Untangling Polygons and Graphs

Josef Cibulka

Received: 25 June 2008 / Revised: 14 February 2009 / Accepted: 14 February 2009 /

Published online: 6 March 2009

© Springer Science+Business Media, LLC 2009

Abstract Untangling is a process in which some vertices in a drawing of a planar graph are moved to obtain a straight-line plane drawing. The aim is to move as few vertices as possible. We present an algorithm that untangles the cycle graph C_n while keeping $\Omega(n^{2/3})$ vertices fixed.

For any connected graph G , we also present an upper bound on the number of fixed vertices in the worst case. The bound is a function of the number of vertices, maximum degree, and diameter of G . One consequence is that every 3-connected planar graph has a drawing δ such that at most $O((n \log n)^{2/3})$ vertices are fixed in every untangling of δ .

Keywords Graph drawing · Planarity · Straight-line drawing · Untangling · Polygons

1 Introduction

Every planar graph can be drawn with each vertex at a prespecified location, so that edges are pairwise noncrossing curves. According to the theorem of Wagner [11], Fáry [4], and Stein [9], every planar graph can be drawn in the plane so that edges are noncrossing straight line segments. However, to obtain such a straight-line plane drawing of a graph with given vertex positions, we may need to move some of the vertices; this process is called *untangling*. It is natural to ask how many vertices can keep their positions during untangling. If a vertex keeps its position, it is called *fixed*, otherwise it is called *free*.

Supported by project MSM0021620838 of the Czech Ministry of Education.

J. Cibulka (✉)

Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic

e-mail: cibulka@kam.mff.cuni.cz

Table 1

Graph class $\mathcal{G}$	Lower bound		Upper bound	
Cycles	$\Omega(n^{2/3})$	Theorem 1	$O((n \log n)^{2/3})$	[7]
Trees	$\sqrt{n/2}$	[2]	$3\sqrt{n} - 3$	[2]
Outerplanar graphs	$\sqrt{n/2}$	[5, 8]	$2\sqrt{n-1} + 1$	[5]
Planar graphs	$\sqrt[4]{(n+1)/2}$	[2]	$\sqrt{n-2} + 1$	[5]

In the following, a *drawing* of a graph G is an injective mapping from $V(G)$ to a set of points in $\mathbb{R}^2$ such that each edge $vw \in E(G)$ is represented by the straight-line segment between the images of v and w . A drawing is *plane* if no two edges cross.

Let G be a graph, and let δ be a drawing of G . We define

$$\begin{aligned} \text{fix}(G, \delta) &= \max_{\beta \text{ plane drawing of } G} |\{v \in V(G) : \beta(v) = \delta(v)\}|, \\ \text{fix}(G) &= \min_{\delta \text{ drawing of } G} \text{fix}(G, \delta). \end{aligned}$$

It is possible to restrict the initial drawings δ in the definition of $\text{fix}(G)$ to those with all vertices on a line, because Kang et al. [6] proved that for some such δ , $\text{fix}(G) = \text{fix}(G, \delta)$.

It is NP-hard to compute $\text{fix}(G, \delta)$ [5, 10] and even to approximate it [5].

At the 5th Czech–Slovak Symposium on Combinatorics in Prague in 1998, Mamoru Watanabe asked whether every polygon on n vertices can be turned into a noncrossing polygon by moving at most εn vertices for some constant $\varepsilon > 0$. This is equivalent to asking whether $\text{fix}(C_n) \geq (1 - \varepsilon)n$ for the cycle graph C_n . Pach and Tardos [7] answered this question in the negative by showing that $\text{fix}(C_n) \in O((n \log n)^{2/3})$. They also proved the lower bound $\text{fix}(C_n) \geq \lfloor \sqrt{n} + 1 \rfloor$. We almost close the gap for cycle graphs by designing an algorithm that always keeps $\Omega(n^{2/3})$ vertices fixed; that is, $\text{fix}(C_n) \in \Omega(n^{2/3})$.

Table 1 summarizes the best known bounds on $\min_{G \in \mathcal{G}, |V(G)|=n} (\text{fix}(G))$ for several graph classes $\mathcal{G}$.

It is known that $\text{fix}(G) \in \Omega(n^{1/4})$ [2] and $\text{fix}(G) \in \Omega(\sqrt{\Delta(G)} + \sqrt{d(G)})$ [5], where $\Delta(G)$ is the maximum degree, and $d(G)$ is the diameter of the given planar graph G on n vertices.

In Sect. 3, we prove an upper bound on $\text{fix}(T)$ for any tree T . It implies that

$$\text{fix}(G) \leq 300\sqrt{n} \log n (\sqrt{\Delta(G)} + \min\{\sqrt[6]{n/\log^2 n}, \sqrt{2d(G)}\})$$

for any connected planar graph G . The upper bound for trees has additional interesting consequences:

- For any 3-vertex-connected planar graph G , $\text{fix}(G) \in O((n \log n)^{2/3})$.
- It holds that $\text{fix}(G) \in O(\sqrt{n}(\log n)^{3/2})$ for any connected planar graph G with both the maximum degree and the diameter in $O(\log n)$ (for example, the complete binary tree).

- If a connected planar graph G satisfies $\text{fix}(G) \geq \gamma n$ for some $\gamma > 0$, then it has a vertex of degree $\Omega(n\gamma^2/\log^2 n)$.

All logarithms in this paper are base 2.

2 Algorithm for Untangling Cycles

Let C_n be the graph with vertices $v_1, v_2, \dots, v_n$ and edges $(v_1, v_2), (v_2, v_3), \dots, (v_n, v_1)$.

Theorem 1 $\text{fix}(C_n) \geq 2^{-5/3}n^{2/3} - O(n^{1/3}) \in \Omega(n^{2/3})$.

Proof Let m be the largest integer such that $m \leq n - 4$ and $(m/16)^{1/3}$ is an integer. Then $m \geq 16(((n-4)/16)^{1/3} - 1)^3 \geq n - O(n^{2/3})$. We define $l := (m/16)^{1/3}$ and $s := 2l$. For the given vertex positions, we fix a horizontal direction so that no two vertices lie on any horizontal or vertical line.

Vertices $v_{m+1}, v_{m+2}, \dots, v_n$ will be free and will make bends on the line between the last and the first fixed vertex.

Our aim is to select the fixed vertices so that they lie in l horizontal layers, each with s fixed vertices with indices increasing (without loss of generality) from left to right. We will also guarantee that the indices i, j of two fixed vertices v_i, v_j from different layers differ by at least a sufficiently large number so that we have enough free vertices to make bends on the line connecting v_i and v_j .

We start creating layers from top to bottom. Each layer consists of the highest s^2 vertices that are not free and lie below all previously created layers. Using the Erdős–Szekeres lemma [3], we select among their indices a monotone subsequence of length exactly s and free the remaining vertices of the layer. There are two types of layers: if the selected vertices have indices increasing from left to right, the layer is an *increasing layer*, otherwise it is a *decreasing layer*. Then we free every vertex that lies below this layer and whose index differs by at most $2(l - i - 1)$ from the index of some of the selected vertices. Here i is the number of previously created layers of the same type. We stop after creating l layers of the same type.

We need to verify that we have enough vertices for all the layers. Each of the at most $2l$ layers consists of s^2 vertices, and for each of the s selected vertices in the i th increasing or in the i th decreasing layer, we freed at most $4(l - i)$ vertices lying below the layer. The number of considered vertices is thus at most

$$2ls^2 + 2s \sum_{i=1}^l 4(l - i) = 8l^3 + 4sl(l - 1) \leq 16l^3 = m.$$

Without loss of generality, we created l increasing layers. The ls selected vertices in these l layers are our fixed vertices; we call them $u_1 = v_{i_1}, u_2 = v_{i_2}, \dots, u_{ls} = v_{i_{ls}}$, where $i_1 < i_2 < \dots < i_{ls}$. We assign new positions to the free vertices v_j satisfying $i_1 < j < i_{ls}$ in the order of increasing indices:

If u_k and u_{k+1} lie in the same layer, we connect them by a straight line segment and place on it the free vertices $v_{i_{k+1}}, \dots, v_{i_{k+1}-1}$ between u_k and u_{k+1} .

Fig. 1 Joining the increasing sequences

Otherwise, we have at least $2d + 2$ free vertices between u_k and u_{k+1} , where d is the number of increasing layers between the layers of u_k and u_{k+1} —these vertices were freed after selecting the higher of u_k and u_{k+1} . We connect u_k and u_{k+1} by a polygonal path with $2d + 2$ bends. All its segments will be either horizontal or vertical, except for the first one, which goes from u_k almost vertically and slightly to the right to avoid having a common subsegment with the last segment of the path between u_{k-1} and u_k . Between each two consecutive layers, we take a horizontal strip without any fixed vertices and call it the *interlayer region*. We place one vertical segment for each of the $d + 2$ layers between the layers of u_k and u_{k+1} , inclusive. Each vertical segment is placed so that all segments already placed in this layer are to the left of it and all the fixed vertices u_j with $j > k + 1$ in this layer are to the right. Its endpoints lie inside interlayer regions (except for the first and the last segment that have one endpoint in u_k and u_{k+1} , respectively) so that the segment does not cross any already placed horizontal segment. The horizontal segments are placed in the interlayer region to connect pairs of vertical segments. This is always possible because one of the two vertical segments that are to be connected lies to the right of every horizontal segment in this interlayer region and thus can go as deep as necessary into the interlayer region. See Fig. 1.

At the end, we will connect u_{l_s} and u_1 by a line with four bends and place on it all the remaining free vertices.

The number of fixed vertices is $l_s = 2l^2 = 2^{-\frac{5}{3}} m^{\frac{2}{3}} \geq 2^{-\frac{5}{3}} n^{\frac{2}{3}} - O(n^{\frac{1}{3}})$. $\square$

3 Upper Bounds

The *crossing number* $\text{cr}(G)$ of a graph G is the lowest number of crossings in a planar (not necessarily straight-line) drawing of G .

We will use the following lemma of Pach and Tardos:

Lemma 2 (Pach and Tardos [7]) *Let H_b be the graph with vertices $u_1, u_2, \dots, u_t$ and edges $(u_1, u_2), (u_2, u_3), \dots, (u_t, u_1)$. Let H_r be a random Hamiltonian cycle on the same vertex set, that is, its edges are $(u_{\sigma(1)}, u_{\sigma(2)}), (u_{\sigma(2)}, u_{\sigma(3)}), \dots, (u_{\sigma(t)}, u_{\sigma(1)})$, where σ is a random permutation of $\{1, 2, \dots, t\}$.*

Then, for any integers $s < t$ and K , the probability that the crossing number $\text{cr}(H)$ of $H = H_b \cup H_r$ is at most K satisfies

$$\text{Prob}[\text{cr}(H) \leq K] \leq \binom{t}{D}^2 \left(\frac{2t}{s}\right)^D \frac{s^{t-D}}{(t-D)!},$$

where $D := \lfloor 35\sqrt{t(K+t)/s} \rfloor$.

The edges of H_r and H_b are called *red edges* and *black edges*, respectively.

Pach and Tardos [7] used the lemma to prove the upper bound $\text{fix}(C_n) \in O((n \log n)^{2/3})$. This proof proceeds by placing the vertices of the cycle graph C_n on a circle in a random order and assuming that this drawing of C_n can be untangled while keeping some $t = c(n \log n)^{2/3}$ vertices fixed, where c is an appropriate constant. Then they consider the graph H whose edge set is the union of the edges of two cycle graphs H_r and H_b on the t fixed vertices. H_r is drawn as the convex hull of the fixed vertices, and H_b is drawn as the untangled C_n but with all its free vertices considered as bends. The untangling of C_n then gives a (not necessarily straight-line) drawing of $H = H_b \cup H_r$ with not many crossings. Then they use the lemma and show that the probability that the graph H has such a small crossing number is less than $\binom{n}{t}^{-1}$, and thus for some ordering of the vertices of C_n , no t -tuple of vertices can stay fixed while untangling C_n .

Theorem 3 *Let T be a tree on n vertices. Let Δ be the maximum degree of T , and let d be the diameter of T . Then*

$$\text{fix}(T) \leq 300\sqrt{n} \log n (\sqrt{\Delta} + \min\{\sqrt[6]{n/\log^2 n}, \sqrt{d}\}).$$

Proof If $n \leq 90000$, the statement is trivially true, so we will assume $n > 90000$.

We use a modification of the above described approach of Pach and Tardos. The cycle graph H_r can be drawn in the same way, but the complication is that we do not have the noncrossing drawing of the graph H_b . Instead, we take a closed walk on a subgraph of T visiting all the fixed vertices of T in some prescribed order. If T can be untangled, we construct in Lemma 4 a drawing of the graph H_b such that H has not many crossings, and then we use Lemma 2.

Let T be the given tree on n vertices. We select one of its vertices to be the root. Let a *DFS-cycle* be an oriented closed walk that follows some depth-first search (DFS) performed on T starting at the root. A DFS-cycle contains each orientation of every edge exactly once. By a *drawing of a DFS cycle* we will mean a drawing (not necessarily straight-line) of the oriented cycle graph whose vertices and edges correspond to occurrences of vertices and edges on the DFS-cycle. Vertices of the drawing corresponding to a vertex v of T are placed near the position of v . See Fig. 2.

We fix one DFS-cycle and number the vertices in the order in which they were first visited during the depth-first search. Let $v_1, v_2, \dots, v_n$ be the resulting sequence. Then we place all the vertices of T in a random order to the vertices of a convex polygon. Let δ denote this mapping of vertices to points.

Fig. 2 A tree and a drawing of one of its DFS-cycles

Let $u_1 = v_{i_1}, \dots, u_t = v_{i_t}$ for some $i_1 < i_2 < \dots < i_t$. The value of t will be fixed later. Let H_b be the cycle graph on $u_1, u_2, \dots, u_t$ with edges $(u_1, u_2), (u_2, u_3), \dots, (u_t, u_1)$, let H_r be the cycle graph whose edges are the edges of the convex hull of $u_1, u_2, \dots, u_t$, and let $H = H_r \cup H_b$.

Lemma 4 *If the above defined drawing δ of T can be untangled while keeping the vertices $u_1, \dots, u_t$ fixed, then the graph H satisfies*

$$\text{cr}(H) \leq \lfloor 1.5t\Delta + \min\{8n, 2td\} \rfloor,$$

where Δ is the maximum degree of T , and d is the diameter of T .

Proof The cycle H_r is drawn as the convex hull of $u_1, u_2, \dots, u_t$. Thus there are no crossings of pairs of red edges.

Let T' be the smallest subtree of T that spans all the fixed vertices. We take the DFS-cycle on T' that comes from the fixed DFS-cycle on T by omitting vertices outside T' . Notice that the first occurrences of the fixed vertices u_i appear on the DFS-cycle of T' in the same order as on the DFS-cycle of T , that is, in the order of increasing indices. To obtain a drawing of H_b , we construct a drawing of the DFS-cycle on T' and then draw each edge (u_i, u_{i+1}) of H_b as the part of the DFS-cycle between the first occurrences of u_i and u_{i+1} .

We take the plane drawing of T in which $u_1, \dots, u_t$ keep their positions. From this drawing we get a drawing of the DFS-cycle on T' by expanding every vertex of T' as in Fig. 3.

We start by expanding the root of T' . We then expand each vertex v_i when its only expanded neighbor is its *parent* (that is, its unique neighbor on the path in T' to the root). Let p_i denote the parent of v_i ; if v_i is the root, then we define p_i to be the second vertex visited during DFS. We take a point C near v_i on the line through p_i and v_i , but outside the line segment between p_i and v_i . Let $\mathcal{S}$ be the circle centered at C with radius chosen so that v_i lies on $\mathcal{S}$. We expand the vertex v_i into a set of *sub-vertices* that we all place on $\mathcal{S}$. First, we create two sub-vertices: $\hat{v}_i$, which keeps the position of v_i , and $\bar{v}_i$, which is placed near $\hat{v}_i$. Sub-vertices $\hat{v}_i$ and $\bar{v}_i$ are ends of the straight line segments corresponding to the oriented edges (p_i, v_i) and (v_i, p_i) of the DFS-cycle, respectively. If v_i is not the root, the vertex $\bar{v}_i$ is placed so that the segments corresponding to (v_i, p_i) and (p_i, v_i) do not cross. For each neighbor $v_j \neq$

Fig. 3 Expanding the vertex v_i in a plane drawing of T'

p_i of v_i , we create sub-vertices $\hat{v}_i^j$ and $\bar{v}_i^j$ near the intersection of S and the segment between C and v_j . Sub-vertices $\hat{v}_i^j$ and $\bar{v}_i^j$ are ends of the segments corresponding to the oriented edges (v_j, v_i) and (v_i, v_j) of the DFS-cycle, respectively. Finally, we connect pairs of the newly created sub-vertices by segments inside S —if (v_j, v_i) is followed by (v_i, v_k) in the DFS-cycle, then we connect $\hat{v}_i^j$ with $\bar{v}_i^k$ (if $v_j = p_i$, then we define $\hat{v}_i^j := \hat{v}_i$, and similarly if $v_k = p_i$, then $\bar{v}_i^k := \bar{v}_i$). If v_i has degree 2 in T' , then we consider the neighbor $v_j \neq p_i$ and choose the ordering of the sub-vertices $\hat{v}_i^j$ and $\bar{v}_i^j$ on S so that the two segments inside S do not cross. This is possible without introducing any unwanted crossings, because $\hat{v}_i^j$ and $\bar{v}_i^j$ are not yet connected to $\bar{v}_i^j$ and $\hat{v}_i^j$.

Straight line segments of the drawing of the DFS-cycle will be called *black segments*. Black segments inside S will be called *short black segments*, all the other black segments will be *long black segments*.

Next we draw an edge (u_i, u_{i+1}) of H_b as the part of the drawing of the DFS-cycle between $\hat{u}_i$ and $\hat{u}_{i+1}$. Since T' is the smallest subtree of T containing all the fixed vertices, all the leaves of T' are fixed vertices. After visiting a leaf, a DFS visits another leaf after no more than d steps, and thus there are at most d edges between two consecutive first visits of fixed vertices in the DFS cycle. Therefore the drawing of the whole cycle H_b is composed of at most $\min\{4n, 2td\}$ black segments.

Because H_r is drawn as a convex polygon, each straight line segment of H_b crosses it in at most two points (after perturbing ends of the segments that have a common subsegment with H_r). There are thus at most $\min\{8n, 4td\}$ crossings of pairs of edges with different colors.

Since T' is drawn with no crossings, there are no crossings between any long black segment and another black segment. Short black segments cross only short black segments within the same expanded vertex of T' , because no other black segment enters the circle S inside which the short black segment lies. For a vertex v with degree at most 2, the construction guarantees that the short black edges created from v do not cross each other. Since T' has at most t leaves and because the average degree of a vertex in a tree is less than 2, the sum of degrees of vertices with degree at least 3 is less than $3t$, and thus there are at most $3t$ short black segments that cross

some other short black segment. Each vertex of T' has degree at most Δ , and thus each of the at most $3t$ short black segments that cross some other black segment crosses at most $\Delta - 1$ black segments. Therefore the total number of crossings of pairs of black edges is less than $1.5t\Delta$. $\square$

Let

$$\begin{aligned} K &:= \lfloor 1.5t\Delta + \min\{8n, 2td\} \rfloor, \\ t &:= \lceil 300\sqrt{n} \log n (\sqrt{\Delta} + \min\{\sqrt[6]{n/\log^2 n}, \sqrt{d}\}) \rceil, \\ s &:= \left\lceil 35^2 \frac{K+t}{t} \log^2 n \right\rceil. \end{aligned}$$

The theorem will be proven if we show that $\text{fix}(T) < t$. If $t \geq n + 1$, the statement of the theorem holds trivially, therefore we will only consider the values for which $t \leq n$ and thus $\Delta \leq n/(300^2 \log^2 n)$.

We have $s < t$ as required, because $K/t \leq 1.5\Delta + \sqrt{n}/(35 \log n)$.

We also have

$$D = \left\lfloor 35\sqrt{\frac{t(K+t)}{s}} \right\rfloor \leq \frac{t}{\log n}, \quad (1)$$

where the inequality follows from the definition of s .

If T can be untangled while keeping t vertices fixed, then for some t -tuple of its n vertices, the above-defined graph $H = H_b \cup H_r$ has crossing number at most K by Lemma 4. For every t -tuple, H_b is exactly as required by Lemma 2, and H_r goes through the vertices of H_b in a uniformly distributed random order, because at the beginning, we placed the vertices of T in a uniformly distributed random order.

Therefore we can apply Lemma 2 and get:

$$\begin{aligned} \text{Prob}[\text{fix}(T, \delta) \geq t] &\leq \binom{n}{t} \text{Prob}[\text{cr}(H) \leq K] \quad \text{by Lemma 4} \\ &\leq \binom{n}{t} \binom{t}{D}^2 \left(\frac{2t}{s}\right)^D \frac{s^{t-D}}{(t-D)!} \quad \text{by Lemma 2} \\ &\leq \left(\frac{en}{t}\right)^t \left(\frac{et}{D}\right)^{2D} \left(\frac{2t}{s}\right)^D \left(\frac{es}{t(1 - \frac{1}{\log n})}\right)^{t-D} \quad \text{by (1)} \\ &\leq \left(e^2 \frac{ns}{t^2}\right)^t \left(\frac{2et^4}{D^2 s^2}\right)^D \left(1 - \frac{1}{\log n}\right)^{D-t} \\ &\leq \left(e^2 \frac{ns}{t^2}\right)^t \left(\frac{2e^3 t^4}{D^2 s^2}\right)^{\frac{t}{\log n}} \\ &\quad \text{by (1) and } 1 - x \leq e^{-2x} \text{ for } x < 1/2 \\ &\leq \left(4e^2 \frac{ns}{t^2}\right)^t \quad \text{since } D \geq 35t/\sqrt{s} \text{ and } t^2 \leq n^2 = 4^{\log n} \end{aligned}$$

$$\leq \left(8e^2 36^2 \frac{n \log^2 n (\min\{4n, td\} + t\Delta)}{t^3} \right)^t$$

by the definition of s and by $t \leq t \cdot \Delta/2$.

We distinguish two cases:

- (a) If $\sqrt[6]{n/\log^2 n} \geq \sqrt{d}$, then $t = \lceil 300\sqrt{n} \log n(\sqrt{\Delta} + \sqrt{d}) \rceil$, and we can estimate

$$\begin{aligned} \text{Prob}[\text{fix}(T, \delta) \geq t] &\leq \left(8e^2 36^2 \frac{n \log^2 n (\Delta + d)}{t^2} \right)^t \\ &\leq \left(\frac{8e^2 36^2}{300^2} \frac{\Delta + d}{(\sqrt{\Delta} + \sqrt{d})^2} \right)^t \\ &< 0.9^t. \end{aligned}$$

- (b) Otherwise $t = \lceil 300\sqrt{n} \log n(\sqrt{\Delta} + \sqrt[6]{n/\log^2 n}) \rceil$ and

$$\begin{aligned} \text{Prob}[\text{fix}(T, \delta) \geq t] &\leq \left(8e^2 36^2 \frac{n \log^2 n (t\Delta + 4n)}{t^3} \right)^t \\ &\leq \left(\frac{8e^2 36^2}{300^2} \frac{t\Delta + 4n}{t(\sqrt{\Delta} + n^{\frac{1}{6}}(\log n)^{-\frac{1}{3}})^2} \right)^t \\ &\leq \left(\frac{8e^2 36^2}{300^2} \frac{t\Delta + 4n}{t\Delta + tn^{\frac{1}{3}}(\log n)^{-\frac{2}{3}}} \right)^t \\ &\leq \left(0.9 \frac{t\Delta + 4n}{t\Delta + 300n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}n^{\frac{1}{3}}(\log n)^{-\frac{2}{3}}} \right)^t \\ &< 0.9^t. \end{aligned}$$

Using the probabilistic method, we conclude that for some drawing δ of T , it holds that $\text{fix}(T, \delta) < t$. $\square$

Corollary 5

1. For every connected planar graph G with maximum degree Δ and diameter d ,

$$\text{fix}(G) \leq 300\sqrt{n} \log n(\sqrt{\Delta} + \min\{\sqrt[6]{n/\log^2 n}, \sqrt{2d}\}).$$

2. There is a constant c such that for every 3-vertex-connected planar graph G ,

$$\text{fix}(G) \leq c(n \log n)^{\frac{2}{3}}.$$

3. *There is a constant c such that for every constant $\gamma > 0$, any spanning tree of any connected planar graph G on n vertices with $\text{fix}(G) \geq \gamma n$ has a vertex with degree at least*

$$\Delta \geq c \frac{n\gamma^2}{\log^2 n}.$$

4. *There is a constant c such that for every constant b , every connected planar graph G on n vertices with both the maximum degree and the diameter at most $b \log n$ satisfies*

$$\text{fix}(G) \leq c\sqrt{bn \log^3 n}.$$

Proof All claims are based on the simple observation that adding edges to a graph H never increases $\text{fix}(H)$, and thus if T is a spanning tree of G , then $\text{fix}(G) \leq \text{fix}(T)$. Part 3 is then straightforward and Part 2 follows from a theorem of Barnette [1], which says that every planar 3-vertex-connected graph has a spanning tree with maximum degree three.

To prove Parts 1 and 4, we now prove the well-known fact that any graph G has a spanning tree T with diameter at most $2d(G)$. Fix any vertex v of G and run a breadth-first search from it. All vertices lie at distance at most $d(G)$ from v , and thus the diameter of the breadth-first search tree is at most $2d(G)$. $\square$

Acknowledgements I am grateful to Alexander Wolff for a careful reading of an earlier version of the paper and many useful suggestions. Thanks to the referees for their comments that helped to improve the presentation of the paper.

References

1. Barnette, D.: Trees in polyhedral graphs. *Can. J. Math.* **18**, 731–736 (1966)
2. Bose, P., Dujmović, V., Hurtado, F., Langerman, S., Morin, P., Wood, D.: A polynomial bound for untangling geometric planar graphs. *Discrete Comput. Geom.* (2008). doi:[10.1007/s00454-008-9125-3](https://doi.org/10.1007/s00454-008-9125-3)
3. Erdős, P., Szekeres, G.: A combinatorial problem in geometry. *Compos. Math.* **2**, 463–470 (1935)
4. Fáry, I.: On straight line representation of planar graphs. *Acta Univ. Szeged, Acta Sci. Math.* **11**, 229–233 (1948)
5. Goaoc, X., Kratochvíl, J., Okamoto, Y., Shin, C.-S., Spillner, A., Wolff, A.: Untangling a planar graph. *Discrete Comput. Geom.* (2009) doi:[10.1007/s00454-008-9130-6](https://doi.org/10.1007/s00454-008-9130-6)
6. Kang, M., Pikhurko, O., Ravsky, A., Schacht, M., Verbitsky, O.: Obfuscated drawings of planar graphs. [0803.0858](https://arxiv.org/abs/0803.0858)
7. Pach, J., Tardos, G.: Untangling a polygon. *Discrete Comput. Geom.* **28**(4), 585–592 (2002)
8. Ravsky, A., Verbitsky, O.: On collinear sets in straight line drawings. [0806.0253](https://arxiv.org/abs/0806.0253)
9. Stein, S.K.: Convex maps. *Proc. Am. Math. Soc.* **2**, 464–466 (1951)
10. Verbitsky, O.: On the obfuscation complexity of planar graphs. *Theor. Comput. Sci.* **396**(1–3), 294–300 (2008)
11. Wagner, K.: Bemerkungen zum Vierfarbenproblem. *Jahresber. Deutsch. Math. Verein.* **46**, 26–32 (1936)