An Improvement to “A Note on Euclidean Ramsey Theory”

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Abstract We characterize those subsets $Y \subseteq \mathbb{R}^n$ such that for every infinite $X \subseteq \mathbb{R}^n$, there is a red/blue coloring of $\mathbb{R}^n$ having no monochromatic red set similar to X and no monochromatic blue set similar to Y .

Keywords Euclidean Ramsey theory

Consider the finite-dimensional Euclidean space $\mathbb{R}^n$. If $X, Y \subseteq \mathbb{R}^n$, then we write

$$\mathbb{R}^n \longrightarrow (X, Y)$$

if for any $A \subseteq \mathbb{R}^n$, there is $X' \subseteq A$ similar to X or there is $Y' \subseteq \mathbb{R}^n \setminus A$ similar to Y . If $\mathbb{R}^n \rightarrow (X, Y)$ is false, then we write $\mathbb{R}^n \not\rightarrow (X, Y)$.

An *affine subspace* of $\mathbb{R}^n$ is a translate of a subspace of $\mathbb{R}^n$. For example, a line is just a 1-dimensional affine subspace. If $Y \subseteq \mathbb{R}^n$, then let $\text{aff}(Y)$ be the affine subspace generated by Y . In general, $|Y| \geq \dim(\text{aff}(Y)) + 1$.

The purpose of this paper is to prove the following theorem, which improves the principal result of [7].

Theorem *Suppose that $2 \leq n < \omega$ and $Y \subseteq \mathbb{R}^n$. The following are equivalent:*

- (1) *For every infinite $X \subseteq \mathbb{R}^n$, $\mathbb{R}^n \not\rightarrow (X, Y)$.*
- (2) *Either $\text{aff}(Y) = \mathbb{R}^n$ or $|Y| \geq \dim(\text{aff}(Y)) + 2$.*

The implication (1) $\Rightarrow$ (2) is Theorem E of [7]. That half of (2) $\Rightarrow$ (1) that assumes $\text{aff}(Y) = \mathbb{R}^n$ is the principal result (Theorem D) of [7]. It suffices to prove the other half of (2) $\Rightarrow$ (1) in which it is assumed that $|Y| \geq \dim(\text{aff}(Y)) + 2$. However,

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the proof we give is a unified proof proving both parts of $(2) \Rightarrow (1)$, using a modification of the method of [7]. The proof in [7] of the implication $(2) \Rightarrow (1)$ of the Theorem in the case that $\text{aff}(Y) = \mathbb{R}^n$ did not apply for $n = 2$. For just this case, there was a separate and unpublished proof due to Komjáth. His proof was based on the proof given by Baumgartner [1] of the special case of $(2) \Rightarrow (1)$ of the Theorem in which Y is an arithmetic progression of length 3 and X is an infinite arithmetic progression. The proof given here includes the $n = 2$ case and, therefore, has both the Baumgartner and Komjáth results as special cases.

For $n = 1$, there is also an equivalence as in the Theorem, but in this case, (2) is simply $|Y| \geq 3$. The aforementioned proof of Baumgartner [1] can easily be modified to a proof of this $n = 1$ case. However, this equivalence is also an immediate consequence of the Theorem. (To be precise, although Baumgartner's initial motivation in [1] was to answer the question of Erdős, which was just this $n = 1$ case, he ultimately proved a stronger result: *If V is an arbitrary vector space over the rationals $\mathbb{Q}$, then there is $A \subseteq V$ such that A contains no infinite arithmetic progressions and $V \setminus A$ contains no arithmetic progressions of length 3.*)

Generally, the method of proof of the Theorem follows the lines of the proof in [7]. There, we utilized a countable coloring $\chi : \mathbb{R}^n \rightarrow \omega$ with the property that there are no distinct points $a, b, c \in \mathbb{R}^n$ that are the vertices of an isosceles triangle such that $\chi(a) = \chi(b) = \chi(c)$. The existence of such a χ was first proved in [6]. Earlier, it had been proved that if $Y \subseteq \mathbb{R}^n$ is such that $|Y| = 3$, then there is a coloring $\chi : \mathbb{R}^n \rightarrow \omega$ such that whenever $Y' \subseteq \mathbb{R}^n$ is similar to Y , then $|\{\chi(a) : a \in Y'\}| \geq 2$. Here we will make use of the following improvement (Corollary 2.4): *if $Y \subseteq \mathbb{R}^n$ is a nonempty finite set, then there is a coloring $\chi : \mathbb{R}^n \rightarrow \omega$ such that whenever $Y' \subseteq \mathbb{R}^n$ is similar to Y , then $|\{\chi(a) : a \in Y'\}| \geq |Y| - 1$.*

We begin with Sect. 1, which is pretty much a review of the basic definitions from [7], culminating in the definition of an acceptable coloring. A couple of fundamental properties of acceptable colorings of $\mathbb{R}^n$ are discussed in Sect. 2. In Sect. 3, we formally define what similarities are, and then some of the connections between them and acceptable colorings. Finally, in Sect. 4, we complete the proof of the Theorem.

1 Preliminaries

We always think of $\mathbb{R}$ as the real-closed ordered field $\mathbb{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$. If $X, Y \subseteq \mathbb{R}$, then $X < Y$ iff $x < y$ for all $x \in X$ and $y \in Y$.

In this section, we consider a fixed, but arbitrary, countable, real-closed subfield $\mathbb{F} \subseteq \mathbb{R}$ and a fixed, but arbitrary, transcendence basis $\mathcal{T} \subseteq \mathbb{R}$ for $\mathbb{R}$ over $\mathbb{F}$, and we arrange that $\mathcal{T}$ is a subset of the closed unit interval $[0, 1]$. We also consider a fixed n , where $2 \leq n < \omega$.

If $D \subseteq \mathbb{R}$ and $R \subseteq \mathbb{R}^k$, then R is D -definable if it is definable by a first-order formula in the language of $\mathbb{R}$ using parameters from $\mathbb{F} \cup D$, and R is definable if it is $\emptyset$ -definable. A fundamental fact that will be repeatedly used is:

$$(*) \begin{cases} \text{If } U \subseteq \mathbb{R}^k \text{ is a connected open set, } \{t_0, t_1, \dots, t_{k-1}\} < \mathcal{T}, \\ \text{and } F : U \rightarrow \mathbb{R} \text{ is a definable analytic function such that} \\ F(t_0, t_1, \dots, t_{k-1}) = 0, \text{ then } F \text{ is identically 0 on } U. \end{cases}$$

If $R \subseteq \mathbb{R}^k$ is $\mathbb{R}$ -definable, then $\text{supp}(R)$ is the unique smallest $T \subseteq \mathcal{T}$ such that R is T -definable. If $a = (a_0, a_1, \dots, a_{k-1}) \in \mathbb{R}^k$, then we write $\text{supp}(a)$ instead of $\text{supp}(\{a\})$. We will mainly be interested in $\text{supp}(a)$ for $a \in \mathbb{R}^n$. When we write $T = \{t_0, t_1, \dots, t_{m-1}\}_{<}$, it is to be understood that $t_0 < t_1 < \dots < t_{m-1}$.

If $I \subseteq \mathbb{R}$ and $k < \omega$, then I is an open *dyadic* interval with *norm* $v(I) = k$ if there is an integer i such that I is the open interval $(i/2^k, (i+1)/2^k)$. A subset $B \subseteq \mathbb{R}^m$ is an m -*box* of norm $v(B) = k$ if there are open dyadic intervals $I_0 < I_1 < \dots < I_{m-1}$, each of norm k , such that $B = I_0 \times I_1 \times \dots \times I_{m-1}$. We let $\mathcal{C}$ be the set of definable analytic functions $f : B \rightarrow \mathbb{R}^n$, where B is a box. If $a \in \mathbb{R}^n$, then $f \in \mathcal{C}$ is an *acceptable color* for a if $\text{supp}(a) = \{t_0, t_1, \dots, t_{m-1}\}_{<}$, $B = \text{dom}(f)$ is an m -box such that $\langle t_0, t_1, \dots, t_{m-1} \rangle \in B$, and $f(t_0, t_1, \dots, t_{m-1}) = a$. If $f \in \mathcal{C}$, then let $v(f) = v(\text{dom}(f))$. If $A \subseteq \mathbb{R}^n$, then an *acceptable coloring* of A is a function $\chi : A \rightarrow \mathcal{C}$ such that $\chi(a)$ is an acceptable color for every $a \in A$.

2 Acceptable Colorings

Throughout this section we assume that $\chi : \mathbb{R}^n \rightarrow \mathcal{C}$ is an acceptable coloring.

Lemma 2.1 *If distinct $a, b, c \in \mathbb{R}^n$ and $r \in \mathbb{F}$ are such that $\|a - b\| = r\|b - c\|$ and $\chi(b) = \chi(c)$, then $v(\chi(b)) < v(\chi(a))$.*

Proof For a contradiction, suppose that $\|a - b\| = r\|b - c\|$, $f = \chi(a)$, $g = \chi(b) = \chi(c)$, and $v(g) \geq v(f)$. Let

$$\begin{aligned}\text{supp}(a) &= \{t_0, t_1, \dots, t_{m-1}\}_{<}, \\ \text{supp}(b) &= \{u_0, u_1, \dots, u_{k-1}\}_{<}, \\ \text{supp}(c) &= \{v_0, v_1, \dots, v_{k-1}\}_{<}.\end{aligned}$$

Let $B_1 = \text{dom}(f) = I_0 \times I_1 \times \dots \times I_{m-1}$ and $B_2 = \text{dom}(g) = J_0 \times J_1 \times \dots \times J_{k-1}$. Choose this counterexample so as to minimize $k + m$ (where we are allowing χ to be replaced by another acceptable coloring). By taking the restriction of f to an m -box $B \subseteq B_1$ such that $\langle t_0, t_1, \dots, t_{m-1} \rangle \in B$ and $v(B) = v(g)$, we can assume that $v(f) = v(g)$.

Let $F : B_1 \times B_2 \times B_2 \rightarrow \mathbb{R}$ be such that

$$F(x, y, z) = \|f(x) - g(y)\|^2 - r^2 \|g(y) - g(z)\|^2.$$

Clearly, F is a definable analytic function such that $F(t, u, v) = 0$.

Since $b \neq c$ and $\chi(b) = \chi(c)$, it must be that $\text{supp}(b) \neq \text{supp}(c)$, so we can let $\ell < k$ be such that $u_\ell \neq v_\ell$.

For each $i < m$ such that $I_i \neq J_\ell$, replace $t_i \in I_i$ with the midpoint of I_i thereby obtaining t' from t . Also, replace each u_i, v_i such that $\ell \neq i < k$ with the midpoint of J_i thereby obtaining u', v' from u, v , respectively. Then, $F(t', u', v') = 0$ by $\circledast$. Letting $a' = f(t')$, $b' = g(u')$, and $c' = g(v')$, we have a counterexample such that $|\text{supp}(a')| \leq |\text{supp}(b')| = |\text{supp}(c')| = 1$. Thus by the minimality of $m + k$, we can assume that $k \leq m = 1$ and then $u_0 \neq v_0$.

Suppose that $\text{supp}(a) = \emptyset$. Then $\|a - g(u_0)\|^2 - r^2\|g(u_0) - g(v_0)\|^2 = 0$, so from $\circledast$ it follows that $\|a - g(u_0)\|^2 - r^2\|g(u_0) - g(u_0)\|^2 = 0$. Thus, $g(u_0) = a$, so by $\circledast$ again, g is constantly a . But then $\text{supp}(g) = \emptyset$, which is a contradiction.

So suppose that $k = 1$. Then, $t_0 \in \{u_0, v_0\}$, as otherwise we could replace t_0 by the midpoint of I_0 and then we would have the previous case. Without loss of generality, assume that $t_0 = u_0$. Thus, $\|f(u_0) - g(u_0)\|^2 - r^2\|g(u_0) - g(v_0)\|^2 = 0$, so from $\circledast$ it follows that $\|f(u_0) - g(u_0)\|^2 - r^2\|g(u_0) - g(u_0)\|^2 = 0$, so that $a = f(u_0) = g(u_0) = b$, which is a contradiction. $\square$

The following corollary is Lemma 2.1 of [7] and the main result of [6].

Corollary 2.2 *If distinct $a, b, c \in \mathbb{R}^n$ and $r \in \mathbb{F}$ are such that $\|a - b\| = r\|b - c\|$, then $\chi(a) \neq \chi(b)$ or $\chi(b) \neq \chi(c)$.*

The next corollary of Lemma 2.1 will not be needed but is included here because of its independent interest.

Corollary 2.3 *Let $a, b, c, d \in \mathbb{R}^n$ be distinct points such that $\chi(a) = \chi(b)$ and there are $r, s \in \mathbb{F}$ such that $\|a - b\| = r\|b - c\|$ and $\|c - d\|$ is one of $s\|d - a\|, s\|a - c\|$, or $s\|b - c\|$. Then, $\chi(c) \neq \chi(d)$.*

Proof Let $f = \chi(a) = \chi(b)$ and suppose that $g = \chi(c) = \chi(d)$. Then Lemma 2.1 implies that $v(f) < v(g)$ and $v(g) < v(f)$. $\square$

A precursor to Corollary 2.2 is the result from [5] that if $Y \subseteq \mathbb{R}^n$ and $|Y| = 3$, then $\mathbb{R}^n$ has countable coloring ψ such that if $Y' \subseteq \mathbb{R}^n$ is similar to Y , then $|\{\psi(x) : x \in Y'\}| \geq 2$. This result had improved earlier results in [2–5]. The next corollary is a further strengthening of this result.

Corollary 2.4 *Let $Y \subseteq \mathbb{F}^n$ be nonempty and finite. If $Y' \subseteq \mathbb{R}^n$ is similar to Y , then $|\{\chi(x) : x \in Y'\}| \geq |Y| - 1$.*

3 Affine Transformations

In this section we investigate the connection between similarities, and more generally affine maps, and acceptable colorings.

Let $e_0, e_1, \dots, e_{n-1} \in \mathbb{R}^n$ be the standard basis for $\mathbb{R}^n$; thus, $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ where the unique 1 has exactly i preceding 0's. Also, let $e_n = (0, 0, \dots, 0)$. A function $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an *affine map* if whenever $x = (x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n$, then

$$\alpha(x) = x_0\alpha(e_0) + x_1\alpha(e_1) + \dots + x_{n-1}\alpha(e_{n-1}) + \alpha(e_n).$$

If α is a bijective affine map, then it is an *affine transformation*. If there is a positive $c \in \mathbb{R}$ such that $\|\alpha(x) - \alpha(y)\| = c\|x - y\|$ for every $x, y \in \mathbb{R}^n$, then α is a *similarity*. If $Y, Y' \subseteq \mathbb{R}^n$, then Y and Y' are similar iff there is a similarity α such that $Y' = \alpha(Y)$.

Acceptable colorings were defined only for $\mathbb{R}^n$, but they could easily have been defined for any $\mathbb{R}^k$. We will need to consider acceptable colorings of $\mathbb{R}^{n(n+1)}$. By identifying $\mathbb{R}^{n(n+1)}$ with $(\mathbb{R}^n)^{n+1}$, we can consider acceptable colors for elements of $(\mathbb{R}^n)^{n+1}$. We will refer to acceptable colors for elements of $(\mathbb{R}^n)^{n+1}$ as *types*. Each type can be identified with a certain element of $\mathcal{C}^{n+1}$; thus, the set of types is a subset of $\mathcal{C}^{n+1}$. If α is an affine map and τ is a type, then we say that τ is a type of α (or α has type τ) if τ is an acceptable color for $\langle \alpha(e_0), \alpha(e_1), \dots, \alpha(e_n) \rangle \in (\mathbb{R}^n)^{n+1}$. Thus, it makes sense to refer to $\nu(\tau)$. Notice that a consequence of $\oplus$ is that if τ is a type of α and β , then α is an affine transformation iff β is, and α is a similarity iff β is.

Lemma 3.1 *Let τ be a type and $a \in \mathbb{F}^n$. Then there is $f \in \mathcal{C}$ such that $\nu(f) = \nu(\tau)$ and whenever α is an affine map that has type τ , then f is an acceptable color for $\alpha(a)$.*

Proof Let $\text{dom}(\tau) = B = I_0 \times I_1 \times \dots \times I_{m-1}$ and $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{F}^n$. Then define $F : (\mathbb{R}^n)^{n+1} \rightarrow \mathbb{R}^n$ such that

$$F(x_0, x_1, \dots, x_n) = a_0x_0 + a_1x_1 + \dots + a_{n-1}x_{n-1} + x_n.$$

Hence, whenever $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine map, then

$$F(\alpha(e_0), \alpha(e_1), \dots, \alpha(e_n)) = \alpha(a).$$

Clearly, F is a definable analytic function.

Define $G : B \rightarrow \mathbb{R}^n$ by

$$G(x_0, x_1, \dots, x_{m-1}) = F(\tau(x_0, x_1, \dots, x_{m-1})),$$

so that G is a definable analytic function.

Consider any affine map α that has type τ , and let $\text{supp}(\alpha) = \{t_0, t_1, \dots, t_{m-1}\}_{<}$. Then $G(t_0, t_1, \dots, t_{m-1}) = \alpha(a)$, so $\text{supp}(\alpha(a)) \subseteq \text{supp}(\alpha)$. Let $J(\alpha) = \{j_0, j_1, \dots, j_{s-1}\}_{<} = \{j < m : t_j \in \text{supp}(\alpha)\}$, and then let $B = I_{j_0} \times I_{j_1} \times \dots \times I_{j_{s-1}}$ and let $g : B \rightarrow \mathbb{R}^n$ be defined by $g(x_0, x_1, \dots, x_{s-1}) = G(z_0, z_1, \dots, z_{m-1})$, where $z_{j_i} = x_i$ if $i < s$ and z_j is the midpoint of I_j if $j \notin \{j_0, j_1, \dots, j_{s-1}\}_{<}$. Therefore, g is an acceptable color for $\alpha(a)$. Thus, $G(t_0, t_1, \dots, t_{m-1}) = g(t_{j_0}, t_{j_1}, \dots, t_{j_{s-1}})$.

Now let β be another affine map having type τ , and let $\text{supp}(\beta) = \{u_0, u_1, \dots, u_{m-1}\}_{<}$. Then $\beta(a) = G(u_0, u_1, \dots, u_{m-1})$. By $\oplus$, $G(u_0, u_1, \dots, u_{m-1}) = g(u_{j_0}, u_{j_1}, \dots, u_{j_{s-1}})$ so that $\beta(a) = g(u_{j_0}, u_{j_1}, \dots, u_{j_{s-1}})$. Thus, $J(\beta) \subseteq J(\alpha)$, so, by symmetry, $J(\beta) = J(\alpha)$. Therefore, g is also an acceptable type for $\beta(a)$. $\square$

If $\tau \in \mathcal{C}^{n+1}$ is a type, then, in the light of Lemma 3.1, we let $\Psi_\tau : \mathbb{F}^n \rightarrow \mathcal{C}$ be the function such that whenever $a \in \mathbb{F}^n$ and α has type τ , then $\Psi_\tau(a)$ is the unique acceptable color f for $\alpha(a)$ such that $\nu(f) = \nu(\tau)$.

Lemma 3.2 *Let $\tau \in \mathcal{C}^{n+1}$ be a type of an affine transformation, and let $f \in \mathcal{C}$. Then, there are only finitely many $a \in \mathbb{F}^n$ such that $\Psi_\tau(a) = f$.*

Proof Let $\text{dom}(\tau) = B = I_0 \times I_1 \times \cdots \times I_{m-1}$. As we saw in the proof of Lemma 3.1, if $a \in \mathbb{F}^n$ and α is an affine transformation, then $\text{supp}(\alpha(a)) \subseteq \text{supp}(\alpha)$.

We claim: If $j_0 < j_1 < \cdots < j_{s-1} < m$, then there is at most one $a \in \mathbb{F}^n$ for which there is α having type τ such that f is an acceptable type for $\alpha(a)$, and $\text{supp}(\alpha(a)) = \{v_0, v_1, \dots, v_{s-1}\}_{<}$, where $v_i \in I_{j_i}$ for $i < s$.

To prove this claim, let both a, b have this property. Thus, there are α, β having type τ such that f is an acceptable color for $\alpha(a)$, $\beta(b)$, and

$$\begin{aligned}\text{supp}(\alpha) &= \{t_0, t_1, \dots, t_{m-1}\}_{<}, \\ \text{supp}(\beta) &= \{u_0, u_1, \dots, u_{m-1}\}_{<}, \\ \text{supp}(\alpha(a)) &= \{t_{j_0}, t_{j_1}, \dots, t_{j_{s-1}}\}_{<}, \\ \text{supp}(\alpha(b)) &= \{u_{j_0}, u_{j_1}, \dots, u_{j_{s-1}}\}_{<}.\end{aligned}$$

Let $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{F}^n$, and define $F : (\mathbb{R}^n)^{n+1} \rightarrow \mathbb{R}^n$ such that

$$F(x_0, x_1, \dots, x_n) = a_0 x_0 + a_1 x_1 + \cdots + a_{n-1} x_{n-1} + x_n.$$

Hence, whenever $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine map (especially letting $\gamma = \alpha$ or $\gamma = \beta$), then

$$F(\gamma(e_0), \gamma(e_1), \dots, \gamma(e_n)) = \gamma(a).$$

Clearly, F is a definable analytic function.

Define $G : B \rightarrow \mathbb{R}^n$ by $G(x_0, x_1, \dots, x_{m-1}) = F(\tau(x_0, x_1, \dots, x_{m-1}))$, so that G is a definable analytic function. Then

$$G(t_0, t_1, \dots, t_{m-1}) = \alpha(a) = f(t_{j_0}, t_{j_1}, \dots, t_{j_{s-1}}).$$

Therefore,

$$\begin{aligned}\beta(a) &= F(\beta(e_0), \beta(e_1), \dots, \beta(e_n)) \\ &= F(\tau(u_0, u_1, \dots, u_{m-1})) \\ &= G(u_0, u_1, \dots, u_{m-1}) \\ &= f(u_{j_0}, u_{j_1}, \dots, u_{j_{s-1}}) \quad (\text{by } \circledast) \\ &= \beta(b).\end{aligned}$$

Therefore, $a = b$, proving the claim.

We have just seen that each such a is determined by $j_0 < j_1 < \cdots < j_{s-1} < m$. Therefore, there are at most $\binom{m}{s}$ such that $a \in \mathbb{F}^n$. $\square$

4 Completing the Proof

In this final section, we complete the proof of the Theorem. Throughout this section, we consider fixed $X, Y \subseteq \mathbb{R}^n$, where X is infinite and either $\text{aff}(Y) = \mathbb{R}^n$ or $|Y| \geq$

$\dim(\text{aff}(Y)) + 2$. Clearly, we can assume that X is countable and that Y is minimal with the required property. We let $|Y| = m + 2$, so m also is fixed throughout this section. Thus, either $m = n - 1$ and $\text{aff}(Y) = \mathbb{R}^n$ or else $2 \leq m = \dim(\text{aff}(Y)) \leq n - 1$.

We let $\mathbb{F} \subseteq \mathbb{R}$ be a countable real-closed field such that $X, Y \subseteq \mathbb{F}^n$ and then continue with the all the previous assumptions.

In both cases (namely, $\text{aff}(Y) = \mathbb{R}^n$ and $\dim(\text{aff}(Y)) < n$), there is a common feature that Y has. Let $Y = \{y_0, y_1, \dots, y_m, y_{m+1}\}$. First, suppose that $\dim(\text{aff}(Y)) < n$. Then there are unique $c_0, c_1, \dots, c_m \in \mathbb{R}$ such that $c_0 + c_1 + \dots + c_m = 1$ and $y_{m+1} = c_0 y_0 + c_1 y_1 + \dots + c_m y_m$. Then $0 \neq c_i \in \mathbb{F}$ for each $i \leq m$. Let $F : (\mathbb{R}^n)^{m+1} \rightarrow \mathbb{R}^n$ be the definable analytic function such that

$$F(x_0, x_1, \dots, x_m) = c_0 x_0 + c_1 x_1 + \dots + c_m x_m.$$

Then

$$\alpha(y_{m+1}) = F(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_m))$$

for every similarity (even any affine map) α .

Next, suppose that $\dim(\text{aff}(Y)) = n$. Then there are two definable analytic functions $F, G : (\mathbb{R}^n)^{m+1} \rightarrow \mathbb{R}^n$ such that

$$\alpha(y_{m+1}) \in \{F(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_m)), G(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_m))\}$$

for every similarity α . (See [7, Lemma 2.5].)

It was noted in the previous section that $\otimes$ implies that if α, β are affine maps having type σ , then α is a similarity iff β is. Here we note a generalization of this. An affine map α is a *similarity on Y* if there is a positive $c \in \mathbb{R}$ such that $\|\alpha(x) - \alpha(y)\| = c\|x - y\|$ for every $x, y \in Y$. Then, if σ is a type of both α, β , then α is a similarity on Y iff β is. Hence, we can say unambiguously that σ is a *similarity type on Y* .

Lemma 4.1 *Let $k < \omega$, $\ell \leq m + 1$, $Y = \{y_0, y_1, \dots, y_m, y_{m+1}\}$, and let $f_0, f_1, \dots, f_{\ell-1} \in \mathcal{C}$ be such that $v(f_i) \leq k$. Then there are only finitely many $f \in \mathcal{C}$ for which there is a type σ that is a similarity type on Y such that $v(\sigma) = k$, $\Psi_\sigma(y_i) \subseteq f_i$ for $i < \ell$ and $\Psi_\sigma(y_i) = f$ for $\ell < i \leq m + 1$.*

Proof Fix $k < \omega$, $\ell \leq m + 1$, $Y = \{y_0, y_1, \dots, y_m, y_{m+1}\}$, and $f_0, f_1, \dots, f_{\ell-1} \in \mathcal{C}$. Without loss of generality, suppose that $v(f_0) = v(f_1) = \dots = v(f_{\ell-1}) = k$.

$\ell < m$. By Corollary 2.4 there is no such type σ , for if α were an affine map of type σ , then α would be a similarity on Y and f would be an acceptable color for each of $\alpha(y_{m-1}), \alpha(y_m)$ and $\alpha(y_{m+1})$.

$\ell = m$. Again, there is no such type σ , for if α were an affine map of type σ , then α would be a similarity on Y , f would be an acceptable color for each of $\alpha(y_m)$ and $\alpha(y_{m+1})$ and $v(f) = k = v(f_0)$, contradicting Lemma 2.1.

$\ell = m + 1$. Let σ be a type of α such that α is a similarity on Y , $v(\sigma) = k$, and $\Psi_\sigma(y_i) = f_i$ for $i \leq m$. Let $f = \sigma(y_{m+1})$. We know that there is a definable analytic $F : (\mathbb{R}^n)^{m+1} \rightarrow \mathbb{R}^n$ such that $F(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_m)) = \alpha(y_{m+1})$. For $i \leq m + 1$, let $t_i = (t_{i0}, t_{i1}, \dots, t_{ir_i-1}) \in \mathbb{R}^{r_i}$ be such that $\text{supp}(\alpha(y_i)) = \{t_{i0}, t_{i1}, \dots, t_{ir_i-1}\}$. Let $B_i = \text{dom}(f_i)$, and let $G : B_0 \times B_1 \times \dots \times B_m \rightarrow \mathbb{R}^n$ be

such that $G(x_0, x_1, \dots, x_m) = F(f(x_0), f(x_1), \dots, f(x_m))$. Thus, G is a definable analytic function such that $G(t_0, t_1, \dots, t_m) = f(t_{m+1}) = \alpha(y_{m+1})$. Let $\text{dom}(G) = I_0 \times I_1 \times \dots \times I_{r-1}$, where $r = r_0 + r_1 + \dots + r_m$. Each I_i is such that $v(I_i) = k$. For each $s < r$, let $t'_s = t_{m+1,j}$ if $t_{m+1,j} \in I_s$, and let t'_s be the midpoint of I_s otherwise. (Since $v(f) = k$, this is well defined.) Then $f(t_{m+1,0}, t_{m+1,1}, \dots, t_{m+1,r_{m+1}-1}) = G(t'_0, t'_1, \dots, t'_{r_1})$. Thus, there are only finitely many possible such f . $\square$

We now make a crucial definition.

Definition 4.2 We say that ρ is *good* if $\rho : \mathcal{C}_0 \rightarrow \{0, 1\}$ is a function such that the following hold:

- $\mathcal{C}_0$ is a finite subset of $\mathcal{C}$.
- Whenever $f, g \in \mathcal{C}$ are distinct and $f \subseteq g \in \mathcal{C}_0$, then $f \notin \mathcal{C}_0$.
- Whenever $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity and $\varphi : \alpha(Y) \rightarrow \mathcal{C}_0$ is acceptable, then there is $y \in Y$ such that $\rho(\varphi(\alpha(y))) = 0$.

Lemma 4.3 Suppose that ρ is good and that $\tau \in \mathcal{C}^{n+1}$ is the type of a similarity such that $v(\tau) \geq v(f)$ for all $f \in \text{dom}(\rho)$. Then there exists $x \in X$ such that $\rho' = \rho \cup \{(\Psi_\tau(x), 1)\}$ is good.

Proof Let $\mathcal{C}_0 = \text{dom}(\rho)$, which is finite since ρ is good. Let $\mathcal{C}_1$ be the set of those $f \in \mathcal{C}$ such that $v(f) \leq v(\tau)$ and for some $g \in \mathcal{C}_0$, there is $h \in \mathcal{C}$ such that $f \cup g \subseteq h$. Clearly, $\mathcal{C}_0 \subseteq \mathcal{C}_1$, and $\mathcal{C}_1$ is finite. Let $X_1 = \{a \in X : \Psi_\tau(a) \in \mathcal{C}_1\}$, which is finite by Lemma 3.2.

Let $\mathcal{C}_2$ be the set of all $f \in \mathcal{C}$ for which there is a type σ of a similarity such that $v(\sigma) = v(\tau)$ and $\Psi_\sigma(y) \in \mathcal{C}_1 \cup \{f\}$ whenever $y \in Y$. Since ρ is good, Lemma 4.1 implies that $\mathcal{C}_2$ is finite. Let $X_2 = \{a \in X : \Psi_\tau(a) \in \mathcal{C}_2\}$, which is finite by Lemma 3.2.

Since X is infinite and $X_1, X_2 \subseteq X$ are finite, we pick some $x \in X \setminus (X_1 \cup X_2)$. Let $f = \Psi_\tau(x)$, $\rho' = \rho \cup \{(f, 1)\}$ and $\mathcal{C}'_0 = \mathcal{C}_0 \cup \{f\}$.

Clearly, $f \notin \mathcal{C}_0 = \text{dom}(\rho)$, so $\rho' : \mathcal{C}'_0 \rightarrow \{0, 1\}$ is a well-defined function. We now prove that ρ' is good.

For a contradiction, suppose that ρ' is not good. Obviously, $\mathcal{C}'_0$ is finite. Also, it is clear that neither $f \subseteq g$ nor $g \subseteq f$ for $g \in \mathcal{C}_0$. Thus, there is $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that is a similarity on Y (or even a similarity) and there is an acceptable $\varphi' : \alpha(Y) \rightarrow \mathcal{C}'_0$ such that $\rho'(\varphi'(\alpha(y))) = 1$ for all $y \in Y$.

Now, if α were such that it had a type σ such that $v(\sigma) = v(\tau)$, then we would arrive at a contradiction as follows. We have that $\varphi'(\alpha(y)) = \Psi_\sigma(y)$ for all $y \in Y$. But then $f \in \mathcal{C}_2$ and $x \in X_2$, contradicting the choice of f .

We will next show that, even though α may not have such a type σ , another choice of an affine map will.

Let $f_j = \theta(e_j)$ for $j \leq n$. Let $I_0 < I_1 < \dots < I_{r-1}$ be those intervals occurring as factors of $f_0, f_1, \dots, f_n$, and let $B = I_0 \times I_1 \times \dots \times I_{r-1}$. Let $(t_0, t_1, \dots, t_{r-1}) \in B \cap \mathcal{T}^r$, and then let $a_j = (\theta(a_j))(t_{i_0}, t_{i_1}, \dots, t_{i_{k_j-1}})$, where the subsequence $0 \leq i_0 < i_1 < \dots < i_{k_j-1}$ is the unique one such that $(t_{i_0}, t_{i_1}, \dots, t_{i_{k_j-1}}) \in \text{dom}(f_j)$. Now let $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the affine map such that $\beta(e_j) = a_j$ for $j \leq n$. By $\circledast$, either β

is a similarity on Y or else $\beta(y_0) = \beta(y_1) = \cdots = \beta(y_{m+1})$. However, if $\beta(y_0) = \beta(y_1) = \cdots = \beta(y_{m+1})$, then $\oplus$ implies that $f_0 = f_1 = \cdots = f_n = f$, a contradiction to Corollary 2.2. Thus, β is a similarity on Y such that $\beta(y) = \alpha(y)$ for all $y \in Y$, and β has a type σ such that $v(\beta) = v(\tau)$. $\square$

We now complete the proof of the Theorem. Let $g_0, g_1, g_2, \dots$ be a list of all the elements of $\mathcal{C}$, and let $\tau_0, \tau_1, \tau_2, \dots$ be a list of all types of similarities. We recursively construct an increasing sequence $\rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \cdots$ of good functions. Let $\rho_0 = \emptyset$, which clearly is good.

Stage $2k + 1$: Let $\ell_{2k} < \omega$ be the least ℓ such that $v(g_k) \leq \ell$ and also $v(g) \leq \ell$ whenever $g \in \text{dom}(\rho_{2k})$. Let $\mathcal{D}$ be the set of those $f \subseteq g_k$ such that $v(f) = \ell$ and for no $g \in \text{dom}(\rho_{2k})$ is $f \subseteq g$. Then let $\rho_{2k+1} = \rho_{2k} \cup \{(f, 0) : f \in \mathcal{D}\}$. Clearly, ρ_{2k+1} is good.

Stage $2k + 2$: Let $\ell_{2k+1} < \omega$ be the least ℓ such that $v(\tau_k) \leq \ell$ and also $v(g) \leq \ell$ whenever $g \in \text{dom}(\rho_{2k+1})$. Apply Lemma 4.3 successively several times, once for each $\tau \subseteq \tau_k$ such that $v(\tau) = \ell$. This results in a good $\rho_{2k+2} \supseteq \rho_{2k+1}$ such that whenever $\tau \subseteq \tau_k$ and $v(\tau) = \ell$, then there is $x \in X$ such that $\Psi_\tau(x) \in \text{dom}(\rho_{2k+2})$ and $\rho_{2k+2}(\Psi_\tau(x)) = 1$.

Let $\rho = \bigcup_{k < \omega} \rho_k$. Then $\rho : \mathcal{C}_0 \rightarrow \{0, 1\}$ for some $\mathcal{C}_0 \subseteq \mathcal{C}$. The odd stages of the construction guarantee that for each $x \in \mathbb{R}^n$, there is $f \in \mathcal{C}_0$ that is acceptable for x . Since each ρ_k is good, this f is unique. Thus, there is a unique acceptable coloring $\chi : \mathbb{R}^n \rightarrow \mathcal{C}_0$. Now define $A = \{x \in \mathbb{R}^n : \rho(\chi(x)) = 0\}$. We show that this A verifies $\mathbb{R}^n \not\rightarrow (X, Y)$. We must prove: (1) there is no $X' \subseteq A$ that is similar to X ; and (2) there is no $Y' \subseteq \mathbb{R}^n \setminus A$ similar to Y . Let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a similarity.

- (1) Let $X' = \alpha(X)$. Let τ_k be a type of α , and notice that $\ell_{2k+1} \geq v(\tau_k)$. Let $\tau \subseteq \tau_k$ be such that $v(\tau) = \ell_{2k+1}$ and τ is a type of α . Then, there is $x \in X$ such that $\Psi_\tau(x) \in \text{dom}(\rho_{2k+2})$ and $\rho_{2k+2}(\Psi_\tau(x)) = 1$. Then $\alpha(x) \in X'$ and $\rho(\alpha(x)) = \rho_{2k+2}(\alpha(x)) = \rho_{2k+2}(\Psi_\tau(x)) = 1$, so $\alpha(x) \notin A$.
- (2) Let $Y' = \alpha(Y)$ and $\varphi = \chi|_{Y'}$. Then, for large enough $k < \omega$, $\varphi(A') \subseteq \text{dom}(\rho_k)$. Since ρ_k is good, there is $y' \in Y'$ such that $\rho(\chi(y')) = \rho_k(\varphi(y')) = 0$. Therefore, $y' \in A$.

This completes the proof of the Theorem.

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