Three-Dimensional Polyhedra Can Be Described by Three Polynomial Inequalities

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Abstract Bosse et al. conjectured that for every natural number $d \geq 2$ and every d -dimensional polytope P in $\mathbb{R}^d$, there exist d polynomials $p_1(x), \dots, p_d(x)$ satisfying $P = \{x \in \mathbb{R}^d : p_1(x) \geq 0, \dots, p_d(x) \geq 0\}$. We show that every three-dimensional polyhedron can be described by three polynomial inequalities, which confirms the conjecture for the case $d = 3$ but also provides an analogous statement for the case of unbounded polyhedra. The proof of our result is constructive.

Keywords Łojasiewicz's inequality · Polynomial · Polytope · Semi-algebraic set · Theorem of Bröcker and Scheiderer

1 Introduction

A subset S of $\mathbb{R}^d$ ($d \in \mathbb{N}$) is said to be an *elementary closed semi-algebraic set* in $\mathbb{R}^d$ if

$$S = (q_1, \dots, q_m)_{\geq 0} := \{x \in \mathbb{R}^d : q_1(x) \geq 0, \dots, q_m(x) \geq 0\} \quad (1.1)$$

for some $m \in \mathbb{N}$ and polynomials $q_1(x), \dots, q_m(x)$ over $\mathbb{R}$. Every elementary closed semi-algebraic set S in $\mathbb{R}^d$ can be represented by $d(d+1)/2$ polynomial inequalities, i.e., there exist polynomials $p_1(x), \dots, p_n(x)$ with $n \leq d(d+1)/2$ such that

$$S = (p_1, \dots, p_n)_{\geq 0}. \quad (1.2)$$

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This follows from the well-known result of Bröcker and Scheiderer; see [4, Sects. 6.5 and 10.4], [1, p. 143]. However, the arguments of Bröcker and Scheiderer are highly nonconstructive, so that not much is known about the relationship of the polynomials $q_1(x), \dots, q_m(x)$ and $p_1(x), \dots, p_n(x)$. One can consider the following more specific problem. Given a special class $\mathcal{S}$ of elementary closed semi-algebraic sets in $\mathbb{R}^d$, find the minimal n such that every semi-algebraic set S from $\mathcal{S}$ can be represented by n polynomials $p_1(x), \dots, p_n(x)$ and, moreover, find algorithms for determination of $p_1(x), \dots, p_n(x)$ from $q_1(x), \dots, q_m(x)$; see also [3, 9].

Results on representations of semi-algebraic sets are relevant for several research areas including polynomial and semi-definite optimization; see [10, 11]. In this manuscript we are concerned with polynomial representations of polyhedra. A nonempty subset P of $\mathbb{R}^d$ is said to be a *polyhedron* if P is the intersection of a finite number of closed halfspaces; see [8, 15]. Furthermore, a nonempty set P is called a *polytope* if it is a convex hull of a finite point set in $\mathbb{R}^d$. It is well known that polytopes can be characterized as bounded polyhedra. The main result of the paper is the following theorem.

Theorem 1.1 *Let $d \in \{2, 3\}$. Then for every d -dimensional polyhedron P in $\mathbb{R}^d$, there exist d polynomials $p_0(x), \dots, p_{d-1}(x) \in \mathbb{R}[x]$ with $P = (p_0, \dots, p_{d-1})_{\geq 0}$.*

Bosse, Grötschel, and Henk [6] conjectured that, for every $d \geq 2$, d polynomial inequalities suffice for representing a d -dimensional polytope. Theorem 1.1 confirms the above-mentioned conjecture for $d \leq 3$. The case where $d = 2$ and P is bounded is essentially due to Bernig; see [5]. For unbounded P , Theorem 1.1 is new for both $d = 2$ and $d = 3$. The conjecture was confirmed for simple polytopes of any dimension and a certain class of semi-algebraic sets containing all simple polytopes; see [2, 3]. After completing the proof of Theorem 1.1, Ludwig Bröcker communicated to the authors a complete positive solution of the above-mentioned conjecture; cf. [7]. We point out that the arguments of Bröcker and our Proposition 7.3 also imply that all d -polyhedra (bounded and unbounded) can be described by d polynomial inequalities.

We emphasize that the proof of Theorem 1.1 is constructive in the sense that we indicate explicit formulas for $p_0, \dots, p_{d-1}$ (for more details, see also Sect. 2 below). However our construction has the undesirable property that the maximal degree of the polynomials $p_0, \dots, p_{d-1}$ constructed in the proof depends not only on the combinatorial structure of P (say, the number of facets and the dimension) but also on its metric structure (angles between faces and distances between vertices); furthermore we were also unable to find explicit upper bounds on the maximal degree of $p_0, \dots, p_{d-1}$. It remains open whether Theorem 1.1 can be strengthened by requiring that the degrees of $p_0, \dots, p_{d-1}$ are at most n with n depending only on the dimension and the number of facets of P . It should be noticed however that such bounds are possible if we allow somewhat more than d polynomials in the representation of P . For example, as a consequence of the theorem of Bröcker and Scheiderer (which is valid for an arbitrary real closed field), in fact, one can show that, for every d -polytope P in $\mathbb{R}^d$, there exist $p_1, \dots, p_{d(d+1)/2} \in \mathbb{R}[x]$ such that $P = (p_1, \dots, p_{d(d+1)/2})_{\geq 0}$ and the degrees of $p_1, \dots, p_{d(d+1)/2}$ are at most n with n depending only on d and the

number of facets of P . This follows directly from the fact that the theorem of Bröcker and Scheiderer holds in an arbitrary real closed field (see [4, Chap. 1] for the definition of real closed fields) and from Theorem 5.1 in [13]. Analogously, if one could extend Theorem 1.1 to polytopes in an arbitrary real closed field, as a corollary, one would immediately get a strengthening of Theorem 1.1 with combinatorial bounds on the maximal degree of $p_0, \dots, p_{d-1}$. Thus, we also ask whether Theorem 1.1 can be extended to arbitrary real closed fields.

The paper is organized as follows. Section 2 contains a sketch of the construction of the polynomials $p_0(x), \dots, p_{d-1}(x)$ from Theorem 1.1 and indicates some proof ideas. Sections 3 and 4 present preliminaries from real algebraic geometry and convex geometry, respectively. In Sect. 5 we introduce and study a special family of neighborhoods of a given convex polytope, which is used in the main proof. Section 6 is concerned with approximation of polytopes and polyhedral cones by algebraic surfaces. In Sect. 7 we prove Theorem 1.1.

2 A Sketch of the Construction

We give a sketch of the construction of the polynomials $p_0(x), \dots, p_{d-1}(x)$ from Theorem 1.1. Standard notions from convexity that are used in this section are introduced in Sect. 3 or can be found in the monographs [8, 14, 15].

Let $d \in \{2, 3\}$ and P be a d -dimensional polyhedron in $\mathbb{R}^d$. Assume that P is bounded. Since the two-dimensional case was settled by Bernig [5], we may assume that $d = 3$. Let m be the number of facets of P . Then $P = \{x \in \mathbb{R}^d : q_1(x) \geq 0, \dots, q_m(x) \geq 0\}$ for appropriate affine functions $q_1(x), \dots, q_m(x)$. We define

$$p_2(x) := q_1(x) \cdot \dots \cdot q_m(x).$$

Further on, we construct a strictly concave polynomial $p_0(x)$ vanishing on each vertex of P and such that the distance between P and $(p_0)_{\geq 0}$ is small enough; see also Fig. 1.¹ With each vertex v of P we associate a polynomial $b_v(x)$ such that

Fig. 1 Regular octahedron and an approximation of the octahedron by a strictly concave polynomial surface

¹The figures in this subsection illustrate the construction for the case of the regular octahedron $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_1| + |\xi_2| + |\xi_3| \leq 1\}$. Since octahedra are not simple polytopes, they are not covered by the main result of [2] on representation of simple polytopes by d polynomial inequalities.

Fig. 2 Polyhedron P and the surface given by the equation $b_v(x) = 0$

Fig. 3 The surface given by the equation $f_v(x)^{2k} b_v(x) = 0$

$(b_v)_{\geq 0} = B(v) \cup (2v - B(v))$, where $B(v)$ is a pointed convex cone with apex at v such that every edge of P incident with v lies in the boundary of $B(v)$; see Fig. 2. We define $p_1(x)$ as a “weighted combination” of the polynomials $b_v(x)$ with v ranging over the set of all vertices of P . More precisely, we set

$$p_1(x) := \sum_v f_v(x)^{2k} b_v(x),$$

where v ranges over all vertices of P , $k \in \mathbb{N}$ is sufficiently large, and, for every vertex v of P , $f_v(x)$ is the polynomial given by

$$f_v(x) := \prod_{\substack{i=1, \dots, m \\ q_i(v) \neq 0}} q_i(x),$$

see also Fig. 3.

By construction $P \subseteq (p_0, p_1, p_2)_{\geq 0}$. We have $p_0(x) < 0$ for all x sufficiently far away from P . Further on, the region of all x satisfying $p_2(x) < 0$ is contiguous with all edges of P . Finally, if G is a vertex or an edge of P , x_0 is a point in the relative interior of G , and v and w are two vertices of P with $v \in G$ and $w \notin G$, respectively, then, for x satisfying $p_2(x) \geq 0$ and converging to x_0 , the term $f_v(x)^{2k} b_v(x)$ dominates over $f_w(x)^{2k} b_w(x)$, provided that k is sufficiently large. The latter observation is used for analysis of the properties of $(p_1)_{\geq 0}$.

The geometry behind the construction is illustrated by the diagram in Fig. 4, where $P_J := \{x \in \mathbb{R}^3 : p_j(x) \geq 0 \forall j \in J\}$ for $J \subseteq \{1, 2, 3\}$ with $J \neq \emptyset$, and an arrow between two sets indicates the inclusion relation.

The case of unbounded d -polyhedra P can be reduced to the case of bounded ones using some “homogenization” idea; for details, see Sect. 7.2.

Fig. 4 Diagram illustrating representation of a three-dimensional polytope by three polynomial inequalities

3 Preliminaries from Real Algebraic Geometry

For information on real algebraic geometry and, in particular, the geometry of semi-algebraic sets, we refer to [1, 4]. If x is a variable in $\mathbb{R}^k$ ($k \in \mathbb{N}$), then $\mathbb{R}[x]$ stands for the class of k -variate polynomials over $\mathbb{R}$. Since $\mathbb{R}$ is a field of characteristic zero, we do not distinguish between real polynomials and real-valued polynomial functions. In particular, we can view affine functions as polynomials of degree at most one. A subset A of $\mathbb{R}^d$ is said to be *semi-algebraic* if

$$A = \bigcup_{i=1}^n \{x \in \mathbb{R}^d : f_{i,1}(x) > 0, \dots, f_{i,s_i}(x) > 0, g_i(x) = 0\}$$

for some $n, s_1, \dots, s_n \in \mathbb{N}$ and $f_{i,j}(x), g_i(x) \in \mathbb{R}[x]$ with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, s_i\}$.

A real-valued function $f(x)$ defined on a semi-algebraic set $A \subseteq \mathbb{R}^d$ is said to be a *semi-algebraic function* if the graph of $f(x)$ is a semi-algebraic subset of $\mathbb{R}^{d+1}$. The following theorem presents Łojasiewicz's Inequality; see [12], [4, Corollary 2.6.7].

Theorem 3.1 ([12]) *Let A be a bounded and closed semi-algebraic set in $\mathbb{R}^d$. Let $f(x)$ and $g(x)$ be continuous, semi-algebraic functions on A satisfying*

$$\{x \in A : f(x) = 0\} \subseteq \{x \in A : g(x) = 0\}.$$

Then there exist $n \in \mathbb{N}$ and $\lambda > 0$ such that $|g(x)|^n \leq \lambda |f(x)|$ for every $x \in A$.

4 Preliminaries from Convexity

For information on convex bodies and polytopes, we refer to [8, 14, 15]. The cardinality of a set is denoted by $|\cdot|$. The origin, Euclidean norm, and scalar product in $\mathbb{R}^d$ are denoted by o , $\|\cdot\|$, and $\langle \cdot, \cdot \rangle$, respectively. We endow $\mathbb{R}^d$ with its Euclidean topology. By $\mathbb{B}^d(c, \rho)$ we denote the closed Euclidean ball in $\mathbb{R}^d$ with center at $c \in \mathbb{R}^d$ and radius $\rho > 0$. The notations *aff*, *conv*, *int*, and *relint* stand for the affine hull, convex hull, interior, and relative interior, respectively. If X and Y are nonempty sets in $\mathbb{R}^d$, we set $X \pm Y := \{x \pm y : x \in X, y \in Y\}$. The *Hausdorff distance* $\text{dist}(X, Y)$ between nonempty compact sets $X, Y \subseteq \mathbb{R}^d$ is defined by

$$\text{dist}(X, Y) := \min\{\rho \geq 0 : X \subseteq Y + \mathbb{B}^d(o, \rho), Y \subseteq X + \mathbb{B}^d(o, \rho)\}$$

see also [14, p. 48]. It is known that $\text{dist}(X, Y)$ is a metric on the space of nonempty compacts sets in $\mathbb{R}^d$.

If for a convex set $X \subseteq \mathbb{R}^d$ and a point $x_0 \in \mathbb{R}^d$, one has $x_0 + \alpha(x - x_0) \in X$ for every $\alpha \geq 0$ and $x \in X$, then X is said to be a *convex cone* with *apex* at x_0 . Given a nonempty subset X of $\mathbb{R}^d$, we introduce the *conical hull* $\text{cone}(X)$ of X as the set of all possible linear combinations $\lambda_1 x_1 + \cdots + \lambda_m x_m$ with $m \in \mathbb{N}$, $\lambda_i \geq 0$, and $x_i \in X$ for $i \in \{1, \dots, m\}$. Clearly, $\text{cone}(X)$ is a convex cone with apex at the origin.

The notions of polyhedron and polytope were defined in the introduction. Polyhedra (resp. polytopes) of dimension n are referred to as *n-polyhedra* (resp. *n-polytopes*). A subset of $\mathbb{R}^d$ which is both a cone and polyhedron is said to be a *polyhedral cone*. A polyhedron is said to be *line-free* if it does not contain a straight line. The set $\text{rec}(P) := \{u \in \mathbb{R}^d : P + u \subseteq P\}$ is said to be the *recession cone* of P . A line-free polyhedral cone is said to be *pointed*. Every polyhedron P can be represented by

$$P = Q + L + C, \tag{4.1}$$

where Q is a polytope, L is a linear subspace of $\mathbb{R}^d$, and C is a pointed polyhedral cone; see [15, Sect. 1.5]. For (4.1), one necessarily has $L + C = \text{rec}(P)$. If $o \notin \text{aff } P$, we introduce

$$\text{hom}(P) := \text{cone}(P) \cup \text{rec}(P).$$

It is known that $\text{hom}(P)$ is a polyhedral cone; see [15, pp. 44–45]. Furthermore, it can be verified that if P is line-free, then $\text{hom}(P)$ is pointed.

For a convex polytope P in $\mathbb{R}^d$, we introduce the *support function* and *exposed face* in direction u by the equalities

$$h(P, u) := \max\{\langle x, u \rangle : x \in P\} \quad \text{and} \quad F(P, u) := \{x \in P : \langle x, u \rangle = h(P, u)\},$$

respectively. Given a polytope P in $\mathbb{R}^d$, $\mathcal{F}(P)$ denotes the class of all faces of P , and, for $i \in \{-1, \dots, d\}$, $\mathcal{F}_i(P)$ stands for the class of all i -faces of P (i.e., faces of dimension i). We recall that, for every point x of P , there exists a unique nonempty face G of P with $x \in \text{relint } P$. Consequently, P is the disjoint union of the relative interiors of all nonempty faces of P . A face G of an n -polytope P in $\mathbb{R}^d$ is said to be a *facet* of P if G has dimension $n - 1$. If G is a face of P , we define

$$\mathcal{F}_i(G, P) := \{F \in \mathcal{F}_i(P) : G \subseteq F\}.$$

By $\text{vert}(P)$ we denote the set of all *vertices* of P (i.e., the set of 0-dimensional faces). If F is a facet of P , by $u_F(P)$ we denote the unit normal of P at the facet F which is parallel to $\text{aff}(P)$ and satisfies $F(P, u_F(P)) = F$. More generally, if G is a proper face of P , we define

$$u_G(P) := \left(\sum_{F \in \mathcal{F}_{n-1}(G, P)} u_F(P) \right) / \left\| \sum_{F \in \mathcal{F}_{n-1}(G, P)} u_F(P) \right\|,$$

where $n := \dim(P)$. It is not hard to see that $F(P, u_G(P)) = G$.

With every facet F of P we associate the affine function

$$q_F(P, x) := \frac{h(P, u_F(P)) - \langle u_F(P), x \rangle}{\text{diam}(P)},$$

where $\text{diam}(P) := \max\{\|x - y\| : x, y \in P\} = \max\{\|x - y\| : x, y \in \text{vert}(P)\}$. By construction, $0 \leq q_F(P, x) < 1$ for every $x \in P$ with the equality $q_F(P, x) = 0$ if and only if $x \in F$. For $v \in \text{vert}(P)$, the *supporting cone* $S(P, v)$ of P at v is given by

$$S(P, v) := \text{cone}(P - v),$$

see also [14, p. 70]. If $\dim P = d$, then

$$S(P, v) = \{x \in \mathbb{R}^d : \langle u_F(P), x \rangle \leq 0 \ \forall F \in \mathcal{F}_{d-1}(v, P)\}$$

and by this

$$v + S(P, v) = \{x \in \mathbb{R}^d : q_F(P, x) \leq 0 \ \forall F \in \mathcal{F}_{d-1}(v, P)\}.$$

For every $v \in \text{vert}(P)$, we define the hyperplane

$$H_v(P) := \{x \in \mathbb{R}^d : \langle x, u_v(P) \rangle = -1\}$$

and the polytope

$$P_v := S(P, v) \cap H_v(P).$$

It can be shown that $\text{cone}(P_v) = S(P, v)$. Hence the cone

$$S_\rho(P, v) := \text{cone}((P_v + \mathbb{B}^d(o, \rho)) \cap H_v(P))$$

with $\rho > 0$ and $v \in \text{vert}(P)$ can be viewed as an “outer approximation” of the supporting cone with the parameter ρ controlling the quality of the approximation. For every $v \in \text{vert}(P)$, one has

$$P \setminus \{v\} \subseteq \text{int}(v + S_\rho(P, v)). \quad (4.2)$$

For a polytope P in $\mathbb{R}^d$ and $x \in \text{aff } P$, we introduce the notation

$$\mathcal{F}^-(P, x) := \{F \in \mathcal{F}_{d-1}(P) : q_F(P, x) \leq 0\}.$$

The class $\mathcal{F}^-(P, x)$ be interpreted as the set of all facets of the polytope P which are visible from x .

In Theorem 1.1 and the remaining statements dealing with algorithms, a polyhedron is assumed to be given by a system of affine inequalities (the so-called *H-representation*), and a polynomial by a list of its coefficients. It is not difficult to show that there exists an algorithm that takes a polytope P and constructs all functions $q_F(P, x)$ where F is a facet of P .

5 A Family of Neighborhoods of a 3-polytope

Let $\varepsilon > 0$, $\rho > 0$, and P be a 3-polytope in $\mathbb{R}^3$. We set $U_P(P, \varepsilon, \rho) := P$ and $U_F(P, \varepsilon, \rho) := \{x \in \mathbb{R}^3 : \mathcal{F}^-(P, x) = \{F\}\}$ for $F \in \mathcal{F}_2(P)$. Furthermore, for $I \in \mathcal{F}_1(P)$ and $v \in \text{vert}(P)$, we define

$$U_I(P, \varepsilon, \rho) := (I + \mathbb{B}^3(o, \varepsilon)) \cap \{x \in \mathbb{R}^3 : \mathcal{F}_2(I, P) \subseteq \mathcal{F}^-(P, x)\} \\ \cap \left(\bigcap_{v \in \text{vert}(P)} S_\rho(P, v) \right), \quad (5.1)$$

$$U_v(P, \varepsilon, \rho) := \mathbb{B}^3(v, \varepsilon) \setminus (v + \text{int } S_\rho(P, v)), \quad (5.2)$$

see also Figs. 5 and 6. We also introduce the region

$$U'_v(P, \varepsilon, \rho) := \mathbb{B}^3(v, \varepsilon) \setminus ((v + \text{int } S_\rho(P, v)) \cup (v - \text{int } S_\rho(P, v))),$$

which is a subset of $U_v(P, \varepsilon, \rho)$; see Fig. 7. With P we associate the set

$$U(P, \varepsilon, \rho) := \bigcup_{G \in \mathcal{F}(P) \setminus \{\emptyset\}} U_G(P, \varepsilon, \rho).$$

Fig. 5 The regions $U_I(P, \varepsilon, \rho)$ with $I \in \mathcal{F}_1(P)$

Fig. 6 The regions $U_v(P, \varepsilon, \rho)$ with $v \in \text{vert}(P)$

Fig. 7 The regions $U'_v(P, \varepsilon, \rho)$ with $v \in \text{vert}(P)$

The main statement of this section is Proposition 5.1, whose formulation is intuitively clear. However, we are not aware of any short proof of it.

Proposition 5.1 *Let P be a 3-polytope in $\mathbb{R}^3$. Then $P \subseteq \text{int } U(P, \varepsilon, \rho)$ for all sufficiently small $\rho > 0$ and all $\varepsilon > 0$.*

The proof of Proposition 5.1 is based on a number of auxiliary statements. The proofs of simple Lemmas 5.2, 5.3, and 5.4 are omitted.

Lemma 5.2 *Let P be a d -polytope in $\mathbb{R}^d$, G a nonempty face of P , and $x \in \text{relint } G$. Then there exists $\varepsilon > 0$ such that every $y \in \mathbb{B}^d(x, \varepsilon)$ satisfies $\mathcal{F}^-(P, y) \subseteq \mathcal{F}_{d-1}(G, P)$*

If a point x outside P is sufficiently close to P , then all facets of P visible from x have a vertex in common. This is stated in a more formal way in the following lemma.

Lemma 5.3 *Let P be a d -polytope in $\mathbb{R}^d$. Then for some $\varepsilon > 0$ and every $x \in P + \mathbb{B}^d(o, \varepsilon)$, there exists $v \in \text{vert}(P)$ satisfying $\mathcal{F}^-(P, x) \subseteq \mathcal{F}_{d-1}(v, P)$.*

Lemma 5.4 *Let P be a d -polytope in $\mathbb{R}^d$ with $o \in \text{vert}(P)$, and let $F \in \mathcal{F}_{d-1}(o, P)$. Then $G := \text{aff } F \cap P_o$ is a facet of P_o . Furthermore, for some $\alpha > 0$, one has*

$$q_F(P, x) = \alpha \cdot q_G(P_o, x) \quad \forall x \in H_o(P).$$

Lemma 5.5 *Let P be a 3-polytope in $\mathbb{R}^3$. Then there exists $\rho > 0$ such that, for all $v \in \text{vert}(P)$ and $x \in (v + S_\rho(P, v)) \setminus \{v\}$, one can find $I \in \mathcal{F}_1(v, P)$ satisfying*

$$\mathcal{F}^-(P, x) \cap \mathcal{F}_2(v, P) \subseteq \mathcal{F}_2(I, P). \quad (5.3)$$

Proof Consider an arbitrary $v \in \text{vert}(P)$. Replacing P by an appropriate translation, we assume that $v = o$. By Lemma 5.3 applied to P_o , one can choose $\rho > 0$ such that, for every $x \in H_o(P) \cap (P_o + \mathbb{B}^3(o, \rho))$, there exists $w \in \text{vert}(P_o)$ satisfying $\mathcal{F}^-(P_o, x) \subseteq \mathcal{F}_1(w, P_o)$. Take an arbitrary $x \in S_\rho(P, o) \setminus \{o\}$. Then $\langle x, u_o(P) \rangle < 0$, and we introduce

$$y := -\frac{x}{\langle x, u_o(P) \rangle} \in H_o(P) \cap (P_o + \mathbb{B}^3(o, \rho)).$$

By the choice of ρ , there exists $w \in \text{vert}(P_v)$ with $\mathcal{F}^-(P_o, y) \subseteq \mathcal{F}_1(w, P_o)$. In view of Lemma 5.4, the latter inclusion implies $\mathcal{F}^-(P, x) \cap \mathcal{F}_2(o, P) \subseteq \mathcal{F}_2(I, P)$, where $I \in \mathcal{F}_1(o, P)$ is such that $H_o(P) \cap \text{cone}(I) = \{w\}$. This shows (5.3). $\square$

Now we are ready to prove the main statement of this section.

Proof of Proposition 5.1 We pick an arbitrary $x \in P$ and show that $x \in \text{int } U(P, \varepsilon, \rho)$. By G we denote the unique face of P with $x \in \text{relint } G$. If $G = P$, then $x \in \text{int } P$, and by this $x \in \text{int } U(P, \varepsilon, \rho)$. If G is a facet of P , then, in view of Lemma 5.2, all points sufficiently close to x lie in $P \cup U_G(P, \varepsilon, \rho)$. If G is an edge of P , we show that

$$x \in \text{int} \left(P \cup \left(\bigcup_{F \in \mathcal{F}_2(G, P)} U_F(P, \varepsilon, \rho) \right) \cup U_G(P, \varepsilon, \rho) \right). \quad (5.4)$$

In fact, it is easy to see that, for every $v \in \text{vert}(P)$, one has $x \in v + \text{int } S_\rho(P, v)$ and $x \in \text{int}(G + \mathbb{B}^3(o, \varepsilon))$. Furthermore, applying Lemma 5.2, we obtain

$$x \in \text{int} \{ y \in \mathbb{R}^3 : \mathcal{F}^-(F, y) \subseteq \mathcal{F}_2(G, P) \}.$$

Thus, (5.4) is fulfilled, and by this $x \in \text{int } U(P, \varepsilon, \rho)$.

It remains to consider the case $x = v$ for some $v \in \text{vert}(P)$. We assume that ρ is small enough so that the assertion of Lemma 5.5 is fulfilled. We show that $x \in \text{int } U(P, \varepsilon, \rho)$ by contradiction. Assume that there exists a sequence $(x_n)_{n=1}^\infty$ converging to x and such that $x_n \notin U(P, \varepsilon, \rho)$ for every $n \in \mathbb{N}$. Take an arbitrary $\mathcal{F}^* \subseteq \mathcal{F}_2(P)$ such that $\mathbb{N}^* := \{n \in \mathbb{N} : \mathcal{F}^-(P, x_n) = \mathcal{F}^*\}$ is infinite. By Lemma 5.2, for all sufficiently large $n \in \mathbb{N}$, one has $\mathcal{F}^-(P, x_n) \subseteq \mathcal{F}_2(v, P)$. Hence $\mathcal{F}^* \subseteq \mathcal{F}_2(v, P)$. If $|\mathcal{F}^*| = 1$, then $x_n \in U(P, \varepsilon, \rho)$ for every $n \in \mathbb{N}^*$. If $|\mathcal{F}^*| \geq 3$, then for all sufficiently large n , one has $x_n \notin v + S_\rho(P, v)$, and by this $x_n \in \mathbb{B}^3(v, \varepsilon) \setminus (v + \text{int } S_\rho(P, v)) = U_v(\varepsilon, \rho)$. In fact, if we assume the contrary, i.e., $x_n \in v + S_\rho(P, v)$ for all sufficiently large $n \in \mathbb{N}^*$, then, by (5.3), we see that for some $I \in \mathcal{F}_1(v, P)$, one has $\mathcal{F}^* \cap \mathcal{F}_2(v, P) \subseteq \mathcal{F}_2(I, P)$. In view of $\mathcal{F}^* \subseteq \mathcal{F}_2(v, P)$, the latter implies $\mathcal{F}^* \subseteq \mathcal{F}_2(I, P)$. Hence $|\mathcal{F}^*| \leq 2$, a contradiction. We consider the remaining case $|\mathcal{F}^*| = 2$. In view of (4.2), one has $x_n \in S_\rho(P, w)$ for all sufficiently large n and

every $w \in \text{vert}(P) \setminus \{v\}$. It is easy to see that, for all sufficiently large n , one has $x_n \in \mathbb{B}^3(v, \varepsilon)$. Hence, for all large n , one has $x_n \in U_v(P, \varepsilon, \rho)$ if $x \notin v + \text{int } S_\rho(P, v)$, and, taking into account the assertion of Lemma 5.5 and the definition of $U_I(P, \varepsilon, \rho)$, one has $x_n \in U_I(P, \varepsilon, \rho)$ for some $I \in \mathcal{F}_1(v, P)$, provided that $x \in v + S_\rho(P, v)$. Consequently, for all large n , one has $x_n \in U(P, \varepsilon, \rho)$. This yields the assertion of the proposition. $\square$

6 Approximation of Polytopes and Polyhedral Cones

A polynomial $f \in \mathbb{R}[x]$ of degree n is *homogeneous* if $f(\alpha x) = \alpha^n f(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Consider a hyperplane H in $\mathbb{R}^d$ with $o \notin H$. Then

$$H = \{x \in \mathbb{R}^d : \langle x, u \rangle = 1\} \quad (6.1)$$

with $u \in \mathbb{R}^d \setminus \{o\}$ uniquely determined by H . Consider a polynomial $f \in \mathbb{R}[x]$ of degree n . The *homogeneous continuation* of $f(x)|_H$ (i.e., the restriction of $f(x)$ to H) to $\mathbb{R}^d$ is defined as the polynomial $\tilde{f} \in \mathbb{R}[x]$ uniquely determined by $\tilde{f}(x) = \langle x, u \rangle^n f(x / \langle x, u \rangle)$ with $x \in \mathbb{R}^d \setminus H$. Clearly, $\tilde{f}$ is a homogeneous polynomial.

We say that there is an algorithm that constructs a sequence $(p_l(x))_{l=1}^\infty$ of polynomials from the given input if there exists an algorithm which constructs $p_l(x)$ from $l \in \mathbb{N}$ and the given input. Theorem 6.1 and Proposition 6.3 are the main statements of this section. The former is concerned with a special type of approximation of polytopes, and the latter with approximation of supporting cones of polytopes. Given a polytope P in $\mathbb{R}^d$, we introduce the following conditions on a sequence $(g_l(x))_{l=1}^\infty$ of polynomials from $\mathbb{R}[x]$.

$\mathcal{A}(P)$: The Hausdorff distance from P to $\{x \in \text{aff } P : g_l(x) \geq 0\}$ converges to zero as $l \rightarrow \infty$.

$\mathcal{I}(P)$: For every $v \in \text{vert}(P)$ and $l \in \mathbb{N}$, one has $g_l(v) = 0$.

$\mathcal{C}(P)$: For every $l \in \mathbb{N}$, the function $g_l(x)|_{\text{aff } P}$ is strictly concave.

$\mathcal{H}(P)$: For every $l \in \mathbb{N}$, one has

$$\{x \in \mathbb{R}^d : \tilde{g}_l(x) \geq 0, \langle x, u \rangle \geq 0\} = \text{cone}(\{x \in \text{aff } P : g_l(x) \geq 0\}),$$

where $\tilde{g}_l(x)$ is a homogeneous continuation of $g_l(x)|_{\text{aff } P}$.

The notations $\mathcal{A}$, $\mathcal{I}$, $\mathcal{C}$, and $\mathcal{H}$ are derived from the words “approximation”, “interpolation”, “concavity”, and “homogeneity”, respectively. Condition $\mathcal{H}(P)$ makes sense only for the case where $o \notin \text{aff } P$.

Theorem 6.1 *Let P be a polytope in $\mathbb{R}^d$. Then the following statements hold true.*

- I. *There exists a sequence $(g_l(x))_{l=1}^\infty$ of polynomials from $\mathbb{R}[x]$ that satisfy $\mathcal{A}(P)$, $\mathcal{I}(P)$, $\mathcal{C}(P)$.*
- II. *For every sequence $(g_l(x))_{l=1}^\infty$ satisfying $\mathcal{A}(P)$, $\mathcal{I}(P)$, $\mathcal{C}(P)$, there exists $\rho > 0$ with*

$$(v - S_\rho(P, v)) \cap \{x \in \text{aff } P : g_l(P, x) \geq 0\} = \{v\} \quad \forall v \in \text{vert}(P) \quad \forall l \in \mathbb{N}. \quad (6.2)$$

III. If $\dim(P) = d - 1$ and $o \notin \text{aff } P$, there exists a sequence $(g_l(x))_{l=1}^\infty$ of polynomials from $\mathbb{R}[x]$ that satisfy $\mathcal{A}(P)$, $\mathcal{I}(P)$, $\mathcal{C}(P)$, $\mathcal{H}(P)$.

Proof Let $n := \dim(P)$.

Part I. For $l \in \mathbb{N}$, we define $g_l(x)$ by

$$g_l(x) := 1 - \sum_{v \in \text{vert}(P)} y_{v,l} \left(\frac{1}{|\mathcal{F}_{n-1}(v, P)|} \sum_{F \in \mathcal{F}_{n-1}(v, P)} (1 - q_F(P, x))^{2(l+l_0)} \right)^{2(l+l_0)}, \quad (6.3)$$

where $l_0 \in \mathbb{N}$ and $y_{v,l} \in \mathbb{R}$. In [2] it was shown that an appropriate $l_0 \in \mathbb{N}$ and scalars $y_{v,l}$ with $v \in \text{vert}(P)$ and $l \in \mathbb{N}$ can be constructed such that the sequence $(g_l(x))_{l=1}^\infty$ satisfies $\mathcal{I}(P)$ and $\mathcal{A}(P)$. By construction, $(g_l(x))_{l=1}^\infty$ also satisfies condition $\mathcal{C}(P)$.

Part II. Without loss of generality we may assume that $n = d$. Consider an arbitrary $v \in \text{vert}(P)$. Replacing P by an appropriate translation, we assume that $v = o$. Let $\rho'_o := \min\{\|x - y\| : x \in P, y \in (-P_o)\}$. We show that there exists $l'_o \in \mathbb{N}$ such that

$$\forall l \geq l'_o \quad \forall x \in (-P_o + \mathbb{B}^d(o, \rho'_o/2)) \cap (-H_o(P)) : \quad g_l(x) < 0. \quad (6.4)$$

Assume the contrary. Then there exist sequences $(l_k)_{k=1}^\infty$, $(x_k)_{k=1}^\infty$, and $(y_k)_{k=1}^\infty$ such that $l_k \in \mathbb{N}$, $x_k \in (-H_o(P))$, $y_k \in -P_o$, $\|x_k - y_k\| \leq \rho'_o/2$, and $g_{l_k}(x_k) \geq 0$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \text{dist}(P, \{x \in \text{aff } P : g_{l_k}(x) \geq 0\}) \\ &\geq \min_{x \in P} \|x - x_k\| \geq \min_{x \in P} \|x - y_k\| - \|x_k - y_k\| \\ &\geq \rho'_o - \|x_k - y_k\| \geq \rho'_o/2 > 0, \end{aligned}$$

a contradiction to the assumption that $(g_l(x))_{l=1}^\infty$ satisfies $\mathcal{A}(P)$. Let us fix l'_o satisfying (6.4). Next we show that, for every $l \in \mathbb{N}$ and every $x \in -P_o$, one has $g_l(x) < 0$. Assume the contrary, i.e., $g_l(x) \geq 0$. Since $(g_l(x))_{l=1}^\infty$ satisfies $\mathcal{C}(P)$ and $\mathcal{I}(P)$, we deduce that $g_l(y) > 0$ for every $y \in P \setminus \text{vert}(P)$. For a sufficiently small $\varepsilon > 0$, one has $-\varepsilon x \in P \setminus \text{vert}(P)$. Then o is a convex combination of $-\varepsilon x$ and y (with nonzero coefficients). Hence, from the strict convexity of $g_l(x)$ and from $g_l(-\varepsilon x) > 0$ and $g_l(x) \geq 0$ we deduce that $g_l(o) > 0$, a contradiction. Hence $g_l(x) < 0$ for every $l \in \mathbb{N}$ and every $x \in -P_o$, and therefore we may fix $\rho_{o,l} > 0$ such that $(-P_o + \mathbb{B}^d(o, \rho'_{o,l})) \cap (-H_o)$ is disjoint with $\{x \in \text{aff } P : g_{l_k}(x) \geq 0\}$. Applying the above arguments to all vertices of P , we determine quantities $\rho_{v,l}$ and ρ'_v with $v \in \text{vert}(P)$ and $l \in \mathbb{N}$.

The assertion follows by setting

$$l'_v := \max\{l'_v : v \in \text{vert}(P)\}, \quad \rho_v := \min\{\rho_{v,1}, \dots, \rho_{v,l'}, \rho'_v/2\}$$

for every $v \in \text{vert}(P)$ and taking $\rho := \min\{\rho_v : v \in \text{vert}(P)\}$.

Part III. We assume that $o \notin \text{aff } P$. Without loss of generality we also assume that $n = d - 1$. For $l \in \mathbb{N}$, the homogeneous continuation $\tilde{g}_l(x)$ of $g_l(x)|_{\text{aff } P}$ can be

expressed by

$$\begin{aligned} \tilde{g}_l(x) &:= \langle x, u \rangle^{4(l+l_0)^2} - \sum_{v \in \text{vert}(P)} y_{v,l} \left(\frac{1}{|\mathcal{F}_{d-2}(v, P)|} \right. \\ &\quad \times \sum_{F \in \mathcal{F}_{d-2}(v, P)} \left(\langle x, u \rangle - \tilde{q}_F(P, x) \right)^{2(l+l_0)} \Big)^{2(l+l_0)}, \end{aligned} \quad (6.5)$$

where $\tilde{q}_F(P, x)$ is a homogeneous continuation of $q_F(P, x)|_{\text{aff } P}$, i.e.,

$$\tilde{q}_F(P, x) := \frac{h(P, u_F(P)) \langle u, x \rangle - \langle u_F(P), x \rangle}{\text{diam}(P)}.$$

If $\langle x, u \rangle = 0$, then (6.5) amounts to

$$\tilde{g}_l(x) := - \sum_{v \in \text{vert}(P)} y_{v,k} \left(\frac{1}{|\mathcal{F}_{d-2}(v, P)|} \sum_{F \in \mathcal{F}_{d-2}(v, P)} \left(\frac{\langle u_F(P), x \rangle}{\text{diam}(P)} \right)^{2(l+l_0)} \right)^{2(l+l_0)}. \quad (6.6)$$

Thus, if $\langle x, u \rangle = 0$, we have $\tilde{g}_l(x) \leq 0$ with equality if and only if $x = o$. Directly from the definition of homogeneous continuation it follows

$$\{x \in \mathbb{R}^d : \langle x, u \rangle > 0, \tilde{g}_l(x) \geq 0\} \cup \{o\} = \text{cone}(\{x \in H : g_l(x) \geq 0\}). \quad (6.7)$$

Equalities (6.6) and (6.7) yield the assertion. $\square$

We shall need the following observation.

Lemma 6.2 *The degree of every nonzero concave polynomial in $\mathbb{R}[x]$ is even.*

Proposition 6.3 *Let P be a 3-polytope in $\mathbb{R}^3$, and let $v \in \text{vert}(P)$. Then there exists a sequence of polynomials $(b_{v,l}(x))_{l=1}^\infty$ satisfying the following conditions:*

- I. *For every $l \in \mathbb{N}$, the polynomial $b_{v,l}(x)$ is of even degree*
- II. *For every $l \in \mathbb{N}$, the polynomial $b_{v,l}(x + v)$ is homogeneous.*
- III. *For all $\rho > 0$, $l \in \mathbb{N}$, $I \in \mathcal{F}_1(v, P)$, and $x \in v + S_\rho(P, v)$ satisfying $\mathcal{F}^-(P, x) = \mathcal{F}_2(I, P)$, one has $b_{v,l}(x) \leq 0$ with equality if and only if $x \in \text{cone}(I - v) + v$.*
- IV. *For every $l \in \mathbb{N}$, the set*

$$B_l(P, v) := \{x \in \mathbb{R}^3 : \langle x - v, u_v(P) \rangle \leq 0, b_{v,l}(x) \geq 0\} \quad (6.8)$$

is a convex cone which has apex at v and satisfies $P \subseteq B_l(P, v)$.

- V. *For every $\rho > 0$,*

$$\begin{aligned} (v + S(P, v)) \cup (v - S(P, v)) &\subseteq (b_{v,l})_{\geq 0} \subseteq \{v\} \\ &\cup (v + \text{int } S_\rho(P, v)) \cup (v - \text{int } S_\rho(P, v)) \end{aligned}$$

if $l \in \mathbb{N}$ is sufficiently large.

Proof Replacing P by an appropriate translation, we may assume that $v = o$. Applying Theorem 6.1(III), we determine a sequence $(g_{o,l}(x))_{l=1}^{\infty}$ which satisfies $\mathcal{A}(P_o)$, $\mathcal{I}(P_o)$, and $\mathcal{C}(P_o)$, respectively. We define $b_{o,l}(x) := \tilde{g}_{o,l}(x - v)$, where $\tilde{g}_{o,l}(x)$ is the homogeneous continuation of $g_{o,l}(x)|_{H_o(P)}$.

The part of Statement I follows Lemma 6.2. Statements II follows by construction.

Let us show Statement III. Consider an arbitrary $\rho > 0$ and $x \in S_{\rho}(P, v)$ with $\mathcal{F}^{-}(P, x) = \mathcal{F}_2(I, P)$. From the definition of $S_{\rho}(P, v)$ it follows that $\langle x, u_o(P) \rangle < 0$, and by this $y := \frac{x}{\langle x, u_o(P) \rangle} \in H_o(P)$. The intersection of $\text{cone}(I)$ and $H_o(P)$ is a vertex of P_o , which we denote by w . Since $\mathcal{F}^{-}(P, x) = \mathcal{F}_2(I, P)$, taking into account Lemma 5.4, we obtain $\mathcal{F}^{-}(P_o, y) = \mathcal{F}_2(w, P_o)$. Consequently, $y \in w - S(P_o, w)$. Then, by Theorem 6.1(II), we get $g_{o,l}(y) \leq 0$ with equality if and only if $y = w$. Consequently, $\tilde{g}_{o,l}(x) \leq 0$ with equality if and only if $x \in S_{\rho}(P, o) \cap \text{aff } I = \text{cone}(I)$.

Statement IV follows directly from Theorem 6.1(III).

It remains to show Statement V. Consider an arbitrary $\rho > 0$. Since $(g_{o,l}(x))_{l=1}^{\infty}$ satisfies $\mathcal{I}(P_o)$ and $\mathcal{C}(P_o)$, in view of the definition of $b_{o,l}(x)$, we obtain

$$S(P, o) \cup (-S(P, o)) \subseteq (b_{o,l})_{\geq 0}.$$

By construction, $b_{o,l}(x) = b_{o,l}(-x)$ for every $x \in \mathbb{R}^3$. Consequently, $(b_{o,l})_{\geq 0} = B_l(P, o) \cup (-B_l(P, o))$. Since $(g_{o,l}(x))_{l=1}^{\infty}$ satisfies $\mathcal{A}(P_o)$, $\mathcal{I}(P_o)$, and $\mathcal{C}(P_o)$, if l is large enough, the relation

$$(b_{o,l})_{\geq 0} \subseteq S_{\rho/2}(P, o) \cup (-S_{\rho/2}(P, o))$$

holds true. Obviously,

$$S_{\rho/2}(P, o) \cup (-S_{\rho/2}(P, o)) \subseteq \{o\} \cup \text{int } S_{\rho}(P, o) \cup (-\text{int } S_{\rho}(P, o)),$$

and we arrive at (6.8). $\square$

7 Proof of the Main Result

7.1 The Case of Bounded Polyhedra

The following theorem implies Theorem 1.1 for the case where P is a d -polytope with $d \leq 3$. (As mentioned in the introduction, the case $d = 2$ is due to Bernig.)

Theorem 7.1 *Let $d \in \{2, 3\}$ and P be a d -dimensional polytope in $\mathbb{R}^d$. Let $(g_m(x))_{m=1}^{\infty}$ be a sequence of polynomials satisfying conditions $\mathcal{A}(P)$, $\mathcal{I}(P)$, $\mathcal{C}(P)$. Then there exist polynomials $p_1(x), \dots, p_{d-1}(x) \in \mathbb{R}[x]$ such that $P = (p_0, \dots, p_{d-1})_{\geq 0}$ and*

$$p_0(x) = g_m(x) \tag{7.1}$$

for some $m \in \mathbb{N}$.

Proof We consider the case $d = 3$. First, we show the existence of polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$ satisfying $P = (p_0, p_1, p_2)_{\geq 0}$, and then we show that these polynomials are constructible. Obviously, there exists $\varepsilon_0 > 0$ such that

$$\forall F \in \mathcal{F}_{d-1}(P) \forall G \in \mathcal{F}(P) : F \cap G = \emptyset \implies \text{aff } F \cap (G + \mathbb{B}^d(o, \varepsilon_0)) = \emptyset. \quad (7.2)$$

In view of Theorem 6.1(II), condition (6.2) is fulfilled for all sufficiently small $\rho > 0$. We also assume that ρ is sufficiently small so that (5.3) in Lemma 5.5 is fulfilled. For $v \in \text{vert}(P)$ and $l \in \mathbb{N}$, let the polynomials $b_{v,l}(x)$ be as in Proposition 6.3. Fix $\alpha_l > 0$ satisfying

$$|b_{w,l}(x)| \leq \alpha_l \quad \forall x \in P + \mathbb{B}^3(o, \varepsilon_0) \quad \forall w \in \text{vert}(P). \quad (7.3)$$

We define

$$\begin{aligned} p_0(x) &:= g_m(x), \\ p_1(x) &:= \sum_{v \in \text{vert}(P)} f_v(x)^{2k} b_{v,l}(x), \end{aligned} \quad (7.4)$$

$$p_2(x) := \prod_{F \in \mathcal{F}_2(P)} q_F(P, x), \quad (7.5)$$

where

$$f_v(x) := \prod_{F \in \mathcal{F}_2(P) \setminus \mathcal{F}_2(v, P)} q_F(P, x) \quad (7.6)$$

for every $v \in \text{vert}(P)$, and the parameters $k, m \in \mathbb{N}$ will be fixed later. It will be shown that, for a sufficiently large k and m , we have $P = (p_0, p_1, p_2)_{\geq 0}$. Let us first show that there exists $\varepsilon \in (0, \varepsilon_0]$ such, for all sufficiently large $k \in \mathbb{N}$, we have

$$U(P, \varepsilon, \rho) \cap (p_1, p_2)_{\geq 0} \subseteq P \cup \left(\bigcup_{v \in \text{vert}(P)} (v - \text{int } S_\rho(P, v)) \right). \quad (7.7)$$

Let us consider an arbitrary $x \in U(P, \varepsilon_0, \rho)$.

Case 1: $x \in U_P(P, \varepsilon_0, \rho) = P$. Clearly, x belongs to the left- and right-hand sides of (7.7).

Case 2: $x \in U_F(P, \varepsilon_0, \rho)$ for some $F \in \mathcal{F}_2(P)$. We have $p_2(x) \leq 0$ with equality if and only if $x \in F$. Consequently, $U_F(P, \varepsilon_0, \rho) \cap (p_2)_{\geq 0} \subseteq P$.

Case 3: $x \in U_I(P, \varepsilon_0, \rho)$ for some $I \in \mathcal{F}_1(P)$. If $x \in I$ and $v \in \text{vert}(P) \setminus \text{vert}(I)$, then at least one of the two facets F from $\mathcal{F}_2(I, P)$ satisfies $v \notin F$, which yields $f_v(x) = 0$. Consequently,

$$\max_{v \in \text{vert}(P) \setminus \text{vert}(I)} |f_v(x)| = 0 \quad \forall x \in I. \quad (7.8)$$

We show that there exists $\beta_I > 0$ is such that

$$\beta_I \leq \max_{w \in \text{vert}(I)} |f_w(x)| \quad \forall x \in U_I(P, \varepsilon_0, \rho). \quad (7.9)$$

Choose $x \in U_I(P, \varepsilon_0, \rho)$. If $x \in \text{vert}(I)$, then $f_w(x) > 0$ for $w = x$. If $x \in \text{relint } I$, then $f_w(x) > 0$ for every $w \in \text{vert}(I)$. Now assume that $x \in U_I(P, \varepsilon_0, \rho) \setminus I$. We fix arbitrary $w \in \text{vert}(I)$ and $F \in \mathcal{F}_2(P) \setminus \mathcal{F}_2(w, P)$. Consider the subcase $F \cap I = \emptyset$. By the definition of $U_I(P, \varepsilon_0, \rho)$, we have $x \in I + \mathbb{B}^3(\rho, \varepsilon_0)$, and by (7.2), we obtain $q_F(P, x) \neq 0$. Consider the subcase $F \cap I \neq \emptyset$. We denote by v the endpoint of I distinct from w . Then $F \in \mathcal{F}_2(v, P)$. Let us show that $q_F(P, x) \neq 0$ by contradiction. Assume the contrary, i.e., that $q_F(P, x) = 0$. Then $F \in \mathcal{F}^-(P, x)$. Furthermore, $\mathcal{F}_2(I, P) \subseteq \mathcal{F}^-(P, x)$ by the definition of $U_I(P, \varepsilon_0, \rho)$. Taking into account the above relations together with $\mathcal{F}_2(I, P) \subseteq \mathcal{F}_2(v, P)$ and $F \in \mathcal{F}_2(v, P)$, we deduce $\mathcal{F}_2(I, P) \cup \{F\} \subseteq \mathcal{F}^-(P, x) \cap \mathcal{F}_2(v, P)$. Hence, $|\mathcal{F}^-(P, x) \cap \mathcal{F}_2(v, P)| \geq 3$, which implies that (5.3) cannot be fulfilled, a contradiction. Summarizing, we get

$$\max_{w \in \text{vert}(I)} |f_w(x)| > 0 \quad \forall x \in U_I(P, \varepsilon_0, \rho),$$

which yields the existence of β_I satisfying (7.9). By Proposition 6.3(III),

$$b_{w,I}(x) = 0 \quad \forall w \in \text{vert}(I) \quad \forall x \in I. \quad (7.10)$$

If $x \in \text{aff } I \setminus I$, one has $x \notin S_\rho(P, w)$ for some vertex of $w \in \text{vert}(I)$. Consequently, $U_I(P, \varepsilon_0, \rho) \cap \text{aff } I = I$. By Proposition 6.3(III) we get

$$b_{w,I}(x) < 0 \quad \forall w \in \text{vert}(I) \quad \forall x \in U_I(P, \varepsilon_0, \rho) \setminus I. \quad (7.11)$$

We also notice that $\max_{v \in \text{vert}(P) \setminus \text{vert}(I)} |f_v(x)|$ and $b_{w,I}(x)$ are semi-algebraic functions and $U_I(P, \varepsilon_0, \rho)$ is a semi-algebraic set. Hence, taking into account (7.8), (7.10), and (7.11) and applying Theorem 3.1, we obtain the existence of $k_I \in \mathbb{N}$ and $\gamma_I > 0$ such that

$$\max_{v \in \text{vert}(P) \setminus \text{vert}(I)} f_v(x)^{2k_I} \leq \gamma_I \min_{w \in \text{vert}(I)} |b_{w,I}(x)| \quad (7.12)$$

for every $x \in U_I(P, \varepsilon_0, \rho)$. In view of (7.8), we can choose $\varepsilon_I \in (0, \varepsilon_0]$ such that, for every $x \in U_I(\varepsilon_I, \rho)$, one has

$$\max_{v \in \text{vert}(P) \setminus \text{vert}(I)} |f_v(x)| \leq \frac{\beta_I}{2}. \quad (7.13)$$

We also assume that k is large enough so that the inequality

$$\alpha_I |\text{vert}(P)| \gamma_I 2^{-2k} \left(\frac{2}{\beta_I} \right)^{2k_I} \leq \frac{1}{2} \quad (7.14)$$

is fulfilled.

Then, for every $x \in U_I(P, \rho, \varepsilon_I)$ and every $k \in \mathbb{N}$ with $k \geq k_I$, we obtain

$$\begin{aligned}
 & \left| \sum_{v \in \text{vert}(P) \setminus \text{vert}(I)} f_v(x)^{2k} b_{v,I}(x) \right| \\
 & \stackrel{(7.3)}{\leq} \alpha_I \sum_{v \in \text{vert}(P) \setminus \text{vert}(I)} f_v(x)^{2k} \\
 & \leq \alpha_I |\text{vert}(P)| \max_{v \in \text{vert}(P) \setminus \text{vert}(I)} f_v(x)^{2k} \\
 & \stackrel{(7.12)}{\leq} \alpha_I |\text{vert}(P)| \gamma_I \min_{w \in \text{vert}(I)} |b_{w,I}(x)| \max_{v \in \text{vert}(P) \setminus \text{vert}(I)} f_v(x)^{2(k-k_I)} \\
 & \stackrel{(7.13)}{\leq} \alpha_I |\text{vert}(P)| \gamma_I \left(\frac{\beta_I}{2} \right)^{2(k-k_I)} \min_{w \in \text{vert}(I)} |b_{w,I}(x)| \\
 & \stackrel{(7.9)}{\leq} \alpha_I |\text{vert}(P)| \gamma_I 2^{-2k} \left(\frac{2}{\beta_I} \right)^{2k_I} \min_{w \in \text{vert}(I)} |b_{w,I}(x)| \max_{w \in \text{vert}(I)} f_w(x)^{2k} \\
 & \stackrel{(7.14)}{\leq} \frac{1}{2} \min_{w \in \text{vert}(I)} |b_{w,I}(x)| \max_{w \in \text{vert}(I)} f_w(x)^{2k} \\
 & \leq \frac{1}{2} \min_{w \in \text{vert}(I)} |b_{w,I}(x)| \sum_{w \in \text{vert}(I)} f_w(x)^{2k} \\
 & \leq \frac{1}{2} \sum_{w \in \text{vert}(I)} f_w(x)^{2k} |b_{w,I}(x)|.
 \end{aligned}$$

Consequently, for x and k as above, we get

$$\begin{aligned}
 p_1(x) & \leq \sum_{w \in \text{vert}(I)} f_w(x)^{2k} b_{w,I}(x) + \left| \sum_{v \in \text{vert}(P) \setminus \text{vert}(I)} f_v(x)^{2k} b_{v,I}(x) \right| \\
 & \leq \sum_{w \in \text{vert}(I)} f_w(x)^{2k} b_{w,I}(x) + \frac{1}{2} \sum_{w \in \text{vert}(I)} f_w(x)^{2k} |b_{w,I}(x)| \\
 & \stackrel{(7.11)}{=} \frac{1}{2} \sum_{w \in \text{vert}(I)} f_w(x)^{2k} b_{w,I}(x) \stackrel{(7.11)}{\leq} 0
 \end{aligned}$$

with equality $p_1(x) = 0$ if and only if $x \in I$.

Case 4: $x \in U'_v(P, \varepsilon_0, \rho)$ for some $v \in \text{vert}(P)$. By Proposition 6.3(V) and the definition of $U'_v(P, \varepsilon_0, \rho)$ we have

$$b_{v,I}(x) < 0 \quad \forall x \in U'_v(P, \varepsilon_0, \rho) \setminus \{v\}. \quad (7.15)$$

In view of (7.2), the definition of $f_v(x)$, and the inclusion $U'_v(P, \varepsilon_0, \rho) \subseteq \mathbb{B}^3(v, \varepsilon_0)$, there exists $\beta_v > 0$ such that

$$f_v(x) \geq \beta_v \quad \forall x \in U'_v(P, \varepsilon_0, \rho). \quad (7.16)$$

On the other hand,

$$\max_{w \in \text{vert}(P) \setminus \{v\}} |f_w(v)| = 0. \quad (7.17)$$

Notice that $\max_{w \in \text{vert}(P) \setminus \{v\}} |f_w(x)|$ and $b_{v,l}(x)$ are semi-algebraic function and that $U'_v(P, \varepsilon_0, \rho)$ is a semi-algebraic set. Thus, taking into account (7.15) and (7.17) and applying Theorem 3.1, we find $\gamma_v > 0$ and $k_v \in \mathbb{N}$ such that

$$\max_{w \in \text{vert}(P) \setminus \{v\}} |f_w(x)|^{2k_v} \leq \gamma_v |b_{v,l}(x)| \quad (7.18)$$

for every $x \in U'_v(P, \varepsilon_0, \rho)$. In view of (7.17), we can choose $\varepsilon_v \in (0, \varepsilon_0]$ such that

$$\max_{w \in \text{vert}(P) \setminus \{v\}} |f_w(x)| \leq \frac{\beta_v}{2} \quad (7.19)$$

for every $x \in U'_v(P, \varepsilon_v, \rho)$. We assume that k is large enough so that

$$\alpha_l |\text{vert}(P)| \gamma_v \left(\frac{\beta_v}{2} \right)^{-2k_v} 2^{-2k} \leq \frac{1}{2}. \quad (7.20)$$

Then, for every $x \in U'_v(P, \varepsilon_v, \rho)$ and $k \geq k_v$, as above, we obtain

$$\begin{aligned} & \left| \sum_{w \in \text{vert}(P) \setminus \{v\}} f_w(x)^{2k} b_{w,l}(x) \right| \\ & \stackrel{(7.3)}{\leq} \alpha_l |\text{vert}(P)| \max_{w \in \text{vert}(P) \setminus \{v\}} |f_w(x)|^{2k} \\ & \stackrel{(7.18)}{\leq} \alpha_l |\text{vert}(P)| \gamma_v |b_{v,l}(x)| \max_{w \in \text{vert}(P) \setminus \{v\}} |f_w(x)|^{2(k-k_v)} \\ & \stackrel{(7.19)}{\leq} \alpha_l |\text{vert}(P)| \gamma_v \left(\frac{\beta_v}{2} \right)^{2(k-k_v)} |b_{v,l}(x)| \\ & \stackrel{(7.16)}{\leq} \alpha_l |\text{vert}(P)| \gamma_v \left(\frac{\beta_v}{2} \right)^{2(k-k_v)} (\beta_v)^{-2k} f_v(x)^{2k} |b_{v,l}(x)| \\ & = \alpha_l |\text{vert}(P)| \gamma_v \left(\frac{\beta_v}{2} \right)^{-2k_v} 2^{-2k} f_v(x)^{2k} |b_{v,l}(x)| \\ & \stackrel{(7.20)}{\leq} \frac{1}{2} f_v(x)^{2k} |b_{v,l}(x)|. \end{aligned}$$

Hence, taking into account (7.15) and (7.16), we obtain that, for k satisfying (7.20) and all $x \in U'_v(P, \varepsilon_v, \rho)$, we have $p_1(x) \leq 0$ with equality if and only if $x = v$.

Consequently, for x and $k \geq k_v$, as above, we obtain

$$\begin{aligned} p_1(x) &\leq f_v(x)^{2k} b_{v,l}(x) + \left| \sum_{w \in \text{vert}(P) \setminus \{v\}} f_w(x)^{2k} b_{w,l}(x) \right| \\ &\leq f_v(x)^{2k} b_{v,l}(x) + \frac{1}{2} f_v(x)^{2k} |b_{v,l}(x)| \\ &\stackrel{(7.15)}{=} \frac{1}{2} f_v(x)^{2k} b_{v,l}(x) \stackrel{(7.15)}{\leq} 0 \end{aligned}$$

with equality $p_1(x) = 0$ if and only if $x = v$.

Case 5: $x \in U_v(P, \varepsilon_0, \rho) \setminus U'_v(P, \varepsilon_0, \rho)$ for some $v \in \text{vert}(P)$. By the definition of $U_v(P, \varepsilon_0, \rho)$ and $U'_v(P, \varepsilon_0, \rho)$, we easily see that $x \in v - \text{int } S_\rho(P, v)$. Thus, $U_v(P, \varepsilon_0, \rho) \setminus U'_v(P, \varepsilon_0, \rho) \subseteq v - \text{int } S_\rho(P, v)$.

By means of the arguments given in the above five cases we verified that (7.7) holds if $k \geq k_v$, $k \geq k_I$, $\varepsilon \leq \varepsilon_v$, $\varepsilon \leq \varepsilon_I$ for all $v \in \text{vert}(P)$ and $I \in \mathcal{F}_1(P)$ and that inequalities (7.14) and (7.20) are fulfilled. By Proposition 5.1, there exists $\delta > 0$ such that $P + \mathbb{B}^3(o, \delta) \subseteq U(P, \varepsilon, \rho)$. By condition $\mathcal{A}(P)$, we can choose $m \in \mathbb{N}$ such that $(p_0)_{\geq 0} \subseteq P + \mathbb{B}^3(o, \delta)$. Thus, for m , as above, we obtain

$$\begin{aligned} (p_0, p_1, p_2)_{\geq 0} &= (p_0, p_1, p_2)_{\geq 0} \cap (P + \mathbb{B}^3(o, \delta)) \subseteq (p_0, p_1, p_2)_{\geq 0} \cap U(P, \varepsilon, \rho) \\ &\stackrel{(7.7)}{\subseteq} (p_0)_{\geq 0} \cap \left(P \cup \bigcup_{v \in \text{vert}(P)} (v - \text{int } S_\rho(P, v)) \right) \\ &= ((p_0)_{\geq 0} \cap P) \cup \left((p_0)_{\geq 0} \cap \bigcup_{v \in \text{vert}(P)} (v - \text{int } S_\rho(P, v)) \right) \\ &\stackrel{(6.2)}{=} P \cup \text{vert}(P) = P. \end{aligned}$$

Thus, $(p_0, p_1, p_2)_{\geq 0} = P$. □

7.2 The Case of Unbounded Polyhedra

The following is a modified version of Łojasiewicz's Inequality.

Lemma 7.2 *Let A be a bounded and closed semi-algebraic set in $\mathbb{R}^d$. Let $f(x)$ and $g(x)$ be continuous semi-algebraic functions on A satisfying*

$$\{x \in A : f(x) = 0\} \subseteq \{x \in A : g(x) = 0\}.$$

Then there exists $n' \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n'$, there is a constant $\lambda > 0$ satisfying $|g(x)|^n \leq \lambda |f(x)|$ for every $x \in A$.

Proof By Theorem 3.1 we can fix $n' \in \mathbb{N}$ and $\lambda' > 0$ such that $|g(x)|^{n'} \leq \lambda' |f(x)|$ for every $x \in A$. Let $\mu \geq 0$ be an upper bound of $|g(x)|$ on A . Then, for every $n \in \mathbb{N}$ with $n \geq n'$, we have $|g(x)|^n \leq \mu^{n-n'} \lambda' |f(x)|$, so that we may set $\lambda := \mu^{n-n'} \lambda'$. □

Proposition 7.3 *Let $d \in \mathbb{N}$ and $d \geq 2$. Assume that, for every $n = 1, \dots, d$ and every n -polytope P in $\mathbb{R}^d$, there exist $p_0(x), \dots, p_{n-1}(x) \in \mathbb{R}[x]$ satisfying*

$$P = \{x \in \text{aff } P : p_0(x) \geq 0, \dots, p_{n-1}(x) \geq 0\}$$

and $p_0(x) > 0$ for all $x \in P \setminus \text{vert}(P)$. Then, for every d -dimensional polyhedron P in $\mathbb{R}^d$, there exist $f_0(x), \dots, f_{d-1}(x) \in \mathbb{R}[x]$ satisfying $P = (f_0, \dots, f_{d-1})_{\geq 0}$.

Proof Consider an arbitrary unbounded d -polyhedron P in $\mathbb{R}^d$. Using the same arguments as in the beginning of the proof of Theorem 1.1, we may restrict ourselves to the case where P is line-free. The set $\text{hom}(P)$ is a $(d+1)$ -dimensional pointed polyhedral cone. Since $\text{hom}(P)$ is pointed, one can determine a hyperplane H' in $\mathbb{R}^{d+1}$ such that $P' := \text{hom}(P) \cap H'$ is bounded and $\text{hom}(P) = \text{cone}(P')$. Let $u' \in \mathbb{R}^{d+1} \setminus \{o\}$ be the normal of H' with $H' = \{x \in \mathbb{R}^{d+1} : \langle x, u' \rangle = 1\}$. Let us consider polynomials $p_0(x), \dots, p_{d-1}(x) \in \mathbb{R}[x]$ such that $P = \{x \in \text{aff } P' : p_0(x) \geq 0, \dots, p_{d-1}(x) \geq 0\}$ and $p_0(x) > 0$ for every $x \in P \setminus \text{vert}(P)$. Without loss of generality we may assume that $p_0(x), \dots, p_{d-1}(x)$ are homogeneous. We define

$$f(x) := \prod_{v \in \text{vert}(P')} (\|x\|^2 \|v\|^2 - \langle x, v \rangle^2).$$

By Lemma 7.2 applied to $f(x)$ and $p_0(x)$ restricted to P' , there exist $\lambda > 0$ and $l \in \mathbb{N}$ satisfying $l \deg f(x) > \deg p_0(x)$ and $\lambda p_0(x) - f(x)^l \geq 0$ for every $x \in P'$ (where $\deg$ stands for degree). Define $f_0(x) := \lambda p_0(x) \langle x, u' \rangle^{k_0} - f(x)^l$, where $k_0 \in \mathbb{N}$ is chosen in such a way that f_0 is homogeneous, i.e., k_0 is determined from the equality

$$k_0 + \deg p_0(x) = l \deg f(x). \quad (7.21)$$

We also set $f_i(x) := p_i(x) \langle x, u' \rangle^{k_i}$ for $i = 1, \dots, d-1$, where $k_1, \dots, k_{d-1} \in \{1, 2\}$ are chosen in such a way that $f_1(x), \dots, f_{d-1}(x)$ have even degrees. By construction, the polynomials $f_0(x), \dots, f_{d-1}(x)$ are homogeneous, have even degrees, and satisfy

$$\text{hom}(P) \cup (-\text{hom}(P)) = \{x \in \mathbb{R}^{d+1} : f_0(x) \geq 0, \dots, f_{d-1}(x) \geq 0\}.$$

Hence, $\{x \in \text{aff } P : f_0(x) \geq 0, \dots, f_{d-1}(x) \geq 0\} = (\text{hom}(P) \cup (-\text{hom}(P))) \cap (\text{aff } P) = P$. $\square$

Theorem 1.1 is a direct consequence of Theorem 7.1 and Proposition 7.3.

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