Discrete Comput Geom (2010) 43: 784–797
DOI 10.1007/s00454-009-9155-5

An Optimization Problem Related to Minkowski's Successive Minima

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Received: 13 November 2008 / Revised: 24 February 2009 / Accepted: 25 February 2009 /
Published online: 21 March 2009

Abstract The purpose of this paper is to establish an inequality connecting the lattice point enumerator of a 0-symmetric convex body with its successive minima. To this end, we introduce an optimization problem whose solution refines former methods, thus producing a better upper bound. In particular, we show that an analogue of Minkowski's second theorem on successive minima with the volume replaced by lattice point enumerator is true up to an exponential factor, whose base is approximately 1.64.

Keywords Successive minima · Lattice points

1 Introduction

Let $\mathcal{K}_0^d$ denote the set of all compact, d-dimensional, 0-symmetric convex bodies for which $0 \in \operatorname{int}(K)$, and let $K \in \mathcal{K}_0^d$ be arbitrary. We denote by $G(K, \Lambda)$ the lattice point enumerator in K with respect to the lattice Λ, i.e., $\#(K \cap \Lambda)$, and let $\operatorname{vol}(K)$ denote the usual d-dimensional Lebesgue measure of K. The ith successive minimum of K with respect to the lattice Λ, denoted by $\lambda_i = \lambda_i(K, \Lambda)$ $(1 \le i \le d)$, is defined as follows:

$$\lambda_i = \inf\{\lambda > 0 \mid \lambda K \cap \Lambda \text{ contains at least } i \text{ linearly independent points}\}.$$

Obviously,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_d < +\infty.$$

Finally, $\det(\Lambda)$ will denote the determinant of a lattice, i.e., the volume of a fundamental parallelotope of Λ.

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In 1896 (see [2]), Minkowski stated and proved his famous two theorems relating the volume of $K \in \mathcal{K}_0^d$ with its successive minima. The first theorem is:

Theorem 1.1 *Let $K \in \mathcal{K}_0^d$. Then the following inequality holds*:

$$\mathrm{vol}(K) \le \det(\Lambda)\left(\frac{2}{\lambda_1}\right)^d.$$

Minkowski himself used the above to prove a lower bound on the discriminant of a number field. In particular, he deduced that there is no nontrivial algebraic extension of $\mathbb{Q}$ that is unramified at all primes. This is a key ingredient in many deep theorems in number theory. It should be noted that Theorem 1.1 is more widely known in the following equivalent form: If $K \in \mathcal{K}_0^d$ satisfies $\mathrm{vol}(K) \ge 2^d \det(\Lambda)$, then it contains a nontrivial lattice point.

Minkowski's second theorem on successive minima is a stronger result:

Theorem 1.2 *Let $K \in \mathcal{K}_0^d$. Then the following inequality holds*:

$$\mathrm{vol}(K) \le \det(\Lambda)\prod_{i=1}^{d}\frac{2}{\lambda_i}.$$

Besides algebraic number theory, Theorem 1.2 has a wide number of applications in various areas of mathematics, as in Diophantine approximation or adelic geometry of numbers, to name a few, and is considered a very deep result in the geometry of numbers [2].

In 1993, Betke, Henk, and Wills [1] stated analogues of Minkowski's theorems for the lattice point enumerator, instead of the volume. Their first theorem is the following:

Theorem 1.3 *Let $K \in \mathcal{K}_0^d$. Then the following inequality holds*:

$$G(K,\Lambda) \le \left[\frac{2}{\lambda_1}+1\right]^d.$$

Here, as usual, $[x]$ denotes the integer part of x. An analogue to the second theorem was proven only for the planar case, $d=2$, being trivial for $d=1$:

Conjecture 1.1 *Let $K \in \mathcal{K}_0^d$. Then the following inequality holds*:

$$G(K,\Lambda) \le \prod_{i=1}^{d}\left[\frac{2}{\lambda_i}+1\right].$$

It should be noted that the conjecture above, if true, would imply Minkowski's second theorem on successive minima, using a simple argument involving the definition of the Riemann integral [1]. Betke, Henk, and Wills proved that Conjecture 1.1

holds roughly up to a factor of $d!$. Later, Henk [3] improved this inequality to

$$G(K,\Lambda) \le 2^{d-1}\prod_{i=1}^{d}\left\lfloor \frac{2}{\lambda_i}+1\right\rfloor. \tag{1}$$

Examining Henk's proof leads us to an optimization problem, in particular, finding a better upper bound for the constant C_d defined below:

Definition 1.1 Let C_d denote the least positive constant such that, for any sequence of d integers $x_1 \le x_2 \le \cdots \le x_d$, there exists a sequence of integers $y_1, y_2, \ldots, y_d$ satisfying:

(a) $x_i \le y_i$ for all i, $1 \le i \le d$.
(b) y_i divides y_{i+1} for all i, $1 \le i \le d-1$.
(c) $\frac{y_1 y_2 \cdots y_d}{x_1 x_2 \cdots x_d} \le C_d$.

In the course of proving Theorem 1.4, Henk essentially proved that $C_d \le 2^{d-1}$. Here, we shall prove a better upper bound, as well as a lower bound:

Proposition 1.1 $2^{(d-1)/2} \le C_d \le (4/e)\cdot 3^{(d-1)/2}$, *and the lower bound is tight.*

Using the method given in the proof of Proposition 1.1, we were able to improve Theorem 1.4:

Theorem 1.4 *Let $K \in \mathcal{K}_0^d$. Then the following inequality holds*:

$$G(K,\Lambda) \le \frac{4}{e}\left(\sqrt[3]{\frac{40}{9}}\right)^{d-1}\prod_{i=1}^{d}\left\lfloor \frac{2}{\lambda_i}+1\right\rfloor.$$

We should note that $4/e \approx 1.47152$ and $\sqrt[3]{40/9} \approx 1.64414$. In Sect. 4, we will prove better inequalities than those described by Theorem 1.4 for the cases $d=3$ and $d=4$:

Proposition 1.2 *Let $K \in \mathcal{K}_0^d$. If $d=3$, we have*

$$G(K,\Lambda) \le 2\prod_{i=1}^{3}\left\lfloor \frac{2}{\lambda_i}+1\right\rfloor$$

and, if $d=4$,

$$G(K,\Lambda) \le \frac{7203}{2375}\prod_{i=1}^{4}\left\lfloor \frac{2}{\lambda_i}+1\right\rfloor$$

($7203/2375 \approx 3.03284$).

The corresponding inequalities derived from Theorem 1.4 for $d = 3$ and $d = 4$ give

$$\frac{4}{e}\left(\frac{40}{9}\right)^{2/3} \approx 3.97781$$

and

$$\frac{4}{e} \cdot \frac{40}{9} \approx 6.54008.$$

2 Henk's Proof

Let us review first the proof of inequality (1); we will simply expand the technique used in the proof in order to obtain a better inequality. The following lemma is needed:

Lemma 2.1 *Let $K \in \mathcal{K}_0^d$, and let $\widetilde{\Lambda} \subset \Lambda$ be two lattices in $\mathbb{R}^d$. Then*

$$G(K, \Lambda) \leq \frac{\det \widetilde{\Lambda}}{\det \Lambda} G\bigl(2K, \widetilde{\Lambda}\bigr). \tag{2}$$

For a proof, see [3].

Proposition 2.1 *Let $0 < x_1 \leq x_2 \leq \cdots \leq x_d$ be d integer numbers. Then there are integers $y_1, y_2, \ldots, y_d$ satisfying $x_i \leq y_i$, $y_i | y_{i+1}$ for all i and*

$$\frac{y_1 y_2 \cdots y_d}{x_1 x_2 \cdots x_d} < 2^{d-1}.$$

Proof It suffices to put $y_1 = x_1$ and inductively construct $x_i \leq y_i < 2x_i$ such that $y_i | y_{i+1}$. Such a construction is possible; assuming that we have constructed $y_1, \ldots, y_k$ satisfying the above requirements, we will construct y_{k+1}. If $x_{k+1} \leq y_k$, we simply set $y_{k+1} = y_k$. Obviously, $x_{k+1} \leq y_{k+1} < 2x_k \leq 2x_{k+1}$. Otherwise, we consider the euclidean division of x_{k+1} by y_k, say $x_{k+1} = m \cdot y_k + r$, where $0 \leq r < y_k$. Then, we set $y_{k+1} = (m+1)y_k$, which again satisfies the desired requirements. □

Proof of inequality (1) We need the following simple fact; if $a^1, \ldots, a^d$ are d linearly independent lattice vectors of Λ, then there is a basis of Λ, say $e^1, \ldots, e^d$ such that for all i with $1 \leq i \leq d$, we have

$$\operatorname{lin}\bigl(a^1, \ldots, a^i\bigr) \subset \operatorname{lin}\bigl(e^1, \ldots, e^i\bigr),$$

where "lin" denotes the linear hull of the mentioned vectors, i.e., the set of all linear combinations with integer coefficients. Furthermore, if the a^i's are such that $a^i \in \lambda_i K$, then

$$\operatorname{int}(\lambda_i K) \cap \Lambda \subset \operatorname{lin}\bigl(0, e^1, \ldots, e^{i-1}\bigr) \cap \Lambda,$$

which follows from the definition of the successive minima.

We set $q_i = [2\lambda_i^{-1} + 1]$. We want to find d integer numbers n_i, $1 \le i \le d$, such that $q_i \le n_i$ and $n_{i+1} | n_i$ for $1 \le i \le d-1$ and

$$\frac{n_1 n_2 \cdots n_d}{q_1 q_2 \cdots q_d} < 2^{d-1}.$$

This is possible by Proposition 2.1. Next, we consider the lattice $\widetilde{\Lambda} \subset \Lambda$ which is generated by the vectors $n_1 e^1, \dots, n_d e^d$. By the above lemma we have

$$G(K, \Lambda) \le G\bigl(2K, \widetilde{\Lambda}\bigr) \prod_{i=1}^{d} n_i < G\bigl(2K, \widetilde{\Lambda}\bigr) \cdot 2^{d-1} \prod_{i=1}^{d} \bigl[2\lambda_i^{-1} + 1\bigr].$$

Thus, it suffices to prove that $G(2K, \widetilde{\Lambda}) = 1$. Assuming otherwise, let g be a nonzero vector that is an element of $2K \cap \widetilde{\Lambda}$, and let k be the largest index of a nonzero coordinate of g. Then, for some $z_i \in \mathbb{Z}$, $1 \le i \le k$, we have

$$g = z_1\bigl(n_1 e^1\bigr) + \cdots + z_k\bigl(n_k e^k\bigr) \in 2K.$$

Since n_k divides $n_1, \dots, n_{k-1}$ and $2/n_k < \lambda_k$, we obtain

$$\frac{1}{n_k} g \in \left(\frac{2}{n_k} K\right) \cap \Lambda \subset \operatorname{int}(\lambda_k K) \cap \Lambda \subset \operatorname{lin}\bigl(0, e^1, \dots, e^{k-1}\bigr) \cap \Lambda,$$

which is a contradiction. Hence, $2K \cap \widetilde{\Lambda} = \{0\}$ and $G(2K, \widetilde{\Lambda}) = 1$, as desired. □

3 An Optimization Problem

We observe that there is an undesired factor of magnitude 2^{d-1}, which is obtained from Proposition 2.1. Can we improve this factor? We are naturally led to Definition 1.1, and we will attempt to give a better upper bound.

In order to obtain an estimate on C_d, we drop the hypothesis on integrality of the x_i's and y_i's; in this setting, $y_i | y_{i+1}$ means that $y_{i+1}/y_i \in \mathbb{Z}$. We call the corresponding constant by c_d:

Definition 3.1 Let c_d denote the least positive constant such that, for any sequence of d positive real numbers $x_1 \le x_2 \le \cdots \le x_d$, there exists a sequence of real numbers $y_1, y_2, \dots, y_d$ satisfying:

(a) $x_i \le y_i$ for all i, $1 \le i \le d$.
(b) $y_{i+1}/y_i \in \mathbb{Z}$ for all i, $1 \le i \le d-1$.
(c) $\frac{y_1 y_2 \cdots y_d}{x_1 x_2 \cdots x_d} \le c_d$.

We will prove later that $c_d \le C_d$. The following nice lemma was proven by Rogers [4]. We provide a proof here for convenience (see also [2], p. 190):

Lemma 3.1 $c_d = 2^{(d-1)/2}$.

Proof For each i, $1 \le i \le d$, we construct the sequence $y_1^i, \dots, y_d^i$ that satisfies

$$y_i^i = x_i, \qquad y_j^i = 2^{a_{ij}} x_i \quad \text{for } j \neq i, \quad \text{where } a_{ij} = -\left\lfloor \log_2 \frac{x_i}{x_j} \right\rfloor.$$

In other words, a_{ij} is the unique integer satisfying

$$x_j \le 2^{a_{ij}} x_i < 2x_j.$$

Therefore,

$$\log_2 \frac{y_j^i}{x_j} = \{\log_2 x_i - \log_2 x_j\}$$

for all j, so

$$\log_2 \frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} = \sum_{j=1}^{d} \{\log_2 x_i - \log_2 x_j\}.$$

Summing over all i, we obtain

$$\sum_{i=1}^{d} \log_2 \frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} = \sum_{i,j=1}^{d} \{\log_2 x_i - \log_2 x_j\}.$$

For any pair (i, j) with $i \neq j$, $\{\log_2 x_i - \log_2 x_j\} + \{\log_2 x_j - \log_2 x_i\} \le 1$ (for $i = j$, it vanishes). Since there are $d(d-1)/2$ such pairs, we get

$$\sum_{i=1}^{d} \log_2 \frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} \le \frac{d(d-1)}{2}.$$

Hence, there is an index i such that

$$\log_2 \frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} \le \frac{d-1}{2},$$

and hence

$$\frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} \le 2^{\frac{d-1}{2}}.$$

Since the increasing sequence $x_1, \dots, x_d$ is arbitrary, we have $c_d \le 2^{(d-1)/2}$. We will show, by an example, that $c_d = 2^{(d-1)/2}$; let $x_i = 2^{(i-1)/d}$. Let $y_1, \dots, y_d$ be an increasing sequence satisfying $x_i \le y_i$ and $y_i | y_{i+1}$ for all i. Dividing all y_i's by an appropriate number, we may assume that $x_i = y_i$ for some i. Since $x_d < 2$, we must have $y_j = x_i$ for all $j \le i$ and of course $y_j \ge 2x_i$ for all $j > i$. Thus,

$$\frac{y_1 \cdots y_d}{x_1 \cdots x_d} \ge 2^{\frac{i-1}{d}} \cdot 2^{\frac{i-2}{d}} \cdots 1 \cdot 2^{\frac{d-1}{d}} \cdot 2^{\frac{d-2}{d}} \cdots 2^{\frac{i}{d}} = 2^{\frac{d-1}{2}}.$$

Since $y_1, \ldots, y_d$ is an arbitrary sequence with the above properties, we finally show that $c_d = 2^{(d-1)/2}$. □

It is a more difficult task to compute C_d exactly; we will provide an upper bound, however.

Proof of Proposition 1.1 The averaging process is slightly different than before; for each integer a with $x_1 \le a < 2x_1$, we construct a sequence $y_1^a, \ldots, y_d^a$ satisfying $y_1^a = a$ and

$$y_i^a = 2^{b_{ai}} a, \quad \text{where } b_{ai} = -[\log_2 a - \log_2 x_i].$$

As before,

$$\log_2 \frac{y_1^a \cdots y_d^a}{x_1 \cdots x_d} = \sum_{i=1}^{d} \{\log_2 a - \log_2 x_i\}.$$

Summing over all a, we obtain

$$\sum_{a=x_1}^{2x_1-1} \log_2 \frac{y_1^a \cdots y_d^a}{x_1 \cdots x_d} = \sum_{a=x_1}^{2x_1-1} \sum_{i=1}^{d} \{\log_2 a - \log_2 x_i\}.$$

For $i = 1$, we obtain

$$\sum_{a=x_1}^{2x_1-1} \left\{\log_2 \frac{a}{x_1}\right\} < x_1 \int_1^2 \log_2 x \, dx.$$

Now let $i > 1$. The following equality holds:

$$\{\log_2 a - \log_2 x_i\} = \left\{\log_2 \frac{a}{x_1}\right\} - \left\{\log_2 \frac{x_i}{x_1}\right\} + \varepsilon,$$

where $\varepsilon = 0$ or 1, depending on whether $\{\log_2 \frac{x_i}{x_1}\} \le \{\log_2 \frac{a}{x_1}\}$ or $\{\log_2 \frac{x_i}{x_1}\} > \{\log_2 \frac{a}{x_1}\}$. If $\{\log_2 \frac{x_i}{x_1}\} = 0$, then we get the same result as in the case $i = 1$. Otherwise, let l be the unique integer satisfying

$$\log_2 \frac{x_1 + l - 1}{x_1} < \left\{\log_2 \frac{x_i}{x_1}\right\} \le \log_2 \frac{x_1 + l}{x_1}.$$

Of course, $1 \le l \le x_1$. Thus, we obtain

$$\begin{aligned}\sum_{a=x_1}^{2x_1-1} \left\{\log_2 \frac{a}{x_i}\right\} &= \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - x_1 \left\{\log_2 \frac{x_i}{x_1}\right\} + l \\ &< \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - x_1 \log_2 \frac{x_1 + l - 1}{x_1} + l\end{aligned}$$

$$\begin{aligned}
&= \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - \log_2 \left(1 + \frac{l-1}{x_1}\right)^{x_1} + l \\
&< \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - \log_2 2^{l-1} + l \\
&= \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} + 1 \\
&= \sum_{a=x_1+1}^{2x_1} \log_2 \frac{a}{x_1}.
\end{aligned}$$

The latter is an upper Riemann sum, multiplied by x_1, for the function $f(x) = \log_2 x$ for the partition

$$1 = \frac{x_1}{x_1} < \frac{x_1+1}{x_1} < \cdots < \frac{2x_1-1}{x_1} < \frac{2x_1}{x_1} = 2.$$

It is a simple task to prove that

$$\frac{1}{x_1} \sum_{a=x_1+1}^{2x_1} \log_2 \frac{a}{x_1}$$

is decreasing in x_1 and converging, of course, to $\int_1^2 \log_2 x\, dx$. Without loss of generality, we may assume that $x_1 \geq 2$; otherwise, we disregard all terms equal to 1, because we can set $y_i = x_i = 1$, and we consider the first term of the sequence $x_1, \ldots, x_d$ which is greater than 1. So, the maximal value of the Riemann sum is

$$\frac{1}{2}\left(\log_2 \frac{3}{2} + \log_2 \frac{4}{2}\right) = \log_2 \sqrt{3},$$

and hence

$$\sum_{a=x_1}^{2x_1-1} \left\{\log_2 \frac{a}{x_i}\right\} < x_1 \log_2 \sqrt{3}.$$

Thus,

$$\sum_{a=x_1}^{2x_1-1} \sum_{i=1}^{d} \{\log_2 a - \log_2 x_i\} < x_1 \left(\int_1^2 \log_2 x\, dx + (d-1) \log_2 \sqrt{3}\right);$$

therefore, there is a number a for which the following inequality holds:

$$\sum_{i=1}^{d} \{\log_2 a - \log_2 x_i\} < 2 - \frac{1}{\ln 2} + (d-1) \log_2 \sqrt{3},$$

so finally

$$\frac{y_1^a \cdots y_d^a}{x_1 \cdots x_d} < \frac{4}{e} \cdot 3^{(d-1)/2},$$

as desired.

As for the other inequality, we will base our arguments on the example at the end of Lemma 3.1, which shows $c_d \geq 2^{(d-1)/2}$. We will actually prove that for all $\delta > 0$, the following inequality holds:

$$C_d > (1-\delta)^{\frac{d(d-1)}{2}} \cdot 2^{\frac{d-1}{2}}.$$

Let $\delta > 0$ be arbitrary, and let M be a positive integer such that

$$M > \frac{1}{\delta \sqrt[d]{2}}.$$

We define then $x_1 = M$, $x_{i+1} = [x_i \sqrt[d]{2}]$ for $1 \leq i \leq d-1$. Let $y_1, \ldots, y_d$ be a sequence of positive integers satisfying $x_i \leq y_i$ and $y_i | y_{i+1}$ for all i and such that the product $y_1 y_2 \cdots y_d$ is minimal. Since $x_d < 2x_1$, we deduce that $y_d = y_1$ or $2y_1$. If $y_1 = y_d$, then by the minimality assumption, $y_i = x_d$ for all i. Otherwise, let i be the maximal index such that $y_i = x_i$ (i.e., $y_1 = y_2 = \cdots = y_i$, $2y_i = y_{i+1} = \cdots = y_d$). Then again, by minimality we have that $y_i = x_i$. So, the sequence $y_1, \ldots, y_d$ has the form

$$\underbrace{x_i, \ldots, x_i}_{i \text{ terms}}, \underbrace{2x_i, \ldots, 2x_i}_{d-i \text{ terms}}$$

for some index i. We will prove that we actually have $i = d$. Indeed, from the definition of the sequence $\{x_i\}_{i=1}^d$ we have that

$$\frac{x_i \cdot \sqrt[d]{2} - 1}{x_i} < \frac{x_{i+1}}{x_i} < \sqrt[d]{2},$$

which implies

$$\sqrt[d]{2} - \frac{1}{M} < \frac{x_{i+1}}{x_i} < \sqrt[d]{2},$$

and since $M > 1/(\delta \sqrt[d]{2})$, we get

$$(1-\delta)\sqrt[d]{2} < \frac{x_{i+1}}{x_i} < \sqrt[d]{2};$$

thus, for $j > i$,

$$(1-\delta)^{j-i} \cdot 2^{\frac{j-i}{d}} < \frac{x_j}{x_i} < 2^{\frac{j-i}{d}}. \tag{3}$$

For $j = d$, the right-hand side becomes

$$\left(\frac{x_d}{x_i}\right)^d < 2^{d-i},$$

or

$$x_d^d < 2^{d-i} x_i^d = \underbrace{x_i \cdots x_i}_{i \text{ terms}} \cdot \underbrace{2x_i \cdots 2x_i}_{d-i \text{ terms}}.$$

So, we proved that $y_i = x_d$ for all i. Using the left-hand side inequalities of (3), for $j = d$, we obtain

$$\prod_{i=1}^{d-1} \frac{x_d}{x_i} > (1-\delta)^{\frac{d(d-1)}{2}} \cdot 2^{\frac{d-1}{2}};$$

hence,

$$C_d > (1-\delta)^{\frac{d(d-1)}{2}} \cdot 2^{\frac{d-1}{2}}$$

for all $\delta > 0$, and thus

$$C_d \geq 2^{\frac{d-1}{2}},$$

completing the proof. □

Proof of Theorem 1.4 We can make a further improvement; let k be the smallest index such that $\lambda_k > 1$. If $k = 1$, then $G(K) = 1$, and the conjecture is verified. If $k > 1$, then we have a reduction to fewer dimensions, namely $k - 1$, because $K \cap \Lambda$ has at most $k - 1$ linearly independent vectors, by the definition of the successive minima. So, if we intersect K and Λ with the linear hull of these vectors, we get a $(k-1)$-dimensional convex body K' and a $(k-1)$-dimensional lattice Λ' such that $\lambda_i(K', \Lambda') \leq 1$ for all i. Furthermore, $G(K, \Lambda) = G(K', \Lambda')$. This shows that we can reduce to the case where all successive minima are less than or equal to 1. In this case, all q_i are at least equal to 3.

Combining this observation with the proof of Proposition 1.1, we can see that we can take $x_1 \geq 3$ for the purposes of our geometric problem. Therefore, the maximal value for the upper Riemann sum

$$\frac{1}{x_1} \sum_{a=x_1+1}^{2x_1} \log_2 \frac{a}{x_1}$$

is obtained for $x_1 = 3$, which is

$$\frac{1}{3}\left(\log_2 \frac{4}{3} + \log_2 \frac{5}{3} + \log_2 \frac{6}{3}\right) = \log_2 \sqrt[3]{\frac{40}{9}}.$$

Therefore, the corresponding constant, under the restriction $x_1 \geq 3$, is less than or equal to

$$\frac{4}{e}\left(\frac{40}{9}\right)^{\frac{d-1}{3}} \approx 1.47152 \cdot 1.64414^{d-1},$$

concluding the proof. □

We continue with proving the following property of an optimal solution:

Lemma 3.2 *Let $x_1 \le x_2 \le \cdots \le x_d$ be a finite sequence of d positive integers. Let $y_1, \ldots, y_d$ be the "optimal" sequence of integers with respect to $x_1, \ldots, x_d$, i.e., satisfying $x_i \le y_i$, $y_i | y_{i+1}$ for all i, such that the product of all y_i is minimal. Then $y_d < 2x_d$.*

Proof Assume otherwise, and let i be the smallest index such that $y_i \ge 2x_i$. If $i = 1$, then

$$\frac{y_1 \cdots y_d}{x_1 \cdots x_d} \ge 2^d,$$

which contradicts the minimality of the product, as it can be seen from Henk's proof. So $i > 1$, and $y_{i-1} < 2x_{i-1}$. Define y_i' to be the least multiple of y_{i-1} exceeding x_i, and inductively for $i < j \le d$, define y_j' to be the least multiple of y_{j-1}' exceeding x_j. Since $y_{i-1} < 2x_{i-1}$, it is not hard to see that $y_i' < 2x_i \le y_i$ and inductively that $y_j' < 2x_j \le y_j$; thus,

$$y_1 \cdots y_{i-1} \cdot y_i' \cdots y_d' < y_1 \cdots y_d,$$

contradicting the minimality of the product $y_1 \cdots y_d$. Thus $y_d < 2x_d$, as desired. □

At this point, we should mention some values of C_d, for small d:

d	2	3	4
C_d	$\sqrt{2}$	250/117	16384/5187

The proof for $C_2 = \sqrt{2}$ is easy; C_3 and C_4 need case-by-case examination, which can finally be reduced to finitely many d-tuples $x_1, \ldots, x_d$ for $d = 3$ or 4. Those were checked by a simple computer program, written in GWBASIC 3.23, verifying the values above. The optimal solutions were obtained from $x_1 = 4$, $x_2 = 9$, $x_3 = 13$ and $y_1 = 5$, $y_2 = 10$, $y_3 = 20$ for $d = 3$, and from $x_1 = 3$, $x_2 = 7$, $x_3 = 13$, $x_4 = 19$ and $y_1 = 4$, $y_2 = 8$, $y_3 = 16$, $y_4 = 32$ for $d = 4$.

4 A Minor Improvement

By reexamining Henk's proof, we can see that we need not require $n_2 | n_1$; indeed, if this index k is either equal to 1 or ≥ 3, then the proof is the same. We need, however, that $n_3 | n_1$. If $k = 2$, we will use the same convexity argument that was used to verify the conjecture for $d = 2$; that is, we may take $e_1 \in \lambda_1 K$ and $e_2 \in \lambda_2 K \setminus \operatorname{int}(\lambda_2 K)$. Then, we can easily deduce that

$$\frac{n_1}{|n_1|} e_1 + \frac{n_2}{|n_2|} e_2 \in \operatorname{int}\bigl(\operatorname{conv}\{0, e_1, e_2, \lambda_2 g\}\bigr)$$

if $n_1 \neq 0$ or

$$\frac{n_2}{|n_2|} e_2 \in \operatorname{int}\bigl(\operatorname{conv}\{0, e_1, e_2, \lambda_2 g\}\bigr)$$

otherwise, contradicting the definition of λ_2. For a proof of these statements, see Theorem 2.2 in [3].

Hence, we may drop the requirement that y_{d-1} divides y_d in the arithmetic problem (replacing it by $y_{d-2}|y_d$):

Definition 4.1 Let C'_d denote the least positive constant such that for any non-decreasing sequence of integers $x_1, x_2, \dots, x_d$, there exists a sequence of integers $y_1, y_2, \dots, y_d$ satisfying:

(a) $x_i \leq y_i$ for all i, $1 \leq i \leq d$.
(b) y_i divides y_{i+1} for all i, $1 \leq i \leq d-2$, and y_{d-2} divides y_d.
(c) $\frac{y_1 y_2 \cdots y_d}{x_1 x_2 \cdots x_d} \leq C'_d$.

This constant obviously satisfies $c_d \leq C'_d \leq C_d$ for $d \geq 3$. The technique in the proof of Proposition 1.1 does not provide a better bound in the general case; however, C'_d is more easily computable for small values of d.

We will also need the following lemma:

Lemma 4.1 *Let $x_1 \leq x_2 \leq \cdots \leq x_d \leq 2x_1$ be a finite sequence of d positive integers. There is always an increasing sequence of d positive integers $y_1, \dots, y_d$ satisfying $y_i | y_{i+1}$, $x_i \leq y_i$ for all i and*

$$\frac{y_1 \cdots y_d}{x_1 \cdots x_d} \leq 2^{\frac{d-1}{2}}.$$

Proof For each i, we construct the sequence $y_1^i, \dots, y_d^i$ satisfying

$$y_1^i = \cdots = y_i^i = x_i, \qquad y_{i+1}^i = \cdots = y_d^i = 2x_i.$$

Obviously, each of these d sequences satisfies $y_j^i | y_{j+1}^i$ and $x_j \leq y_j^i$. Furthermore,

$$\prod_{i=1}^{d} \frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} = 2^{\frac{d(d-1)}{2}},$$

and thus there is at least one i such that

$$\frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} \leq 2^{\frac{d-1}{2}}.$$ □

4.1 The Case $d = 3$

Theorem 4.1 $C'_3 = 2$.

Proof In each of the following cases for x_1, x_2, x_3, we will construct three integers y_1, y_2, y_3 satisfying $x_i \le y_i$, $y_1 | y_2$, $y_1 | y_3$, and

$$\frac{y_1 y_2 y_3}{x_1 x_2 x_3} \le 2.$$

Case 1 $x_2 \ge \frac{3}{\sqrt{2}} \cdot x_1$: We put $y_1 = x_1$, and let y_2, y_3 be the least multiples of x_1 exceeding x_2, x_3, respectively. By hypothesis, we will have $y_i / x_i \le \sqrt{2}$ for $i = 2, 3$, and thus

$$\frac{y_1 y_2 y_3}{x_1 x_2 x_3} \le 2.$$

Case 2 $2x_1 < x_2 < \frac{3}{\sqrt{2}} \cdot x_1$: The existence of an integer strictly between $2x_1$ and $\frac{3}{\sqrt{2}} \cdot x_1$ implies $x_1 \ge 9$. If $x_3 \ge \frac{9}{4} \cdot x_1$, then we put $y_1 = x_1$, and let y_2, y_3 be the least multiples of x_1 exceeding x_2, x_3, respectively. By hypothesis, $y_2/x_2 < 3/2$ and $y_3/x_3 \le 4/3$, and thus

$$\frac{y_1 y_2 y_3}{x_1 x_2 x_3} \le 2.$$

If $x_3 < \frac{9}{4} \cdot x_1$, then if x_3 is even, we put $y_1 = x_3/2$, $y_2 = y_3 = x_3$. We will have

$$\frac{y_1 y_2 y_3}{x_1 x_2 x_3} < \left(\frac{x_3}{2x_1}\right)^2 < \left(\frac{9}{8}\right)^2 < 1.27.$$

If x_3 is odd, we put $y_1 = (x_3 + 1)/2$, $y_2 = y_3 = x_3 + 1$. Then

$$\begin{aligned}\frac{y_1 y_2 y_3}{x_1 x_2 x_3} &< \left(\frac{\frac{9}{4} \cdot x_1 + 1}{2x_1}\right)^2 \cdot \frac{x_3 + 1}{x_3} = \left(\frac{9}{8} + \frac{1}{2x_1}\right) \cdot \left(1 + \frac{1}{x_3}\right) \\ &\le \left(\frac{9}{8} + \frac{1}{18}\right) \cdot \left(1 + \frac{1}{2x_1 + 1}\right) \le \frac{85}{72} \cdot \frac{20}{19} < 2.\end{aligned}$$

Case 3 $\sqrt{2} \cdot x_1 \le x_2 \le 2x_1$: If either $x_3 \le 2x_1$ or $x_3 \ge \frac{3}{\sqrt{2}} \cdot x_1$, then we put $y_1 = x_1$, and let y_2, y_3 be the least multiples of x_1 exceeding x_2, x_3, respectively. By hypothesis, we will have $y_i / x_i \le \sqrt{2}$ for $i = 2, 3$, and thus

$$\frac{y_1 y_2 y_3}{x_1 x_2 x_3} \le 2.$$

Suppose now that $2x_1 < x_3 < \frac{3}{\sqrt{2}} \cdot x_1$. Then, $x_1 \ge 9$, and we put $y_3 = x_3$ if x_3 is even and $y_3 = x_3 + 1$ otherwise. Then, we put $y_1 = y_3/2$ and $y_2 = y_3$. It is not hard to verify that $(y_1 y_3)/(x_1 x_3) < 1.18$ and

$$\frac{y_2}{x_2} < \frac{\frac{3}{\sqrt{2}} \cdot x_1 + 1}{\sqrt{2} \cdot x_1} = \frac{3}{2} + \frac{1}{\sqrt{2} \cdot x_1} < \frac{3}{2} + \frac{1}{12} = \frac{19}{12}.$$

Thus,

$$\frac{y_1y_2y_3}{x_1x_2x_3} < \frac{19}{12} \cdot 1.18 < 2.$$

Case 4 $x_2 < \sqrt{2} \cdot x_1$: If $x_3 < \sqrt{2} \cdot x_2$, then $x_3 < 2x_1$, and we apply Lemma 4.1. If either $\sqrt{2} \cdot x_2 \le x_3 \le 2x_2$ or $x_3 \ge \frac{3}{\sqrt{2}} \cdot x_2$, we put $y_1 = y_2 = x_2$, and let y_3 be the least multiple of x_2 exceeding x_3. Then, $y_1/x_1 < \sqrt{2}$ and $y_3/x_3 \le \sqrt{2}$, and thus

$$\frac{y_1y_2y_3}{x_1x_2x_3} < 2.$$

It only remains to check the case $2x_2 < x_3 < \frac{3}{\sqrt{2}} \cdot x_2$. We will have $x_2 \ge 9$ and $x_1 \ge 7$. As in case 3 above, we can find integers y_2, y_3 such that $x_i \le y_i$, $y_2|y_3$, and

$$\frac{y_2y_3}{x_2x_3} < 1.18.$$

In particular, $y_2/x_2 < 1.18$. Now, we put $y_1 = y_2$. We will have $y_1/x_1 = (y_2/x_2) \cdot (x_2/x_1) < 1.18 \cdot \sqrt{2}$, and thus

$$\frac{y_1y_2y_3}{x_1x_2x_3} < \sqrt{2} \cdot 1.18^2 < 2,$$

completing the proof. □

Theorem 4.1 clearly implies the first part of Proposition 1.2 ($d = 3$).

4.2 The Case $d = 4$

Even though C_4' is more easily computable than C_4, computer tests could not be avoided. The optimal quadruplets were $x_1 = 6$, $x_2 = 15$, $x_3 = 19$, $x_4 = 25$ and $y_1 = 7$, $y_2 = y_3 = 21$, $y_4 = 42$. They yield

$$\frac{y_1y_2y_3y_4}{x_1x_2x_3x_4} = \frac{7203}{2375} \approx 3.032842105263157894736842105263157894736\ldots.$$

This proves the second part of Proposition 1.2 ($d = 4$).

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