$. The path $\mathcal{P}$ is *simple* if the facets $F_0, F_1, \dots, F_l$ are all different. Since $D(\mathcal{C})$ is a graph of valence $\geq d$ at each node, it has at least $d + 1$ nodes ($f_{d-1}(\mathcal{C}) \geq d + 1$), and by the Menger–Whitney Theorem the d -connectedness of $D(\mathcal{C})$ is equivalent to the following claim. (The Menger–Whitney Theorem can be formulated as follows (see [30, p. 171]): A graph is k -connected iff every pair of vertices can be joined by at least k paths with pairwise disjoint interiors. The theorem goes back to K. Menger (see [21] and [22]) and, independently, H. Whitney [34]; see [33]. The theorem is used in the proof of Balinski’s theorem; see [13, pp. 212–213], where it is attributed solely to Whitney, without mentioning his forerunner Menger.)

Claim 3.1 *Given a list $A, B, C_1, \dots, C_{d-1}$ of $d + 1$ distinct facets of $\mathcal{C}$, there is a simple path $\mathcal{P}$ in $D(\mathcal{C})$ that connects A to B and avoids $C_1, \dots, C_{d-1}$.*

Proof of Claim The existence of a simple path $\mathcal{P}$ in $D(\mathcal{C})$ that connects A to B follows by the assumption that $D(\mathcal{C})$ is a connected graph (Condition (D) in Definition 1.3). To prove the claim above we proceed by induction. Assuming we have a simple path $\mathcal{P}$ in $D(\mathcal{C})$ that connects A to B and avoids $C_1, C_2, \dots, C_{k-1}$ (for some $k, 1 \leq k \leq d - 1$), we modify $\mathcal{P}$ to produce a simple path $\mathcal{P}'$ that avoids C_k as well. We shall need the following

Lemma 3.4 *Let J be a $(d - 3)$ -face of $\mathcal{C}$. The subgraph of $D(\mathcal{C})$ spanned by the facets of $\mathcal{C}$ that include J is a disjoint union of simple circuits.*

Proof of Lemma Every facet ($= (d - 1)$ -face) of $\mathcal{C}$ that includes J has exactly two subfacets ($= (d - 2)$ -faces) that include J . Conversely, every subfacet that includes J is included in exactly two facets that include J (consider the no branching Condition (C) in Definition 1.3). In other words, the subgraph under consideration is 2-regular. Since it is finite, it must be a disjoint union of simple circuits. $\square$

Back to the proof of Claim 3.1.

Induction step $k - 1 \rightarrow k$. Let $\mathcal{P} = (F_0, F_1, \dots, F_l)$ be a simple path in $D(\mathcal{C})$ that connects $A (= F_0)$ to $B (= F_l)$ and avoids $C_1, \dots, C_{k-1}$ (for some $1 \leq k \leq d - 1$). If $\mathcal{P}$ avoids C_k as well, put $\mathcal{P}' = \mathcal{P}$. Otherwise assume $C_k = F_j$ for some (unique) j with $1 \leq j \leq l - 1$.

Define $C =_{\text{def}} C_k = F_j$, $G =_{\text{def}} F_{j-1} \cap C$, $H =_{\text{def}} C \cap F_{j+1}$. In the dual graph of the $(d - 1)$ -polytope $\mathcal{C}$ there are $d - 1$ simple paths $\pi_1, \dots, \pi_{d-1}$ with pairwise disjoint interiors that connect the $(d - 2)$ -faces G and H . This follows from the $(d - 1)$ -connectedness of the dual graph of $\mathcal{C}$ (Balinski’s Theorem). Let $\pi = (\Phi_0, \Phi_1, \dots, \Phi_\lambda)$ with $\Phi_0 = G$ and $\Phi_\lambda = H$ be one of those $d - 1$ paths ($\dim(\Phi_i \cap \Phi_{i+1}) = d - 3$ for $0 \leq i \leq \lambda - 1$). π induces a path Π from F_{j-1} to F_{j+1} in $D(\mathcal{C})$ as follows. Each of the $(d - 2)$ -faces Φ_i ($0 \leq i \leq \lambda$) is included in exactly two $(d - 1)$ -faces of $\mathcal{C}$. One of them is C . Call the other one Q_i . In particular, $Q_0 = F_{j-1}$ and $Q_\lambda = F_{j+1}$. For each $i, 0 \leq i < \lambda$, the three $(d - 1)$ -cells Q_i, C and

Q_{i+1} share a $(d-3)$ -face $\Psi_i =_{\text{def}} \Phi_i \cap \Phi_{i+1}$. Thus $\langle Q_i, C, Q_{i+1} \rangle$ is an arc of one of the circuits around $J =_{\text{def}} \Psi_i$, as described in Lemma 3.4 above. The arc complementary to $\langle Q_i, C, Q_{i+1} \rangle$ within that circuit is an arc in $D(C)$ that connects Q_i to Q_{i+1} , avoids C , and uses only $(d-1)$ -cells that include Ψ_i . Concatenating these circumventing arcs from $i = 0$ to $i = \lambda - 1$, we obtain a path in $D(C)$ that connects F_{j-1} to F_{j+1} , and avoids C . We call this path Π . Applying this procedure to the paths $\pi_1, \dots, \pi_{d-1}$, we obtain $d-1$ paths $\Pi_1, \dots, \Pi_{d-1}$ in $D(C)$ that connect F_{j-1} to F_{j+1} and avoid C .

Next we will show (Lemma 3.5 below) that two of these paths never share an internal node ($= (d-1)$ -cell). Since the list $C_1, \dots, C_{d-1}$ of forbidden $(d-1)$ -cells contains, apart from C , only $d-2$ additional cells, it follows that one of the avoiding paths, say Π_v , does not use any of the forbidden $(d-1)$ -cells. Replace the arc $\langle F_{j-1}, C, F_j \rangle$ in $\mathcal{P}$ by Π_v , pass, if necessary, to a simple subpath, and call it $\mathcal{P}'$. $\mathcal{P}'$ is a simple path in $D(C)$ from A to B that avoids $C_1, \dots, C_{k-1}$, and C_k .

Lemma 3.5 *Let π, π' be two of the paths $\pi_1, \dots, \pi_{d-1}$ listed above, that connect $G (= F_{j-1} \cap C)$ with $H (= F_{j+1} \cap C)$ in the dual graph of C (say, $\pi = \pi_m, \pi' = \pi_{m'}, 1 \leq m < m' \leq d-1$). Let Π and Π' be the paths induced by π and π' , respectively, that connect F_{j-1} to F_{j+1} in $D(C)$. Then Π and Π' have no internal node in common.*

Proof Assume $\pi = (\Phi_0, \Phi_1, \dots, \Phi_\lambda)$, where $\Phi_0 = F_{j-1} \cap C, \Phi_\lambda = F_{j+1} \cap C$, and each Φ_i is a $(d-2)$ -face of C . Put $\Psi_i =_{\text{def}} \Phi_i \cap \Phi_{i+1}$ ($i = 0, 1, \dots, \lambda-1$). Each Ψ_i is a $(d-3)$ -face of C . If Q is an internal node of the induced path Π , then $Q \cap C$ is one of the $(d-2)$ -cells $\Phi_1, \dots, \Phi_{\lambda-1}$ or one of the $(d-3)$ -cells $\Psi_0, \dots, \Psi_{\lambda-1}$. The corresponding entities for π' are $\pi' = (\Phi'_0, \dots, \Phi'_{\lambda'})$ ($\Phi'_0 = \Phi_0, \Phi'_{\lambda'} = \Phi_\lambda$), and $\Psi'_i = \Phi'_i \cap \Phi'_{i+1}$ ($i = 0, 1, \dots, \lambda'-1$).

If Q' is an internal node of Π' , then $Q' \cap C$ is one of the cells $\Phi'_1, \dots, \Phi'_{\lambda'-1}$ or $\Psi'_0, \dots, \Psi'_{\lambda'-1}$. To show that $Q \neq Q'$, it suffices to check that the two sets $\{\Phi_i : 1 \leq i \leq \lambda-1\} \cup \{\Psi_i : 0 \leq i \leq \lambda-1\}$ and $\{\Phi'_i : 1 \leq i \leq \lambda'-1\} \cup \{\Psi'_i : 0 \leq i \leq \lambda'-1\}$ are disjoint. In fact, the Φ' -s are different from the Φ -s, since π and π' have no internal node in common. Each Ψ_i is included in exactly two $(d-2)$ -faces of C , which are adjacent nodes of π , and each Ψ'_i is included in exactly two $(d-2)$ -faces of C , which are adjacent nodes of π' . Since π and π' have no internal nodes in common, this implies that $\Psi_i \neq \Psi'_i$. Please pay attention to the possibility that, say, $\lambda = 1$, and $\Psi_0 = F_{j-1} \cap C \cap F_{j+1}$. In this case, we necessarily have $\lambda' > 1$, since $\pi' \neq \pi$. $\square$

This concludes the proof of Claim 3.1 and ends the proof of Theorem 3.1. $\square$

3.3 Upper Bounds for the Polygonal Diameter of $\text{int } \mathcal{C}$ and of $\text{ext } \mathcal{C}$

Definition 3.3 For a set $S \subset \mathbb{R}^d$ and points $a, b \in S$, denote by $\pi_S(a, b)$ the smallest number of edges of a polygonal line that connects a to b within S . ($\pi_S(a, b) =_{\text{def}} \infty$ if no such polygonal line exists.) If S is polygonally connected, then $\pi_S(\cdot, \cdot)$ is an integer valued metric on S . The *polygonal diameter* of S is defined as $\text{poldiam}(S) =_{\text{def}}$

$\sup\{\pi_S(a, b) : a, b \in S\}$. Now we shall establish (almost) tight upper bounds for the polygonal diameter of $\text{int } \mathcal{C}$ and of $\text{ext } \mathcal{C}$, where $\mathcal{C}$ is a $(d-1)$ -PPM in $\mathbb{R}^d$ with n facets. In Sect. 4 below, we shall present examples of polyhedral $(d-1)$ -spheres $\mathcal{C}, \mathcal{C}'$ in $\mathbb{R}^d$ with n facets (for all $d \geq 2$ and $d < n < \infty$) such that $\text{poldiam}(\text{int } \mathcal{C}) = \lfloor \frac{n}{d} \rfloor$, and $\text{poldiam}(\text{ext } \mathcal{C}') = \lceil \frac{n}{d} \rceil$. For $d = 2$, we shall even produce, in a subsequent paper, such examples with $\mathcal{C} = \mathcal{C}'$ (see [25]). The following theorem clearly implies (F) in Sect. 1 above.

Theorem 3.2 (Upper bounds for $\text{poldiam}(\cdot)$) *If $\mathcal{C}$ is a polyhedral $(d-1)$ -pseudo-manifold (i.e., a $(d-1)$ -PPM) with n facets in $\mathbb{R}^d$, then the relations $\text{poldiam}(\text{int } \mathcal{C}) \leq 2 + \lfloor \frac{n-2}{d} \rfloor$ and $\text{poldiam}(\text{ext } \mathcal{C}) \leq 3 + \lfloor \frac{n-2}{d} \rfloor$ hold.*

Proof Assume a, b are two points in the same component ($\text{int } \mathcal{C}$ or $\text{ext } \mathcal{C}$) of $\mathbb{R} \setminus |\mathcal{C}|$. We shall distinguish three possible cases.

Case I: Both a and b are of bounded horizon (with respect to $\mathcal{C}$) (cf. Definition 3.2 above).

Case II: a is of bounded horizon, whereas b is of unbounded horizon (or vice versa).

Case III: Both a and b are of unbounded horizon.

Denote by A (resp., B) the set of those facets F of $\mathcal{C}$ such that a (resp., b) sees a point in $\text{relint } F$ via $\mathbb{R}^d \setminus |\mathcal{C}|$. By Proposition 3.1 above, we know that $\#A \geq d+1$ if a is of bounded horizon, and $\#A \geq 1$ otherwise (Lemma 3.3 above).

If $A \cap B \neq \emptyset$, say $F \in A \cap B$, then a sees a point $a' \in \text{relint } F$ and b sees a point $b' \in \text{relint } F$ via $\mathbb{R}^d \setminus |\mathcal{C}|$. Choose points $a'' \in]a, a'[$ (close to a') and $b'' \in]b, b'[$ (close to b'), such that the segment $[a'', b'']$ runs parallel to F and misses $\mathcal{C}$. Then $\langle a, a'', b'', b \rangle$ is a 3-path that connects a to b via $\mathbb{R}^d \setminus |\mathcal{C}|$. The upper bounds in Theorem 3.2 are all ≥ 3 , except for $\text{int } \mathcal{C}$, when $n = d+1$. But when $n = d+1$, $\text{int } \mathcal{C}$ is the interior of a d -simplex, which is convex. Assume, from now on, that $A \cap B = \emptyset$, and consider each case separately.

Case I: In this case, $\#A \geq d+1$, $\#B \geq d+1$. Let $D = D(\mathcal{C})$ be the dual graph of $\mathcal{C}$. By Theorem 3.1, D is d -connected. Extend D to a graph D'' by adding two (auxiliary) vertices, a and b , connecting a by edges to all members of A and b to all members of B . As one can easily check, the extended graph D'' is again d -connected. Therefore, in D'' we can find d paths $\pi_1, \dots, \pi_d$ that connect a to b and have pairwise disjoint interiors. If we take these paths as short as possible (in particular, diagonal free), then each π_i will visit the set A ($=$ neighbors of a) exactly once. Since $\#A \geq d+1$ and $\#B \geq d+1$, it follows that all these d paths together visit at most $n-2$ vertices of D . The shortest of these paths, say π_1 , is of the form $\langle a, F_1, \dots, F_m, b \rangle$, where $m \leq \lfloor \frac{n-2}{d} \rfloor$, $F_1, \dots, F_m$ are facets of $\mathcal{C}$, F_{i-1} and F_i share a subfacet G_i for $i = 2, \dots, m$, a sees via $\mathbb{R}^d \setminus |\mathcal{C}|$ a point $a' \in \text{relint } F_1$, and b sees via $\mathbb{R}^d \setminus |\mathcal{C}|$ a point $b \in \text{relint } F_m$. If we choose points $g_i \in \text{relint } G_i$ for $i = 2, \dots, m$, then $\langle a, a', g_2, g_3, \dots, g_m, b', b \rangle$ is a polygonal path of $m+2$ edges that connects a to b and runs along $|\mathcal{C}|$. By Lemma 3.2, this path can be pushed away from $|\mathcal{C}|$, thus producing a polygonal path of $m+2$ edges that connects a to b via $\mathbb{R}^d \setminus |\mathcal{C}|$.

Case II: In this case, extend the graph $D = D(\mathcal{C})$ to a graph D' by adding the (auxiliary) vertex a and connecting a by edges to all vertices in A . Fix a vertex

$F' \in B$. D' is again d -connected and contains d paths $\pi_1, \dots, \pi_d$ from a to F' , with pairwise disjoint interiors. If we take these d paths to be diagonal free, then their union misses at least one vertex of A (recall $\#A \geq d+1$). The union of their interiors misses F' as well, and thus uses at most $n-2$ facets of $\mathcal{C}$. The shortest of these paths, say π_1 , is of the form $\langle a, F_1, \dots, F_m, F' \rangle$, where $m \leq \lfloor \frac{n-2}{d} \rfloor$. As in Case I, this path gives rise to a polygonal path $\langle a, a', g_2, \dots, g_m, g_{m+1}, b', b \rangle$, with $a' \in \text{relint } F_1$, $g_i \in \text{relint}(F_{i-1} \cap F_i)$ ($i = 2, \dots, m$), $g_{m+1} \in \text{relint}(F_m \cap F')$, and $b' \in \text{relint } F'$, and thus to a polygonal path of at most $3 + \lfloor \frac{n-2}{d} \rfloor$ edges that connects a to b via $\mathbb{R}^d \setminus |\mathcal{C}|$.

Case III: Denote by U_a (resp., U_b) the set of unit vectors u such that the ray $a + \mathbb{R}^+u$ (resp., $b + \mathbb{R}^+u$) misses $|\mathcal{C}|$. Since $|\mathcal{C}|$ is compact and $a \notin |\mathcal{C}|$, U_a is an open subset of the unit sphere $S^{d-1} \subset \mathbb{R}^d$. We have $U_a \neq \emptyset$, since a is of unbounded horizon, and the same holds for U_b . Choose vectors $u_a \in U_a$ and $u_b \in U_b$ such that $u_b \neq -u_a$. There is a linear function f that satisfies $f(u_a) = f(u_b) = 1$. Put $M =_{\text{def}} \max\{f(x) : x \in |\mathcal{C}|\}$. If $t > M - \min\{f(a), f(b)\}$, then the segment $[a + tu_a, b + tu_b]$ misses $|\mathcal{C}|$, and $\langle a, a + tu_a, b + tu_b, b \rangle$ is a polygonal path of three edges that connects a to b via $\mathbb{R}^d \setminus |\mathcal{C}|$. $\square$

4 Lower Bounds for the Maximal Polygonal Diameter of the Interior and Exterior of a $(d-1)$ -PPM in $\mathbb{R}^d$

For $n \geq d+1 > 0$, define (note that $\text{poldiam}(\cdot)$ is defined in Definition 3.3 above):

$$\text{pd-int}(d, n) =_{\text{def}} \max\{\text{poldiam}(\text{int } \mathcal{P}) : \mathcal{P} \text{ is a } (d-1)\text{-PPM in } \mathbb{R}^d \text{ and } f_{d-1}(\mathcal{P}) = n\}$$

(“pd” stands of “polygonal diameter”) and

$$\begin{aligned} \text{pd-ext}(d, n) =_{\text{def}} \max\{\text{poldiam}(\text{ext } \mathcal{P}) : \mathcal{P} \text{ is a } (d-1)\text{-PPM in } \mathbb{R}^d \\ \text{and } f_{d-1}(\mathcal{P}) = n\}. \end{aligned}$$

The following upper bounds were established in Theorem 3.2 above:

$$\text{pd-int}(d, n) \leq 2 + \left\lfloor \frac{n-2}{d} \right\rfloor, \quad (3)$$

$$\text{pd-ext}(d, n) \leq 3 + \left\lfloor \frac{n-2}{d} \right\rfloor. \quad (4)$$

In this section we produce two $(d-1)$ -PPMs $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^d$, with $f_{d-1}(\mathcal{P}) = f_{d-1}(\mathcal{Q}) = n$, $n \geq d+1 > 0$, such that $\text{poldiam}(\text{int } \mathcal{P}) = \lfloor \frac{n}{d} \rfloor$ and $\text{poldiam}(\text{ext } \mathcal{Q}) = \lceil \frac{n}{d} \rceil$. These examples establish lower bounds $\lfloor \frac{n}{d} \rfloor$ (resp., $\lceil \frac{n}{d} \rceil$) for $\text{pd-int}(d, n)$ (resp., $\text{pd-ext}(d, n)$) which are quite close to the upper bounds given in (3) (resp., (4)) above.

Example 4.1 (Zigzag sleeve) This is an example of a polyhedral $(d-1)$ -sphere $\mathcal{P}$ in $\mathbb{R}^d$ (i.e., combinatorially equivalent to the boundary complex of a convex polytope in $\mathbb{R}^d$) with n ($\geq d+1 > 0$) facets such that $\text{poldiam}(\text{int } \mathcal{P}) = m =_{\text{def}} \lfloor \frac{n}{d} \rfloor$. Assume first that $n \equiv 0 \pmod{d}$, i.e., $n = md$, $m \geq 2$. Let Δ^{d-1} be a $(d-1)$ -simplex in $\mathbb{R}^d$

Fig. 4 Example 4.1 for $m = 5$, $d = 3$

with barycenter at the origin o contained in the relative interior of the unit $(d-1)$ -ball $B^{d-1}(o, 1)$ of $\mathbb{R}^{d-1}$, where $\mathbb{R}^{d-1} =_{\text{def}} \{x = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d : \xi_d = 0\}$, and let $F_1, \dots, F_d$ be the facets of Δ^{d-1} . Define $m+1$ points $a_0, a_1, \dots, a_m$ by $a_v =_{\text{def}} (-1)^v e_{d-1} + v e_d$ ($v = 0, 1, \dots, m$). For $v = 1, \dots, m$, define a d -polytope P_v as follows:

$$P_1 =_{\text{def}} \text{conv}(\{a_0\} \cup (a_1 + \Delta^{d-1})),$$

$$P_v =_{\text{def}} \text{conv}((a_{v-1} + \Delta^{d-1}) \cup (a_v + \Delta^{d-1})) = [a_{v-1}, a_v] + \Delta^{d-1}$$

for $2 \leq v < m$, and for $v = m$ we define $P_m =_{\text{def}} (\text{conv}(a_{m-1} + \Delta^{d-1}) \cup \{a_m\})$. Define also $P =_{\text{def}} \bigcup_{v=1}^m P_v$. P_1 is a d -simplex (a d -pyramid based on $a_1 + \Delta^{d-1}$ apexed at a_0), P_v , $1 < v < m$, is a d -prism on a $(d-1)$ -simplex (the sum of the segment $[a_{v-1}, a_v]$ and the simplex Δ^{d-1}), and P_m is a d -simplex (a d -pyramid based on $a_{m-1} + \Delta^{d-1}$ and apexed at a_m). The union $P = P_1 \cup P_2 \cup \dots \cup P_m$ is piecewise linearly homeomorphic (as a transformation that changes the d -coordinate only, i.e., a “piecewise shear” parallel to $\mathbb{R}^{d-1}$) to a convex polytope which is a $(d-1)$ -prism on a $(d-1)$ -simplex with two d -simplices attached to its two bases, like a pencil sharpened at both ends (see Fig. 4 for $m = 5, d = 3$). (Such a convex polytope may be called a *bi-apexed prism*.)

The boundary complex $\mathcal{P}$ of P consists of $md (= n)$ convex $(d-1)$ -polytopes ($2d$ $(d-1)$ -simplices and $(m-2) \cdot d$ $(d-1)$ -prisms on $(d-2)$ -simplices), listed as follows:

$$\text{conv}(\{a_0\} \cup (a_1 + F_i)), \quad [a_{v-1}, a_v] + F_i \quad (2 \leq v \leq m-1),$$

$$\text{conv}((a_{m-1} + F_i) \cup \{a_m\}),$$

all this for $i = 1, 2, \dots, d$. All these $(d-1)$ -polytopes, together with all their faces, form a $(d-1)$ -PPM, henceforth denoted by $\mathcal{P}$, satisfying $\text{int } \mathcal{P} = \text{int } P$ (here $\text{int } \mathcal{P}$ is the interior of $\mathcal{P}$ in the sense of the Jordan–Brouwer separation theorem (Theorem 1.1 above) and $\text{int } P$ is the (topological) interior of P as a (closed) set in $\mathbb{R}^d$). The following proposition implies that $\text{poldiam}(\text{int } \mathcal{P}) = m$.

Proposition 4.1 *It is possible to proceed from any point $q_0 \in \text{int } B^d(a_0, 1) \cap \text{int } P$ (a 1-*vicinity* of a_0 in $\text{int } P$) to any point $q_m \in \text{int } B^d(a_m, 1) \cap \text{int } P$ (a 1-*vicinity* of a_m in $\text{int } P$) by a polygonal path of m edges, but not by less than this.*

Proof Define $\gamma =_{\text{def}} \langle q_0, a_1, a_2, \dots, a_{m-1}, q_m \rangle$. γ is a polygonal path of m edges in $\text{int } P$ from q_0 to q_m . This proves the first part of the proposition. To prove the second part, we give first an informal argument (see Fig. 4). In order to reach from q_0 to $P_1 \cap P_2 (= a_1 + \Delta^{d-1})$, there must be a decreasing segment (ξ_{d-1} decreases as a function of ξ_d); in order to reach from $P_1 \cap P_2$ to $P_2 \cap P_3 (= a_2 + \Delta^{d-1})$, there must be an increasing segment (ξ_{d-1} increases as a function of ξ_d), etc.; finally, in order to reach from $P_{m-1} \cap P_m$ to q_m , there must be a decreasing [increasing] segment if m is odd [if m is even]. This makes a zigzag of m edges.

Formal proof. Among the polygonal paths from q_0 to q_m in $\text{int } P$, let γ be a polygonal path of minimal number, say n , of edges, and let $p_0, p_1, \dots, p_n$ be its vertices in this order, i.e., $\gamma = \langle p_0 = q_0, p_1, p_2, \dots, p_{n-1}, p_n = q_m \rangle$. (We have to show that $n \geq m$.) Note that the role played by n here is different from that played by $n =_{\text{def}} f_{d-1}(\mathcal{C})$ defined supra.

We will think of γ as a (piecewise linear and continuous) function from $[0, 1]$ to $\mathbb{R}^d$ with $\gamma(0) = q_0$ and $\gamma(1) = q_m$. Since $\gamma(0) = q_0 \neq q_m = \gamma(1)$, γ is not a circuit.

Claim 4.1 γ is 1-1, i.e., γ is a simple path.

Proof of Claim If γ is not 1-1, then it contains a circuit, say $\tilde{\gamma} : [t_1, t_2] \rightarrow \mathbb{R}^d$ with $[t_1, t_2] \subset [0, 1]$, $\gamma(t) = \tilde{\gamma}(t)$ for $t \in [t_1, t_2]$ and $\tilde{\gamma}(t_1) = \tilde{\gamma}(t_2)$. Reduce γ to $[0, t_1] \cup [t_2, 1]$ (i.e., drop the circuit $\tilde{\gamma}$ from γ). The resulting polygonal path connects q_0 to q_m via $\text{int } P$ and it clearly has fewer edges than γ , a contradiction. This proves Claim 4.1. $\square$

For $1 \leq i \leq m-1$, define a vertex q_i of γ as follows: Put $t_i =_{\text{def}} \max\{t \text{ with } 0 \leq t \leq 1 : \gamma_d(t) = i\}$ (t_i is the last time that γ visits the hyperplane $\xi_d = i$). Clearly, $\gamma(t_i) \in a_i + \Delta^{d-1} (= P_i \cap P_{i+1})$, hence

$$\text{sg}(\gamma_{d-1}(t_i)) = (-1)^i, \quad 1 \leq i \leq m-1, \quad (5)$$

and by Claim 4.1 above, $\gamma(t_i) \in [p_{\iota-1}, p_{\iota}[$ for some *unique* $\iota = \iota(i)$, $1 \leq \iota \leq n$. Define

$$q_i =_{\text{def}} \begin{cases} p_{\iota-1} & \text{if } \text{sg}((p_{\iota-1})_{d-1}) = (-1)^i, \\ p_{\iota} & \text{otherwise.} \end{cases}$$

This definition ensures that

$$\text{sg}((q_i)_{d-1}) = (-1)^i (= \text{sg}(a_i)) \quad (6)$$

for $0 \leq i \leq m$. We will show that the vertices q_i , $0 \leq i \leq m$, are all different.

Lemma 4.1 *Let $[u, v]$ be a segment contained in $\text{int } P$ such that $(u)_d = i$ for some i with $1 \leq i \leq m-1$, $(v)_d \neq i$ and $[u, v] \cap G = \emptyset$, where $G =_{\text{def}} \{x \in \mathbb{R}^d : (x)_{d-1} = 0\}$. Then either $i-1 < (v)_d < i$ or $i < (v)_d < i+1$.*

Proof of Lemma Put $G_i =_{\text{def}} \{x \in \mathbb{R}^d : \text{sg}(x)_{d-1} = (-1)^i \text{ or } 0\}$. Note that $u \in a_i + \Delta^{d-1} (= P_i \cap P_{i+1})$, hence G_i is the closed half-space bounded by G which contains u (and v).

The d -polytope $P_i [P_{i+1}]$ is a d -prism if $1 < i \leq m-1$ [$1 \leq i < m-1$], a d -simplex if $i = 1$ [$i = m-1$]; and it is contained in the layer $L_i^- =_{\text{def}} \{x \in \mathbb{R}^d : i-1 \leq (x)_d \leq i\}$ [$L_i^+ =_{\text{def}} \{x \in \mathbb{R}^d : i \leq (x)_d \leq i+1\}$]. The d -polytope $Q_i =_{\text{def}} P_i \cap G_i$ [$Q_{i+1} =_{\text{def}} P_{i+1} \cap G_i$] satisfies

$$\text{int } Q_i \subset \text{int } L_i^- \quad [\text{int } Q_{i+1} \subset \text{int } L_i^+], \quad (7)$$

and it has $d+2$ facets: d facets are subsets of the d facets of $P_i [P_{i+1}]$ whose relative interiors are contained in $\text{int } L_i^-$ [$\text{int } L_i^+$], henceforth called *loxo facets*, as well as the two facets $a_i + \Delta^{d-1} (= P_i \cap P_{i+1})$ and $G \cap P_i$ [$G \cap P_{i+1}$]. If $(v)_d < i$ [$i < (v)_d$], then $[u, v] \cap \text{int } Q_i \neq \emptyset$ [$[u, v] \cap \text{int } Q_{i+1} \neq \emptyset$], and since $[u, v]$ does not intersect any of the d loxo facets of $P_i [P_{i+1}]$ (since $[u, v] \subset \text{int } P$) and it does not intersect the facet $G \cap P_i$ [$G \cap P_{i+1}$] (since $[u, v] \cap G = \emptyset$), it follows that $[u, v] \cap \text{bd } Q_i = u$ [$[u, v] \cap \text{bd } Q_{i+1} = u$], hence $v \in \text{int } Q_i$ [$v \in \text{int } Q_{i+1}$]. It follows from (7) that $i-1 < (v)_d < i$ if $(v)_d < i$ [$i < (v)_d < i+1$ if $i < (v)_d$]. This proves Lemma 4.1. $\square$

Claim 4.2 $(q_0)_d < 1$, $m-1 < (q_m)_d$ and for $1 \leq i \leq m-1$, $i-1 < (q_i)_d < i+1$.

Proof of Claim The first two inequalities follow from $\|q_0 - a_0\| < 1$ and $\|q_m - a_m\| < 1$, respectively. Let $1 \leq i \leq m-1$. If $(q_i)_d = i$, we are done. Assume $(q_i)_d \neq i$. Using the notation introduced above, $\{\gamma(t_i), q_i\} \subset \text{int } G_i$ by (5) and (6), hence $[\gamma(t_i), q_i] \cap G = \emptyset$. Since $\gamma(t_i) \in P_i \cap P_{i+1}$, the segment $[\gamma(t_i), q_i]$ satisfies the conditions of Lemma 4.1 above (with $u = \gamma(t_i)$) and $v = q_i$), hence either $i-1 < (q_i)_d < i$ or $i < (q_i)_d < i+1$, hence $i-1 < (q_i)_d < i+1$. This proves Claim 4.2. $\square$

We show now that $q_i \neq q_j$ for $0 \leq i < j \leq m$. If j and i are of different parities, this follows from (6). Let $j = i + 2\kappa$ for some $\kappa \geq 1$. In view of Claim 4.2,

$$(q_i)_d < i+1 < (q_{i+2})_d < i+3 < (q_{i+4})_d < \dots < i + (2\kappa - 1) < (q_{i+2\kappa})_d,$$

hence $q_i \neq q_{i+2\kappa} (= q_j)$. This concludes the proof of Proposition 4.1. $\square$

It remains to modify Example 4.1 to the case where $n \not\equiv 0 \pmod{d}$, i.e., $n = md + r$, $1 \leq r \leq d-1$. If $r = 1$, truncate the vertex a_m of P by a hyperplane passing in an ε -vicinity of a_m for some sufficiently small $\varepsilon > 0$. The resulting body P' has $md + 1 (= n)$ facets, d new vertices and, clearly, if ε is small enough, then $\text{poldiam}(\text{int } P') = \text{poldiam}(\text{int } P) = m = \lfloor \frac{n}{d} \rfloor$. If $r = 2$, truncate P near a_m as above, and then truncate P' near enough to one of its new vertices; the resulting body P'' has $md + 2 (= n)$ facets and again $\text{poldiam}(\text{int } P'') = m = \lfloor \frac{n}{d} \rfloor$. For $3 \leq r \leq d-1$ make, similarly, r truncations near a_m to obtain a body $P^{(r)}$ with $md + r (= n)$ facets and $\text{poldiam}(\text{int } P^{(r)}) = m = \lfloor \frac{n}{d} \rfloor$.

Example 4.2 (Pyramid with a zigzag cave) Now we will modify the complex of Example 4.1 above to obtain a $(d-1)$ -PPM $\mathcal{Q}$ in $\mathbb{R}^d$ with n ($\geq d+1 > 0$) facets

such that $\text{poldiam}(\text{ext } \mathcal{Q}) = \lceil \frac{n}{d} \rceil$. Again assume first that $n \equiv 0 \pmod{d}$, i.e., $n = md$ ($m \geq 2$). In this case, $\lceil \frac{n}{d} \rceil = m$, so $\mathcal{Q}$ will have md facets, and $\text{poldiam}(\mathcal{Q}) = m$.

Define m d -polytopes $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ as follows:

- (a) $\mathcal{Q}_v =_{\text{def}} P_v$ for $4 \leq v \leq m$.
- (b) $\mathcal{Q}_2 =_{\text{def}} \text{conv}((e_d + M \cdot \Delta^{d-1}) \cup (a_2(\varepsilon) + \Delta^{d-1}))$ with large enough M satisfying

$$\pm e_{d-1} + (e_d + \Delta^{d-1}) \subset \text{relint}(e_d + M \cdot \Delta^{d-1}) \left(\subset H =_{\text{def}} \{x \in \mathbb{R}^d : (x)_d = 1\} \right), \quad (8)$$
 and $a_2(\varepsilon) =_{\text{def}} e_{d-1} + (3 - \varepsilon)e_d$, where ε is a positive number whose smallness is determined below. Since $M \cdot \Delta^{d-1}$ is a homothetic image of Δ^{d-1} and $a_2(\varepsilon) + \Delta^{d-1}$ is a translate of Δ^{d-1} , an easy calculation shows that $e_d + M \cdot \Delta^{d-1}$ is a homothet of $a_2(\varepsilon) + \Delta^{d-1}$ with center of homothety $b =_{\text{def}} \frac{a_2(\varepsilon) - \frac{1}{M}e_d}{1 - \frac{1}{M}}$ (and constant of homothety M). Hence $\mathcal{Q}_2$ is combinatorially equivalent to a d -prism on a $(d-1)$ -simplex (same as $\text{conv}((a_1 + \Delta^{d-1}) \cup (a_2(\varepsilon) + \Delta^{d-1}))$). The facets $G_i =_{\text{def}} \text{conv}((e_d + M F_i) \cup (a_2(\varepsilon) + F_i))$, $1 \leq i \leq d$, of $\mathcal{Q}_2$ (recall that F_i , $1 \leq i \leq d$, are facets of Δ^{d-1}) will be referred to below as the *side facets* of $\mathcal{Q}_2$.
- (c) $\mathcal{Q}_3 =_{\text{def}} \text{conv}((a_2(\varepsilon) + \Delta^{d-1}) \cup (a_3 + \Delta^{d-1}))$. $\mathcal{Q}_3$ is a d -prism on a $(d-1)$ -simplex analogous to P_3 above, where $a_2(\varepsilon)$ plays the role of a_2 in the definition of P_3 . The replacement of a_2 by $a_2(\varepsilon)$ (where $\varepsilon > 0$ is small) flattens $\mathcal{Q}_3$ in the d -axis direction (the d -axis difference between the two bases of the prism is small).
- (d) $\mathcal{Q}_1 =_{\text{def}} \text{conv}((e_d + M \cdot \Delta^{d-1}) \cup \{a_0(L)\})$, where $a_0(L) =_{\text{def}} (L+1)e_d$ and L is a positive number whose largeness is determined below.

First we determine M . Since $\Delta^{d-1} \subset \text{relint } B^{d-1}(o, 1)$ (= the relative interior of the unit $(d-1)$ -ball centered at the origin o in the plane $(x)_d = 0$) and since $o \in \text{relint } \Delta^{d-1}$,

$$\pm e_{d-1} + \Delta^{d-1} \subset \text{relint } B^{d-1}(o, 2) \quad (9)$$

(= the relative interior of the $(d-1)$ -ball of radius 2 centered in the origin in the hyperplane $(x)_d = 0$). Clearly, $B^{d-1}(o, 2) \subset M \cdot \Delta^{d-1}$ for

$$M > \frac{2}{\text{dist}(o, \text{relbd } \Delta^{d-1})}, \quad (10)$$

hence (8) follows from (9) provided M satisfies (10). So put

$$M =_{\text{def}} \frac{2}{\text{dist}(o, \text{relbd } \Delta^{d-1})} + 1.$$

Next we determine ε . (As mentioned above, the smallness of ε determines the flatness of $\mathcal{Q}_3$ in the d -axis direction.)

Put $H =_{\text{def}} \{x \in \mathbb{R}^d : (x)_d = 1\}$, $H^- =_{\text{def}} \{x \in \mathbb{R}^d : (x)_d \leq 1\}$ and $\hat{H}^- = (\text{cone}_b(e_d + M \cdot \Delta^{d-1})) \cap \text{int } H^-$. Our main concern in determining $\varepsilon > 0$ is

Requirement 4.1 (Smallness of $\varepsilon > 0$) For $u \in \hat{H}^-$ and $z^+ \in a_3 + \Delta^{d-1}$ the half open segment $[u, z^+[$ avoids the basis $a_2(\varepsilon) + \Delta^{d-1}$ of $\mathcal{Q}_3$. In other words, it is

impossible to reach from a point in $\hat{H}^-$ a point in the upper basis $a_3 + \Delta^{d-1}$ of Q_3 directly by a segment via the lower basis $a_2(\varepsilon) + \Delta^{d-1}$ of Q_3 .

We will show now the existence of such a (small) positive ε .

Put $G =_{\text{def}} \{x \in \mathbb{R}^d : (x)_{d-1} = 0\}$, $G_- =_{\text{def}} \{x \in \mathbb{R}^d : (x)_{d-1} \leq 0\}$, $G_+ =_{\text{def}} \{x \in \mathbb{R}^d : (x)_{d-1} \geq 0\}$, and $H_3 =_{\text{def}} \{x \in \mathbb{R}^d : (x)_d = 3\}$.

Since $a_3 + \Delta^{d-1} \subset \text{int } G_-$, Requirement 4.1 will be met provided

$$(\text{relint}(\text{cone}_u(z^-))) \cap H_3 \subset G_+ \quad (11)$$

for $u \in \hat{H}^-$ and $z^- \in a_2(\varepsilon) + \Delta^{d-1}$. Consider the (parallel) $(d-2)$ -flats $F_1^{d-2} =_{\text{def}} 3e_d + \text{span}(e_1, e_2, \dots, e_{d-2})$ and $F_0^{d-2} =_{\text{def}} e_d + (M+1)e_{d-1} + \text{span}(e_1, e_2, \dots, e_{d-2})$.

We have $F_0^{d-2} \parallel F_1^{d-2}$, $F_1^{d-2} \subset H_3 \cap G$ and $F_0^{d-2} \subset H$. F_0^{d-2} is a hyperplane of H strictly supporting $e_d + M \cdot \Delta^{d-1} (\subset H)$. Put $I^{d-1} =_{\text{def}} \text{aff}(F_0^{d-2} \cup F_1^{d-2})$. The hyperplane I^{d-1} is the common boundary of two closed half-spaces I_-^{d-1}, I_+^{d-1} ; choose the signs $+, -$ so that $e_d + M \cdot \Delta^{d-1} \subset I_-^{d-1}$, so $H_3 \cap G_+ \subset I_+^{d-1}$. Define $\delta =_{\text{def}} \text{dist}(e_{d-1} + \Delta^{d-1}, G)$; since the two sets involved in this definition are disjoint, $e_{d-1} + \Delta^{d-1}$ is compact and G is closed, $\delta > 0$. Let $\varepsilon > 0$ satisfy

$$\frac{\varepsilon}{\delta} < \frac{2}{M+1}. \quad (12)$$

The difference between F_0^{d-2} and F_1^{d-2} in the d -axis direction is 2, and their difference in the $(d-1)$ -axis direction is $M+1$. It follows that if ε satisfies (12), then $a_2(\varepsilon) + \Delta^{d-1} \subset I_+^{d-1}$, and Condition (11) is met by such an ε . So define $\varepsilon =_{\text{def}} \frac{\delta}{M+1}$ ($= \frac{1}{2} \cdot \frac{2\delta}{M+1}$). With this ε Requirement 4.1 is met. It remains to determine L .

Requirement 4.2 (Largeness of L) $L > 0$ is so large that

$$\bigcup_{v=2}^m Q_v \subset Q_1. \quad (13)$$

The existence of such an L can be easily verified by a glance on Fig. 5. For a formal proof put

$$\begin{aligned} A &=_{\text{def}} \text{conv}((e_{d-1} + e_d + \Delta^{d-1}) \cup (-e_{d-1} + e_d + \Delta^{d-1})), \\ B &=_{\text{def}} \text{conv}(A \cup ((m-1)e_{d+1} + A)), \quad \text{and} \\ \delta_1 &=_{\text{def}} \text{dist}(A, \text{relbd}(e_d + M \cdot \Delta^{d-1})). \end{aligned}$$

A is a $(d-1)$ -polytope, B is a right d -prism of height $m-1$ on A containing $\bigcup_{v=2}^m Q_v$, and $\delta_1 > 0$ since A and $e_d + M \cdot \Delta^{d-1}$ are compact and disjoint ($A \subset \text{relint}(e_d + M \Delta^{d-1})$, by (8)). If $\frac{L}{\text{dist}(o, \text{relbd}(M \cdot \Delta^{d-1}))} > \frac{m-1}{\delta_1}$, then the segment

one edge at least. After this stage there remain to pass through the $m - 1$ “gates” $a_2(\varepsilon) + \Delta^{d-1}, a_3 + \Delta^{d-1}, \dots, a_{m-1} + \Delta^{d-1}$ in this order (see Fig. 5 above) before reaching q_m , and this takes at least $m - 1$ additional edges. The formal proof of this is analogous to the argument given for a similar assertion (Lemma 4.1) in the proof of Proposition 4.1 above, and we skip over the details with just one hint: Requirement 4.1 above (smallness of $\varepsilon > 0$) ensures that it is impossible to reach from $\hat{q}_1 \in \hat{H}^-$ to the “gate” $a_3 + \Delta^{d-1}$ via the “gate” $a_2(\varepsilon) + \Delta^{d-1}$ by just one edge (at least two edges are needed). It follows that a k -gon from q_0 to q_m via $\text{ext } Q$ has at least $1 + (m - 1) = m$ edges. $\square$

This example will be modified now to the remaining case $n \not\equiv 0 \pmod{d}$ as follows. Assume first that $n \equiv 1 \pmod{d}$, i.e., $n = md + 1$ and $\lceil \frac{n}{d} \rceil = m + 1$. Take the (non-convex) body Q described above and truncate its vertex $a_0(L)$ by a hyperplane H' parallel to H (i.e., parallel to $\text{span}(e_1, \dots, e_{d-1})$) passing close to $a_0(L)$; denote by Q' the resulting body. The boundary $\text{bd } Q'$ of Q' is the body of a polyhedral $(d - 1)$ -complex, henceforth denoted by Q' , having $n = md + 1$ facets.

Claim 4.3 $\text{poldiam } Q' = m + 1 (= \lceil \frac{n}{d} \rceil)$.

Proof of Claim Denote by F' the new (horizontal) facet obtained by truncating $a_0(L)$ from Q by H' , let q' be a point in $\text{ext } Q'$ near the barycenter of F' (q'_0 lies “above” F') and let $q \in (\text{int } B^d(a_m, 1) \cap \text{int } Q_m)$ (a 1-vicinity of a_m in $\text{ext } Q'$).

In order to reach from q_0 to q_m by a polygonal path via $\text{ext } Q'$, it is necessary to reach first a point in $\hat{H}^-$ in order to enter the zigzag cave $\bigcup_{v=2}^m Q_v$, and this can be done by not less than two edges, because q_0 lying near and above the barycenter of F' does not “see” directly any point of $\hat{H}^-$ via $\text{ext } Q'$. After reaching a point in $\hat{H}^-$ there remain the $m - 1$ “gates” $a_2(\varepsilon) + \Delta^{d-1}, a_3 + \Delta^{d-1}, \dots, a_{m-1} + \Delta^{d-1}$ to pass through in order to reach q_m , and this takes at least $m - 1$ additional edges. Thus, in analogy with Proposition 4.2 above, it is possible to reach from q_0 to q_m by a polygonal path of $2 + m - 1 = m + 1 = \lceil \frac{n}{d} \rceil$ edges via $\text{ext } Q'$, but not by less than this. This proves Claim 4.3. $\square$

As to the remaining case $n \equiv l \pmod{d}$, $2 \leq l \leq m - 1$, truncate any $l - 1$ vertices of the facet F' of Q' by $l - 1$ hyperplanes passing every one very close to the corresponding truncated vertex. The (non-convex) body $Q^{(l)}$ obtained has $n = md + l$ facets and $\text{poldiam}(\text{ext } Q^{(l)}) = m + 1 = \lceil \frac{n}{d} \rceil$.

This finishes Example 4.2.

5 Two 2-PPMs in $\mathbb{R}^3$ Whose Interiors Cannot Be Tetrahedrized Without Additional Vertices

As said in Remark 1.1, part (5) above, the interior of a simple closed polygon in the plane can be triangulated without additional vertices. The following two examples show that this property does not generalize to a $(d - 1)$ -PPM in $\mathbb{R}^d$ for $d \geq 3$.

Fig. 6 The six basic rectangular panels of Example 5.2

Example 5.1 The first example is the famous Schönhardt's twisted prism over a triangle, which is a simplicial 2-manifold $\mathcal{C}$ in $\mathbb{R}^3$ with six vertices combinatorially equivalent to the boundary complex of the 3-octahedron, first described in [28]. It is obtained from a 3-prism on a triangular basis by a (small), say clockwise, twist of one of the two bases. The three quadrilateral facets are thus broken at their three mutually disjoint diagonals and “bent in” so that the three dihedral angles at these three diagonals become concave (i.e., $> \pi$). The relative interiors of the other three diagonals are thus “pushed out” to lie entirely in $\text{ext } \mathcal{C}$. Clearly, any tetrahedron whose vertices are in $\text{vert } \mathcal{C}$ must have as an edge one of these three “outside diagonals”. It follows that $\text{int } \mathcal{C} \cup \mathcal{C}$ cannot be tetrahedrized without additional vertices. This example was recently generalized in [27] to a twisted prism over any (convex) polygon, and much earlier it was generalized by [4] in a different, yet very simple direction. In [27], one can also find further discussion and references on this subject.

Let us present now another example which turns out to be combinatorially equivalent to the so called rhombicuboctahedron. Later on we shall see that, although Schönhardt's polyhedron is much simpler than our example below, this example is perhaps not devoid of interest because of another tetrahedrization-obstructing property it has over Schönhardt's and its generalizations described above.

Example 5.2 The idea is to put some mutually disjoint 2-dimensional rectangular panels $M_1, \dots, M_k$ in $\mathbb{R}^3$ such that $o \in \mathbb{R}^3 \setminus \bigcup_{i=1}^k M_i$ and every segment $[o, v]$, where v is a vertex of some panel M_i ($1 \leq i \leq k$), meets another panel M_j ($1 \leq j \leq k$, $j \neq i$), i.e., o does not “see” any vertex of $\bigcup_{i=1}^k M_i$, and then to complete $\{M_1, \dots, M_k\}$ to a 2-PPM $\mathcal{C}$ by adding facets without additional vertices (i.e., $\text{vert } \mathcal{C} = \bigcup_{i=1}^k \text{vert } M_i$) so that $o \in \text{int } \mathcal{C}$. The interior $\text{int } \mathcal{C}$ of such a 2-PPM cannot be tetrahedrized without additional vertices since a tetrahedron whose vertices are in $\text{vert } \mathcal{C}$ and whose edges do not meet any panel M_i misses o . Here we take six congruent rectangular panels $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ of size 1×3 in parallel to the coordinates planes, defined as follows (cf. Fig. 6).

Let $\varepsilon > 0$ be small, define $x_{11} = (1\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} + \varepsilon)$, $x_{12} = (1\frac{1}{2}, \frac{1}{2}, \frac{1}{2} + \varepsilon)$, $x_{13} = (-1\frac{1}{2}, \frac{1}{2}, \frac{1}{2} + \varepsilon)$, $x_{14} = (-1\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} + \varepsilon)$, and put $X_1 =_{\text{def}} \text{conv}\{x_{1i} : 1 \leq i \leq 4\}$.

X_1 is a rectangle of size 1×3 parallel to the (x, y) -plane at z -distance $\frac{1}{2} + \varepsilon$ from it, whose long (resp., short) edges run in parallel to the x -axis (resp., y -axis). Let X_2 be the rectangle whose vertices x_{2i} , $1 \leq i \leq 4$, are obtained from the vertices x_{1i} , $1 \leq i \leq 4$, merely by negation of the z -coordinate from $\frac{1}{2} + \varepsilon$ to $-\frac{1}{2} - \varepsilon$. X_2 is a translate of X_1 by $-1 - 2\varepsilon$ in direction parallel to the z -axis, hence the z -distance between X_1 and X_2 is $1 + 2\varepsilon$.

Let Y_1 (resp., Y_2) be the rectangle obtained from X_1 (resp., X_2) by the cyclic permutation of the coordinates $(x, y, z) \rightarrow (y, z, x)$ (a rotation of 120° around the line $\text{aff}((0, 0, 0), (1, 1, 1)))$, and let Z_1 (resp., Z_2) be the rectangle obtained from X_1 (resp., X_2) by the cyclic permutation of the coordinates $(x, y, z) \rightarrow (z, x, y)$. We label the vertices of Y_1 (resp., Y_2) by y_{1i} (resp., y_{2i}) ($1 \leq i \leq 4$) and those of Z_1 (resp., Z_2) by z_{1i} (resp., z_{2i}) ($1 \leq i \leq 4$), as depicted in Fig. 6. The six rectangular panels obtained are mutually disjoint. Clearly, if $\varepsilon > 0$ is small enough (say $\varepsilon = \frac{1}{10}$), then the origin is “hidden” from all their 24 ($= 6 \times 4$) vertices x_{ij} , y_{ij} , z_{ij} ($1 \leq i \leq 2$, $1 \leq j \leq 4$), i.e., every segment between o and one of these vertices meets some panel in its relative interior. The standard unit ($1 \times 1 \times 1$) cube Q of $\mathbb{R}^3$ centered at o depicted in Fig. 6 by broken lines is situated between every pair of parallel panels, and each facet of Q is at distance ε apart from the closest panel. Note that $Y_1 \cup Y_2 \subset \text{conv}(\text{aff } X_1 \cup \text{aff } X_2)$, $X_1 \cup X_2 \subset \text{conv}(\text{aff } Z_1 \cup \text{aff } Z_2)$, and $Z_1 \cup Z_2 \subset \text{conv}(\text{aff } Y_1 \cup \text{aff } Y_2)$. Let $M_1, M_2 \in \{X_1, X_2, Y_1, Y_2, Z_1, Z_2\}$ be two non-parallel panels. To every short edge m_1 of M_1 corresponds the long edge m_2 of M_2 which is closest to it, and vice versa, m_1 is the short edge of M_1 which is closest to m_2 . The lines $\text{aff } m_1$, $\text{aff } m_2$ are parallel, hence $\text{conv}(m_1 \cup m_2)$ is an isosceles trapezoid whose bases are of lengths 1 and 3. While the couple (M_1, M_2) ranges over all couples of non-parallel panels and (m_1, m_2) ranges over all couples of such (short-long) edges, we obtain $6 \cdot 2 = 12$ congruent (spatial) isosceles trapezoids, such as (cf. Fig. 6) $z_{13}z_{12}x_{11}x_{14}$, $x_{21}x_{22}y_{11}y_{14}$, $y_{11}y_{12}z_{21}z_{22}$, etc. Finally define eight triangles as follows. To each vertex v of the central cube Q correspond three of the panels X_i, Y_i, Z_i ($1 \leq i \leq 2$) which are closest to it – each at distance ε from v ; on each of these three panels there is a vertex which is closest to v . The convex hull of these three vertices is a triangle T_v that corresponds to v , e.g., $x_{11}z_{11}y_{13}$ is such a triangle (Fig. 6). The 26 panels thus defined (6 rectangular, 12 trapezoidal and 8 triangular) form the set of facets of a polyhedral 2-pseudomanifold $\mathcal{C}$ in $\mathbb{R}^3$ satisfying $f_0(\mathcal{C}) = 24$, $f_1(\mathcal{C}) = 6 \cdot 4 + 3 \cdot 8 = 48$, and $f_2(\mathcal{C}) = 26$. Note that $f_0(\mathcal{C}) - f_1(\mathcal{C}) + f_2(\mathcal{C}) = 2$, hence $\mathcal{C}$ is spherical. In fact, $\mathcal{C}$ is combinatorially equivalent to the face complex of the rhombicuboctahedron (an Archimedean solid) as one can easily check. Since $o \in \text{int } \mathcal{C}$ and o is “hidden” from every vertex of $\mathcal{C}$ (by some rectangular panel) $\text{int } \mathcal{C}$ cannot be tetrahedrized without additional vertices. End of Example 5.2.

The 2-PPMs $\mathcal{C}$ just described have the following tetrahedrization-obstructing property not shared by Schönhardt’s 2-PPM and its generalizations:

Property A *There is a point $p \in \text{int } \mathcal{C}$ (in our case $p = o$) such that p does not see any vertex of $\mathcal{C}$ via $\text{int } \mathcal{C}$. In other words, p is “hidden” from every vertex of $\mathcal{C}$ by some facet of $\mathcal{C}$.*

Property A implies that every tetrahedron with vertices in $\text{vert } \mathcal{C}$ misses p or intersects $\text{ext } \mathcal{C}$, hence $\text{int } \mathcal{C} \cup \mathcal{C}$ cannot be tetrahedrized without additional vertices. Schönhardt's 2-PPM (and its generalizations) show that this does not go vice versa: it cannot be tetrahedrized without additional vertices, and yet every point in its interior sees some vertex via the interior (as the reader may easily check). In view of Example 5.2 above, the following questions arise:

Question 5.1

- (a) Is there a 2-PPM $\mathcal{C}$ in $\mathbb{R}^3$ having Property A with $\#(\text{vert } \mathcal{C}) < 24$?
- (b) If there is such a 2-PPM, what is the minimal number of its vertices?

6 The Role Played by Jordan's Separation Theorem for Simple Closed Polygons in the Proof of the Eulerian Relation $v - e + f = 2$ for Planar Graphs

As is well known, a connected planar graph $G = \langle V, E \rangle$ embedded in the plane with $v =_{\text{def}} \#V$ vertices and $e =_{\text{def}} \#E$ edges, whose complement $\mathbb{R}^2 \setminus |E|$ has f connectivity components ("domains"), satisfies the Eulerian relation

$$v - e + f = 2. \quad (15)$$

It seems to be much less known that a (rigorous) proof of (15) demands in an *essential* way the full statement of Jordan's Curve Theorem and its variant: a tree embedded in the plane does not separate it. In fact, of the many proofs of (15) encountered in the literature (notably textbooks in discrete mathematics, which are supposed to be rigorous), usually using induction on e , only a small fraction mentions Jordan's theorem at all, and even these mostly just pay lips service to the theorem, deliberately overlooking some delicate points. It is therefore desirable to prove (15) first for the relatively simple case where G is embedded *piecewise linearly* in the plane, because in this case Jordan's theorem is particularly simple to prove, as mentioned in the introduction ("raindrop" proof). As far as we could check, no such treatment appears in the literature. The main stages for such a treatment may go as follows:

- (a) Note first that, when G is a simple closed polygon, (15) reduces to Jordan's theorem.
- (b) Prove that a tree T (piecewise linearly) embedded in the plane does not separate the plane. The proof is by induction on the numbers of edges of T . This proves (15) for the case that G is a tree.
- (c) Let T be a spanning tree of G . Use (b) as an initial induction step (induction on e), and then "put back" the edges of G not appearing in T one by one.
- (d) Use Jordan's theorem for simple closed polygons in the plane to prove that at each stage an edge is added f increases (exactly) by 1, thus keeping (15) in balance.

We may come to this topic in a subsequent paper [26]. What is the natural generalization of (15) for polyhedral $(d - 1)$ -complexes (piecewise linearly embedded) in $\mathbb{R}^d$?

Guided by the model given above for the proof of (15) it seems that the Jordan–Brouwer Separation Theorem for $(d - 1)$ -PPMs enunciated in Theorem 1.1 above will have some role in proving such a generalization.

References

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