Unavoidable Crossings in a Thinnest Plane Covering with Congruent Convex Disks

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Abstract Two convex disks K and L in the plane are said to *cross each other* if the removal of their intersection causes each disk to fall into disjoint components. Almost all major theorems concerning the covering density of a convex disk were proved only for crossing-free coverings. This includes the classical theorem of L. Fejes Tóth (Acta Sci. Math. Szeged 12/A:62–67, 1950) that uses the maximum area hexagon inscribed in the disk to give a significant lower bound for the covering density of the disk. From the early seventies, all attempts of generalizing this theorem were based on the common belief that crossings in a plane covering by congruent convex disks, being counterproductive for producing low density, are always avoidable. Partial success was achieved not long ago, first for “fat” ellipses (A. Heppes in Discrete Comput. Geom. 29:477–481, 2003) and then for “fat” convex disks (G. Fejes Tóth in Discrete Comput. Geom. 34(1):129–141, 2005), where “fat” means of shape sufficiently close to a circle. A recently constructed example will be presented here, showing that, in general, all such attempts must fail. Three perpendiculars drawn from the center of a regular hexagon to its three nonadjacent sides partition the hexagon into three congruent pentagons. Obviously, the plane can be tiled by such pentagons.

To the memory of Professor László Fejes Tóth.

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But a slight modification produces a (non-tiling) pentagon with an unexpected covering property: every thinnest covering of the plane by congruent copies of the modified pentagon must contain crossing pairs. The example has no bearing on the validity of Fejes Tóth's bound in general, but it shows that any prospective proof must take into consideration the existence of unavoidable crossings.

Keywords Covering · Convex disk · Crossing · Density

1 Introduction

The central problem of the theory of packing and covering is to find the most economical ways to pack (i.e., to arrange without overlap) congruent copies of a given disk within a given convex domain or to cover the domain by congruent copies of a given disk. To measure optimality of the arrangements the notion of *density* was introduced. We review the basic notions in Sect. 4, where we also explain the method of cell decomposition. Results of the packing and covering theory are surveyed in the books of Pach and Agarwal [11] and of Brass, Moser, and Pach [2]; see also the survey article [4]. Although many far reaching theorems are known for packing, much less is known about covering. In fact, one of the earliest conjectures concerning covering of the entire plane is still open:

Conjecture (L. Fejes Tóth [7]) If d is the density of a covering of the plane with congruent copies of a convex disk C , then

$$d \geq \frac{\text{Area}(C)}{\text{Area}(h)},$$

where h denotes a hexagon of maximum area inscribed in C .

The above inequality was proved only for a special case, namely under the restriction that the congruent copies *do not cross each other* (two convex discs C and C' *cross each other* if each of the sets $C \setminus C'$ and $C' \setminus C$ is disconnected), see [5–7], and [1]. Because of this special restriction, the theorem is only in part analogous to the similar theorem for packing proved by L. Fejes Tóth in [5].

Removing the troubling “no crossings allowed” restriction is one of the major unsolved problems in this area. In [7], L. Fejes Tóth writes:

... the restriction concerning the number of intersection points of the boundary curves appears to be superfluous. This restriction not only diminishes the elegance of the theorem, but also breaks the analogy between the two theorems. It would be therefore desirable to free the second theorem from this restriction.

The general theorem would imply, for example, that the lowest density among coverings with congruent copies of a given centrally symmetric disk can be attained in a lattice arrangement of the disks. In particular, it would yield that the covering density of every ellipse is $2\pi/\sqrt{27}$, the same as that of the circular disk (see Kershner [10]), which thus far has not been proved in general.

Fig. 1 The example of Heppes:
a crossing occurs in every
covering of the square with
these two congruent hexagons

At first sight it seems indeed that using crossing pairs of congruent convex disks, thereby needlessly wasting area, would be counterproductive if we search for the most economical (i.e., thinnest possible) covering of the plane. It was therefore commonly believed that the condition requiring the absence of crossings was merely a technical one and it was hoped that one could prove the Fejes Tóth conjecture simply by eliminating all crossings through a rearrangement of the members of the covering while not increasing the covering's density.

The first indication of difficulties with such an approach was provided by a simple example of Heppes (unpublished; mentioned and described in [8] and in [12]) (see Fig. 1) and then again by a series of examples of Wegner [12] that presented coverings of *bounded* convex domains, where the desired rearrangements are impossible. On the other hand, recent results of Heppes and G. Fejes Tóth seem to indicate that if the entire plane is to be covered, then such rearrangements might exist after all. Specifically, Heppes [9] proved that the lowest density among coverings of the plane with congruent, sufficiently “fat” ellipses can be attained without using crossing pairs and therefore also in a lattice covering. Later G. Fejes Tóth [3] generalized Heppes's result and proved the same for any r -fat plane convex disk C with $r > 0.933$ (and with $r > 0.741$, for ellipses). (A convex disk is said to be r -fat if it is contained in a unit circle and contains a concentric circle of radius r .)

In this paper we describe a special pentagon, we determine its covering density, and we prove its peculiar property: every minimum density covering of the plane with congruent copies of this pentagon must contain a crossing pair.

2 The Construction and Statement of the Result

Dissect a regular hexagon H of area 3 into three congruent pentagons (of unit area, of course) by cutting along segments drawn from the center of H perpendicularly to three nonadjacent sides of H . The resulting pentagonal tile P_0 has two right angles and three angles measuring 120° each. (By saying that a convex disk is a tile we mean that the plane can be tiled by its congruent copies.) Keeping the vertices at the 120° angles of P_0 fixed, enlarge the pentagon P_0 slightly by moving the vertices at the right angles outside H and so that in the modified pentagon the sides emanating from the relocated vertices change their directions by the same small angle. Denote by ε the increase of the pentagon's area caused by the modification and denote the so enlarged pentagon by P_ε (see Fig. 2a). Obviously, if $\varepsilon > 0$ is sufficiently small,

Fig. 2 The modified pentagons

then P_ε is not a tile. If each of the three pentagons into which H was partitioned is modified in the exactly same way, then the three congruent copies of P_ε cover the regular hexagon H , and they *cross each other*, no matter how small value of ε is chosen.

Extend hexagon H to a hexagonal tiling of the entire plane (this can be done in one way only) and cover all hexagons of the tiling by congruent triples of P_ε (this can be done in many ways, as the covering triple can be oriented in two different ways: varying the orientation or not, from hexagon to hexagon). The result is an apparently thin plane covering, of density $1 + \varepsilon$. In this paper we prove the following:

Theorem *If ε is sufficiently small, then every covering of the plane with congruent copies of the pentagon P_ε is of density at least $1 + \varepsilon$. Furthermore, each of the coverings that attains the minimum density of $1 + \varepsilon$ must contain crossing pentagons.*

3 Properties of the Pentagons P_0 and P_ε

Recalling the notation from the introduction, H refers to the regular hexagon of area 3, P_0 denotes one of the three congruent pentagons into which H is partitioned, and P_ε is the modified pentagon of area $1 + \varepsilon$. Observe that the edge length e of H is slightly greater than 1; $e = \sqrt{2/\sqrt{3}} = 1.07\dots$, to be precise. Temporarily, the parameter ε is a variable that will be fixed later to take on a suitable, sufficiently small positive value. We now introduce additional notation and terminology. P_0 has two right angles and three angles measuring 120° each. The three vertices at the obtuse angles of P_0 will be called the *obtuse vertices*. The obtuse vertex at which two equal length sides meet will be called the *central obtuse vertex*, or the *central vertex*, for brevity. The other two obtuse vertices will be called the *outer obtuse vertices*. The motivation for this terminology is that when H is dissected into three pentagons congruent to P_0 , the central vertex of each of the three pentagons is the one that lies in the center of H and the other six obtuse vertices (two per pentagon) lie on the perimeter

of H . Naturally, these names for vertices can be carried over to the corresponding vertices in any congruent copy of P_0 and in any congruent copy of P_ε as well (see Fig. 2a), without ambiguity. Although we generally assume that $\varepsilon > 0$, we sometimes refer to P_0 as “ P_ε with $\varepsilon = 0$,” especially in arguments involving the limit case.

Since the covering problems with which we are concerned require that density always be finite (minimum possible in particular), every time we consider a covering with congruent convex disks, we generally assume it to be locally finite.

Definition 1 Let C be a compact set, and let n be a positive integer. Given a locally finite family of congruent convex disks, the *first surrounding* of C , denoted by $\text{Surr}^{(1)}(C)$, is the collection of all disks from the family that intersect C , provided that the union $\bigcup \text{Surr}^{(1)}(C)$ of this collection, called the *1-st surrounding domain* of C , contains C in its interior. We say that C is *surrounded once* by this family. The *second surrounding* of C , denoted $\text{Surr}^{(2)}(C)$, is the first surrounding of the first surrounding domain of C , that is, $\text{Surr}^{(2)}(C) = \text{Surr}^{(1)}(\bigcup \text{Surr}^{(1)}(C))$. Inductively, for all $n \geq 2$, we define the n th surrounding of C by $\text{Surr}^{(n)}(C) = \text{Surr}^{(1)}(\bigcup \text{Surr}^{(n-1)}(C))$, and we call the set $\bigcup \text{Surr}^{(n)}(C)$ the *nth surrounding domain* of C . We say that C is *surrounded n times* by its n th surrounding.

Remark 1 Observe that if the given family of disks is a covering of the plane, then, for every $n \geq 1$ and every compact set C , $\text{Surr}^{(n)}(C)$ exists, and for any two disks, there is an integer n such that each of the disks lies in the n th surrounding of the other. Observe also that for any pair of disks K and K' , that are members of the covering, if K' belongs to the n th surrounding of K , then K belongs to the n th surrounding of K' , and if the surrounding domains $\bigcup \text{Surr}^{(n)}(K)$ and $\bigcup \text{Surr}^{(m)}(K')$ intersect, then K' belongs to the $(n + m)$ th surrounding of K .

A tiling (resp. covering) of the plane with congruent copies of a convex disk K will be called a *K-tiling* (resp. *K-covering*), for brevity. As we mentioned before, P_0 -tilings exist: an obvious P_0 -tiling is obtained from an H -tiling by dissecting each copy of H into three copies of P_0 . A triple of copies of P_0 forming a regular hexagon congruent to H will be called a *regular triple*, and every tiling of the plane consisting of regular P_0 -triples will be called a *regular P_0 -tiling*.

Suppose that in a given family of congruent copies of P_0 one of the pentagons is n times surrounded by others. We say that the n th surrounding is *regular* if it is congruent to the n th surrounding of a pentagon in some regular P_0 -tiling of the plane.

Proposition 1 *Suppose that a family of congruent copies of pentagon P_0 is given, no two of which overlap. If one of the pentagons is twice surrounded by some members of the family, then the surrounded pentagon is a member of a regular triple. Moreover, at each of the outer obtuse vertices of the surrounded pentagon, two other pentagons are adjacent with their common outer obtuse vertices.*

Proof Label with (1) the pentagon assumed to be surrounded twice. It is easy to see that the central vertex of pentagon (1) must coincide with two obtuse vertices of other

Fig. 3 Tiling with P_0 -pentagons

pentagons. Proposition 1 asserts that the central vertex of pentagon (1) coincides with the *central* vertices of two other pentagons. Suppose that this is not so, which means that the central vertex of pentagon (1) coincides with one of the outer obtuse vertices of pentagon (2). To fill the empty 90° -corner at pentagons (1) and (2), pentagon (3) of the first surrounding of pentagon (1) can fit there in two ways only. But each of the two possibilities creates an empty corner (see the arrows pointing to such corners in Fig. 3), whose vertex cannot be reached by another copy of P_0 necessary for the second surrounding of (1). The second part of the proposition is proved in a very similar manner as the first one. $\square$

Proposition 1 yields immediately its “expanded” version:

Proposition 2 *Suppose that a family of pentagons congruent to P_0 is given, no two of which overlap. If one of the pentagons is $n + 2$ times surrounded by the others, then its n th surrounding is regular. Moreover, at each of the outer obtuse vertices of each of the pentagons in the n th surrounding, three pentagons are adjacent with their common outer obtuse vertices.*

Corollary 1 *Every P_0 -tiling of the plane is regular.*

Propositions 1 and 2 can be considered as local versions of Corollary 1. We will next show next that structural statements similar to the above, even in local versions, hold for P_0 -coverings (instead of P_0 -tilings) and for P_ε -coverings as well if we assume that the pentagons do not overlap too much and that ε is sufficiently small. We begin with a standard definition from the theory of convex bodies.

Definition 2 Given a convex disk C and a number $\lambda > 0$, the set of points in C whose distance from the boundary of C is at least λ is the *inner parallel domain* of C at distance λ , but we will call it the λ -core of C for brevity. Of course, for the λ -core of C to be nondegenerate, λ must be smaller than the inradius of C .

Lemma 1 *For every sufficiently small value of λ , there exist arbitrarily small values of ε such that the following holds: Given a family of congruent copies of P_ε whose λ -cores do not overlap. If one of the pentagons is surrounded twice by the others, then the surrounded pentagon belongs to a triple of pentagons that have a common point c whose open 3λ -neighborhood contains the central vertices of the three pentagons. Moreover, at each of the outer obtuse vertices of the surrounded pentagon, three pentagons meet so that they have a common point whose 3λ -neighborhood contains an outer obtuse vertex of each of these pentagons.*

Fig. 4 Three 120° angular sectors covering the plane

Proof Assume at the beginning that $\varepsilon = 0$. Let P denote the pentagon that is surrounded, and let v be its central vertex. Observe that the circle C of radius $\frac{1}{2}$ centered at v does not contain any vertex of P other than v . Therefore if λ is sufficiently small, then C must be covered by P and two other pentagons whose central vertices lie in C and whose no other vertex lies in the circle. Were it otherwise, then by taking a sequence of values of λ converging to zero one would obtain a contradiction with Proposition 1.

The homothetic enlargement centered at v and with coefficient $\frac{1}{\lambda}$ turns C into a circle of radius $\frac{1}{2\lambda}$ covered by three 120° angular sectors one of which has its vertex at v , and whose 1-cores do not overlap. Obviously, the three sectors must have a common point. We aim to find an upper bound for the distance from some point in the common part of the three sectors to the vertices of the sectors. Observe that the circle's radius increases without bound as $\lambda \rightarrow 0$.

Now consider the limit case (see Fig. 4a): Three 120° angular sectors, say V_1 , V_2 , V_3 cover the entire plane, while their 1-cores do not overlap. Then it is obvious that the sectors are rotated with respect to each other by 120° angles, and the margin of overlap at their boundaries is of width at most 2. It is also obvious that the three sectors must have a common point; let p be one of them. It is easy to see that p lies in the margin of width 2 in each of the sectors, and it must be contained in the rhombus formed by the union of two equilateral triangles of height 2 at the vertex v of V_1 (see Fig. 4b), otherwise two of the sectors' 1-cores would overlap. Therefore the distance from p to the vertex of V_1 is at most $\frac{4}{\sqrt{3}} = 2.30\dots$. For the same reason the distance from p to each of the other two vertices is bounded above by the same number.

Since the configuration of sectors covering the circle of a radius increasing to infinity approaches that in the above limit case, there exists a sufficiently small $\lambda > 0$ for which the distances in question are smaller than, say 2.6. This means that for $\varepsilon = 0$, the conclusion of our lemma holds with the 2.6λ -neighborhood of the common point. Increasing the coefficient from 2.6 to 3 allows for the same conclusion for sufficiently small $\varepsilon > 0$ as well.

The proof of the second part is very similar to the proof of the first, with the only difference that the value of λ needs to be chosen smaller. $\square$

Definition 3 Let λ and ε be positive numbers such that the conclusion of Lemma 1 holds. Three pentagons congruent to P_ε are said to form a *regular triple* if their central vertices coincide and their outer obtuse vertices form a regular hexagon. A triple of pentagons, each congruent to P_ε , whose λ -cores do not overlap is *semiregular* if they have a common point c whose 3λ -neighborhood contains the central vertices of all three of them.

Observe that if λ and ε are sufficiently small, then the outer obtuse vertices of the three pentagons in a semiregular triple form a convex hexagon.

The following lemma is an “expanded” version of Lemma 1 and follows directly from it:

Lemma 2 *For every sufficiently small value of λ , there exist arbitrarily small values of ε such that for every positive integer n the following holds: Given a family of congruent copies P_ε whose λ -cores do not overlap. If one of the pentagons is surrounded $n + 2$ times by the others, then every pentagon in the n th surrounding belongs to a semiregular triple of pentagons. Moreover, at each of the outer obtuse vertices of each of the pentagons in the n th surrounding, three pentagons meet so that they have a common point whose 3λ -neighborhood contains an outer obtuse vertex of each of these pentagons.*

Definition 4 For a semiregular triple of pentagons congruent to P_ε , a common point c whose 3λ -neighborhood contains the central vertices of all three of them will be called a *central pin* of the semiregular triple, and we will say that the triple is *pinned* by it. Similarly, a copy of P_ε in a semiregular triple can be pinned by *outer pins* at each of its outer obtuse vertices to other copies of P_ε , as in Lemma 2. In each case, one pin pins exactly three pentagons, and its distance from the corresponding three vertices of the pinned pentagons is smaller than 3λ . (The term “pin” is used for comparison with three pieces of paper pinned together to a bulletin board by a single pin through their corners.)

Remark 2 Observe the particular relation between λ and ε in Lemmas 1 and 2 and see that these two numbers cannot be chosen “sufficiently small” independently from each other. To see that λ *must be chosen first*, notice that otherwise, after ε has been chosen, λ could be then chosen much smaller than ε , and then there would not exist a family of pentagons satisfying the assumptions of the lemma. The correct order of setting the values of λ and ε is followed in the proof of Lemma 1: the coefficient of 3 in 3λ in the case $\varepsilon = 0$ is somewhat greater than needed, since the coefficient of $2.7 > \frac{4}{\sqrt{3}}$ would suffice. The rounded up coefficient of 3 allows for replacing P_0 with P_ε for sufficiently small $\varepsilon > 0$.

Remark 3 Consider a covering of the plane with congruent convex disks and the n th surrounding ($n \geq 1$) of one of them. If the diameter of the disks is d , then the distance from a point in the surrounded disk to each point in the n th surrounding is at most $(n + 1)d$. Hence the n th surrounding of a disk is contained in a circle of radius $(n + 1)d$, and therefore the area covered by the n th surrounding is at most $\pi[(n + 1)d]^2$.

Fig. 5 The triple described in Lemmas 3 and 4

By the above remark, a simple area estimate yields the following:

Proposition 3 *Given a convex disk K and a number λ such that the λ -core of K is nondegenerate, for every positive integer i , there is a constant $N_i = N_i(K, \lambda)$ such that in every K -covering, the i th surrounding of any disk contains at most N_i disks whose λ -cores are mutually disjoint.*

The following two lemmas are most crucial in the proof of the theorem. The first one states that the regular P_ε -triple is the (unique) solution to a certain area maximizing problem:

Lemma 3 *Suppose that three congruent copies of the pentagon P_ε form a semiregular triple. Let the six outer obtuse vertices of the three pentagons be labeled clockwise by $v_1, \dots, v_6$ as shown in Fig. 5, and let six vertices $p_1, \dots, p_6$ of a hexagon H_p be given, two in each of the corresponding pentagons, so that the distance between p_i and v_i is smaller than 3λ for $i = 1, \dots, 6$. If ε and λ are sufficiently small, then $\text{Area}(H_p) \leq 3$. Moreover, equality occurs only if the triple is regular and H_p is congruent to the regular hexagon H .*

Proof If $\varepsilon > 0$ and $\lambda > 0$ are sufficiently small, then vertex v_1 is farther away from the segment p_6p_2 than the point p_1 (see Fig. 5). Thus, $\text{Area}(H_p) \leq \text{Area}(v_1p_2p_3p_4p_5p_6)$. Similarly, if the remaining “old” vertices of the hexagon $v_1p_2p_3p_4p_5p_6$ are one-by-one replaced by the corresponding vertices of the hexagon $H_v = v_1v_2v_3v_4v_5v_6$, then the area of the hexagon increases every time. Thus, at the end we have $\text{Area}(H_p) \leq \text{Area}(H_v)$.

Next we prove that if $\text{Area}(H_v)$ is at a maximum, then the common point c of the pentagons coincides with all three central vertices of the pentagons. Assume to the contrary that the central vertex, say w , of the pentagon corresponding to the edge v_2v_3 is distinct from c . We will show that the translation of that pentagon by $\vec{w}\vec{c}$ increases the area of the hexagon H_v . If the hexagon altered by the pentagon’s shift contains the initial one, then, obviously, the area increases. Otherwise we may assume that the altered hexagon contains v_2 but not v_3 . The change in the hexagon’s area caused by the pentagon’s shift is equal to $|\vec{w}\vec{c}| \cdot (\frac{1}{2}\text{pr}(v_1v_2) + \text{pr}(v_2v_3) - \frac{1}{2}\text{pr}(v_4v_3))$, where $\text{pr}(\cdot)$ is the length of the projection of a segment to the line perpendicular to $\vec{w}\vec{c}$. But

Fig. 6 Remark 4 and detail in proof of Lemma 4

if we replace the three pentagons by a regular triple of P_0 -pentagons and apply the same δ -shift to one of them, then, clearly, the above combination of the projections is positive, thus the same holds if $\varepsilon > 0$ and $\lambda > 0$ are sufficiently small. (Although the shift may cause the λ -cores to overlap, this does not affect the rest of the proof.)

The proof is thereby reduced to the case in which all three central vertices of the (not necessarily semiregular) triple coincide with c . Since now the hexagon H_v is inscribed in the circle centered at c and of the same radius as the circle circumscribing the regular hexagon H , the maximum area of H_v is attained if and only if H_v is congruent to H . $\square$

Notice that the area of the hexagon H_p depends continuously on the configuration of the points $p_1, \dots, p_6$. Therefore:

Proposition 4 *If for a sequence of semiregular triples of pentagons, the areas of their corresponding hexagons H_p converge to 3, then the triples' configurations converge to the regular triple. Moreover, a triple for which the area of the corresponding hexagon H_p is sufficiently close to 3 must contain crossing pairs of pentagons.*

Observe that the hexagon $H_p = p_1 p_2 p_3 p_4 p_5 p_6$ in its initial position (as in the above lemma) need not be covered by the three pentagons, even though Fig. 5 shows the case where it is. The reason for that is that ε can be much smaller than λ (and it has to be so in the proof of the theorem), which allows for an “inward dent” in the union of the three pentagons near the middle of the edge $p_1 p_2$ for example, so that the acute tips near that edge end up inside H_p (see Fig. 6a).

The next lemma, just as Lemma 3, gives an inequality that solves an area maximizing problem concerning three congruent copies of P_ε that form a semiregular triple, but instead of maximizing the area of the hexagon associated with the triple, we now compare the area of the hexagon with the sum of the areas of the pentagons' parts contained in the hexagon.

Lemma 4 Suppose that pentagons P' , P'' , and P''' , each congruent to P_ε , form a semiregular triple with a central pin c considered here as the origin. Let the six outer obtuse vertices of the pentagons be labeled clockwise by $v_1, \dots, v_6$. Further, let six points $p_1, \dots, p_6$ be given, two in each of the corresponding pentagons, so that the distance between p_i and v_i is smaller than 3λ for $i = 1, \dots, 6$ as shown in Fig. 5. Denote by H_p the hexagon $p_1 \dots p_6$ and assume that the three pentagons cover $\frac{9}{10}H_p$, the homothetic image of H_p . If λ and ε are sufficiently small, then

$$\text{Area}(H_p) \leq \text{Area}(P' \cap H_p) + \text{Area}(P'' \cap H_p) + \text{Area}(P''' \cap H_p).$$

Moreover, equality occurs only if $\varepsilon = 0$ and the three pentagons form a regular triple.

Remark 4 It is important to notice that in Lemma 4 the condition requiring that $\frac{9}{10}H_p$ should be covered by the three pentagons is essential. Arrange three copies of pentagon P_0 so that their central vertices coincide and one of the pentagons' interior is disjoint from the other two, while the other two overlap leaving α -angular gaps on both sides (see Fig. 6a). The inequality asserted by Lemma 4 is equivalent to $2B + C \leq A$, where A , B , and C denote the areas of the shaded regions as shown in Fig. 6a. If α approaches zero, then $\frac{2B}{A}$ approaches 1 and $\frac{C}{A}$ approaches $\frac{1}{3}$, therefore if α is sufficiently small, then $2B + C > A$, contrary to the asserted inequality.

Proof of Lemma 4 Consider first the special case where $\varepsilon = 0$ and the six points coincide with the outer obtuse vertices of the pentagons, forming the hexagon $H_p = H_v = v_1, \dots, v_6$. Let e denote the edge length of the regular hexagon H of area 3.

Assuming that H_v is not congruent to H , we will modify the arrangement of the pentagons by translating them one-by-one, each time translating a pentagon in the direction parallel to one of its sides of length $\frac{\sqrt{3}}{2}e$ and away from the central vertex. As a pentagon is shifted, its outer obtuse vertices (two of the vertices of H_v) move along with it. During these shifts, the pin c remains fixed, and a pentagon being shifted stops its motion as soon as c becomes its boundary point. Therefore c remains a common point of the pentagons during the entire process. Since c lies in the intersection of P' with the 3λ -neighborhood of the central vertex of P' and remains there after the shift, each of the shifts, and each of their compositions as well, moves points by a distance at most 3λ . (Obviously, any of the shifts may destroy the property that the λ -cores do not overlap as well as the property that the three pentagons cover $\frac{9}{10}H_p$, but this does not interfere with the rest of the proof.)

After at most six translations the three pentagons are brought to a position, call it the *terminal position*, in which their three central vertices coincide at c . During these motions, the vertices of H_v become (constrained) variables, making the area of H_v , the sum

$$P_v = \text{Area}(P' \cap H_v) + \text{Area}(P'' \cap H_v) + \text{Area}(P''' \cap H_v),$$

and other related quantities, continuous functions of these variables.

In order to estimate the changes of areas caused by the shifts, we introduce additional notation. Let H_S denote the area of the hexagon H_v in the starting position, and let H_T be the area of H_v in the terminal position. Similarly, let P_S and P_T denote

the starting and the terminal values of the sum P_v , respectively. Finally, let U_v denote the area in H_v that is not covered by the three pentagons, with U_S and U_T being the starting and terminal values of U_v , respectively.

Consider one of the translations, say the first one, moving one of the pentagons by a distance of δ , while the other two pentagons remain fixed (see Fig. 6b). The increase $\Delta_1(H_v)$ of the area of the hexagon H_v is approximately, within an arbitrarily given small percentage of it, equal to $(\frac{1}{2} + \frac{1}{2} - \frac{1}{4})\delta e = \frac{3}{4}\delta e$ if λ is sufficiently small. (The subscript 1 refers to the first translation; the area computation is very elementary—just a combination of several base-times-height type area formulas that can be easily verified using Fig. 6b). At the same time, the increase $\Delta_1(P_v)$ of the sum of the areas of the parts of pentagons contained in H_v is smaller than $(\frac{1}{8} + \frac{1}{16})\delta e = \frac{3}{16}\delta e$ (we consider only edges of H_v whose change in direction results in increase of area, and we ignore the other edges).

Notice that the difference between the increases $\Delta_1(H_v)$ and $\Delta_1(P_v)$ is at least $\frac{3}{8}\delta e$. Here $\frac{3}{8}\delta e$ is taken instead of the exact difference between the estimates, thus generously compensating for the small percentage in the previous area approximation. This product is one-half of the area within H_v that before the translation was covered and after the translation may have become uncovered due to the shift of the pentagon (see the parallelogram with two opposite sides of size δ at the bottom of the shaded pentagon in Fig. 6b; the area of this parallelogram is $\frac{3}{4}\delta e$). Denoting by $\Delta_1(U_v)$ this uncovered area, we have

$$\Delta_1(H_v) > \Delta_1(P_v) + \frac{\Delta_1(U_v)}{2}. \quad (1)$$

Same inequalities hold for each of the shifts, with corresponding increases $\Delta_1, \dots, \Delta_j$. By adding all of these inequalities we get the following inequality linking the final area increases $\Delta(H_v)$ and $\Delta(P_v)$ with the area $\Delta(U_v)$ uncovered at the end of all shifts:

$$\Delta(H_v) > \Delta(P_v) + \frac{\Delta(U_v)}{2}. \quad (2)$$

Since in the final configuration of the pentagons the three central vertices coincide, we have two cases: either one of the pentagons overlaps with each of the other two, i.e., there is one angular gap between the pentagons, or one of the pentagons is overlapped by neither of the other two, i.e., there are two angular gaps between the pentagons. Although Fig. 6a shows the latter case only, it still illustrates our argument well. Let A denote the area of the union of the overlaps, let B denote the area of the union of angular gaps, and let C denote the area of the union of the triangles of the same kind as the one shown in Fig. 6a.

As we noticed before, as λ approaches 0, $\frac{B}{A}$ approaches 1 and $\frac{C}{A}$ approaches $\frac{1}{3}$. Since obviously $1 - \frac{119}{200} - \frac{1}{3} > 0$, there exists a sufficiently small λ for which

$$A - \frac{119}{200}B - C > 0 \quad (3)$$

holds. Now, since

$$P_T = H_T + A - B - C = H_T - \frac{81}{200}B + A - \frac{119}{200}B - C, \quad (4)$$

we get

$$P_T > H_T - \frac{81}{200}B. \quad (5)$$

Observe now that by the assumption of our lemma,

$$U_T \geq \left(\frac{9}{10}\right)^2 B = \frac{81}{100}B. \quad (6)$$

This is so because, at the initial stage, the three pentagons covered the $\frac{9}{10}$ -homothetic part of B , and now no part of B is covered. Therefore, for the sufficiently small value of λ chosen above, we have:

$$H_T \leq P_T + \frac{U_T}{2}. \quad (7)$$

By subtracting (2) from (7) we obtain

$$H_T - \Delta(H_v) < P_T - \Delta(P_v). \quad (8)$$

But now, in view of $H_T = H_S + \Delta(H_v)$ and $P_T = P_S + \Delta(P_v)$, inequality (8) yields $H_S < P_S$, which is exactly what was to be proved. Observe that equality holds only if H_S is congruent to H_T and $U_T = 0$, that is, if not a single shift was needed and if the initial triple of pentagons was regular.

Finally we prove the existence of a suitable $\varepsilon > 0$: It is easy to see that the argument used above for the case of $\varepsilon = 0$ can be repeated for some $\varepsilon > 0$. Namely, if ε is sufficiently small, then all four bounds change by a sufficiently small percentage and the bounds in the above discussion change by a small enough percentage depending on ε so that this “error” is absorbed at the point where we take the coefficient of $\frac{3}{4}$ instead of the exact one to take care of the first small percentage error. The “error” caused by replacing points p_i with v_i is also absorbed at the same time, provided that we set the value of λ sufficiently small at the beginning of the proof. $\square$

4 Density, the Method of Cells, and Outline of Proof of the Theorem

4.1 Density

We begin with recalling the notation and definitions concerning the density of a covering and the covering density of a disk, commonly used in the theory of packing and covering (see, for instance, [7] or [4]).

A collection $\mathcal{C} = \{C_1, C_2, \dots\}$ of compact sets with nonempty interiors is said to cover a domain D if $D \subset \bigcup_i C_i$. If the domain D is bounded, then the *density of $\mathcal{C}$ relative to D* is defined as

$$d(\mathcal{C}, D) = \frac{1}{\text{Area}(D)} \sum \text{Area}(C_i),$$

where the summation extends over those values of i for which C_i intersects D .

If D is the entire plane, then we define the *upper density* $\overline{d}$ and the *lower density* $\underline{d}$ of the covering $\mathcal{C}$ as follows:

$$\overline{d}(\mathcal{C}) = \limsup_{r \rightarrow \infty} d(\mathcal{C}, B^2(r)) \quad \text{and} \quad \underline{d}(\mathcal{C}) = \liminf_{r \rightarrow \infty} d(\mathcal{C}, B^2(r)),$$

where $B^2(r)$ denotes the circular disk of radius r centered at the origin. If these two densities have the same value, then it is called the *density* $d(\mathcal{C})$ of the covering $\mathcal{C}$.

If we restrict ourselves to coverings $\mathcal{C}$ of the plane with congruent copies of a compact set C bounded by a simple closed curve (called a *topological disk*, or just a *disk*), then we define the *covering density* $\vartheta(C)$ of the disk C as the smallest density of a covering of the plane with congruent copies of C , i.e.,

$$\vartheta(C) = \inf\{d(\mathcal{C})\},$$

where the infimum is taken over all C -coverings $\mathcal{C}$ of the plane that have a (well-defined) density, that is, for which $\overline{d} = \underline{d}$.

It is well known that the infimum is attained and that

$$\vartheta(C) = \min d(\mathcal{C}) = \min \underline{d}(\mathcal{C}),$$

where the second minimum is taken over *all* C -coverings $\mathcal{C}$ of the plane.

In view of the above identity, in order to prove that a certain number is the covering density of a given disk, it suffices to present a covering with that density and then prove that it is smaller than or equal to the lower density of every covering of the plane with congruent copies of the given disk.

4.2 The Cell Method

Suppose that we are given a covering $\mathcal{C}$ of the plane with congruent copies of a disk C and we wish to prove that its lower density is greater than or equal to a certain number. The *method of cells* in its most basic form goes as follows. We cover the plane with cells (they are allowed to overlap) and we assign one disk C from $\mathcal{C}$ to each cell in a one-to-one manner. (Ideally, the disk assigned to the cell covers the cell.) If the ratio between the area of the disk assigned to the cell and the area of the cell is at least as large as the conjectured number, then the density inequality follows. (This ratio is called the *local density* of the cell, or the *cell's density*.) Such simple cell designs and assignments usually do not exist, but there are ways to refine this basic method.

In what follows we describe a major modification of the basic cell method designed to fit the particular problem at hand, which is proving that the covering density of the pentagon P_ε (with a suitably chosen value of $\varepsilon > 0$) is exactly $1 + \varepsilon$. The specific design of the cells will also allow us to prove the unavoidability of crossings.

4.3 Outline of Proof

Let $\mathcal{C}$ be an arbitrary covering of the plane with congruent copies of the pentagon P_ε . As always we assume that the covering is locally finite. We may also assume that the covering is irreducible, that is, that no pentagon is covered by other pentagons.

Part 1 First we choose sufficiently small values of λ and $\varepsilon > 0$ (the latter will depend on the former) and then we show that $1 + \varepsilon$ is smaller than or equal to the lower density of $\mathcal{C}$. At the beginning, a preliminary cell system in the plane is described such that:

- The cells are compact subsets of the plane;
- The cells cover the plane;
- The diameters of the cells have a common upper bound.

Some pentagons may be greatly overlapped by other pentagons, causing the local density to significantly exceed 1. Such “surplus areas” will be partitioned, and their parts will be distributed between the overlapping pentagons. At the same time these pentagons will be clipped to eliminate the overlaps, and the clipped pentagons will be treated as cells. These cells will have a large surplus area. The remaining (unclipped) pentagons will also be considered as cells but, at the beginning, they will have no surplus area. Our task will be to show that the average surplus area per cell will be sufficiently high to conclude that the global density of the covering is at least $1 + \varepsilon$. To this goal, parts of the surplus area will often be transferred from one cell to another. This will be done so that:

- There will be a common upper bound for the distance between the cell that gives some of its surplus to another and the cell that receives it.

While maintaining the above four conditions, some adjustments of the cell design and ways of surplus area distributions will be needed before we could claim that the cells cover the plane and that each cell’s local density is at least $1 + \varepsilon$. At that point we will conclude that $\underline{d}(\mathcal{C}) \geq 1 + \varepsilon$.

Part 2 To complete the proof we show that crossings must occur in every thinnest covering. Considering the cell design used to prove the density bound, at that point it will be evident that if the covering’s density is exactly $1 + \varepsilon$, then a sequence of semiregular triples of pentagons must exist whose configurations converge to the regular triple. Moreover, as it was previously mentioned (see Remark 4), any triple sufficiently close to being regular must have crossing pairs of pentagons.

5 Proof of the Theorem

First choose $\lambda > 0$ and $\varepsilon_0 > 0$ so that each of the Lemmas 2, 3, and 4 holds for the pentagon P_ε with any $\varepsilon \leq \varepsilon_0$. Regarding Lemma 4, we also assume that λ is sufficiently small so that if a semiregular triple is twice surrounded by pentagons (all pentagons in the second surrounding are congruent to P_ε), then the triple of pentagons satisfies the lemma’s requirement that the $\frac{9}{10}$ -homothetic image of the hexagon H_v be covered by the pentagons of the triple. The value of λ is now set for the rest of the proof, while ε will for a while remain variable. The proofs of the lemmas make it clear that ε_0 must be very small, certainly much smaller than $\frac{1}{2}$. Observe that there exists a minimum $\mu > 0$ depending only on λ such that if two copies of P_ε overlap so that their λ -cores intersect, then the common area of the two pentagons is at least μ

(in fact, the size of μ is approximately equal to $4\lambda^2$). Let ε be so small that if the λ -core of P_ε is overlapped by the λ -core of another copy of P_ε , then the area of the overlap between the two pentagons is at least $N\varepsilon$, where the integer N will be determined in the forthcoming part of the proof.

Before continuing, the reader should keep in mind the last subsection of the previous section, called “Outline of proof.” That part is needed for understanding the reasons behind the terms “preliminary cell design,” “surplus area,” and “refinement.”

Let $\mathcal{C}$ be an arbitrary covering of the plane by congruent copies of the pentagon P_ε . We may assume that $\mathcal{C}$ is locally finite and that it is irreducible, that is, that every pentagon covers some area not covered by any of the other pentagons. (It is obvious that every locally finite covering with congruent disks contains an irreducible subcovering.) We sort all pentagons of the covering $\mathcal{C}$ into two types:

Definition 5 A pentagon is *dense*, and will be called a *D-pentagon*, if another pentagon overlaps it by an area greater than $N\varepsilon$. (The large overlap provides sufficient “local density surplus” whose large part can be distributed to some of the other pentagons, explaining the use of the term “dense,” and we use the label “*D*” for “dense” or “donor.”) A pentagon is *thin*, and will be called a *T-pentagon*, if it is not a *D-pentagon*. (Here we use the label “*T*” for “thin” or “taker.”)

5.1 The Preliminary Cell Design and Area Distribution

If two pentagons overlap by an area greater than $N\varepsilon$, then each of them is classified as a *D-pentagon*. As we mentioned before, part of this “large” overlap can be distributed to some of the other pentagons.

Temporarily, consider the collection of *D-pentagons* only. A *component* of the union of the *D-pentagons* is defined as the set of points that are covered by exactly the same subfamily of *D-pentagons*. We use the term “component” because each of them is the same as the closure of one of the connected components of the complement of the union of the boundaries of the *D-pentagons*.

We will clip the *D-pentagons* to eliminate overlaps between them while assuring that the union of the clipped *D-pentagons* is the same as the union of the original *D-pentagons*. The clipped-off pieces are then assigned to the clipped pentagons as “surplus area” in an equitable way.

Consider a component created by exactly i overlapping *D-pentagons* and imagine the pentagons as if lying in separate layers, one above another. Dissect the component by broken lines into i pieces of equal area (see Fig. 7a). Then make a one-to-one assignment between the i pieces and the i pentagons that create the component, taking the piece from the layer of the *D-pentagon* to which it is assigned. Keep the piece assigned to the particular *D-pentagon* attached to it and clip off the other $i - 1$ pieces from the *D-pentagon* to place them on top of the clipped *D-pentagon* in the same manner as one would place poker chips on a card table. Repeat this process with the remaining components contained in the original *D-pentagon*, and after all the clipping the remaining part of the pentagon is called the *clipped D-pentagon*. (Although in Fig. 7b the clipped pentagons are connected, connectivity is not a general requirement for cells. Also, notice that each clipped *D-pentagon* contains its unaltered component consisting of points covered by no other *D-pentagon*.)

Fig. 7 Clipping the D -pentagons**Table 1** The preliminary cell types and the preliminary area assignment

Type of cell	Cell's area	Area assigned to cell
Clipped D -pentagon	$< 1 + \varepsilon$	$\geq \text{cell's area} + 3\varepsilon$
T^+ -pentagon	$1 + \varepsilon$	$\geq 1 + 31\varepsilon$
T^0 -pentagon	$1 + \varepsilon$	$1 + \varepsilon$

Observe that the clipping reduces the area of the original D -pentagon by at least one-half of the overlap (equality occurs only when each overlap's multiplicity is equal to 2), and this amount is equal to the total area of the pieces placed on top of the clipped D -pentagon. We wish to assure that the local density in every clipped D -pentagon be $1 + 2\varepsilon$ or greater. Thus we can designate a surplus area of $\frac{N}{2}\varepsilon - 3\varepsilon$ in every D -pentagon for distribution to some of the other pentagons in the covering. Namely, even if only a surplus of 3ε were left with the pentagon, then the local density in the pentagon would already be at least $\frac{1+3\varepsilon}{1+\varepsilon} > 1 + 2\varepsilon$ (the inequality is satisfied as soon as $\varepsilon < \frac{1}{2}$, which is safely assumed).

Let the integer N in the above definition of ε be $2(30N_4 + 3)$. (For the definition of N_4 , see Proposition 3.) From each clipped D -pentagon's surplus area of $(30N_4 + 3)\varepsilon$ we transfer an area of 30ε to every T -pentagon in its 4th surrounding, still leaving a surplus area of at least 3ε with the D -pentagon. This area distribution is called *the preliminary area distribution*. The coefficient of 30 above is chosen to be large enough to facilitate subsequent modifications of the cell design and area distribution. The reason for choosing the coefficient to be exactly 30 will soon become apparent.

Definition 6 If a surplus area of 30ε is assigned to a T -pentagon, we will call the pentagon an *augmented thin pentagon* or a T^+ -pentagon, for brevity. The remaining, unaugmented T -pentagons will be called T^0 -pentagons.

Table 1 summarizes the preliminary cell types and the preliminary area assignment.

Table 2 The cell types and the area distribution after the first refinement

Type of cell	Cell's area	Area assigned to cell
Clipped D -pentagon	$< 1 + \varepsilon$	$\geq \text{cell's area} + 3\varepsilon$
T^+ -cell	$1 + \varepsilon$	$\geq 1 + 31\varepsilon$
Hexagonal cell	≤ 3	$3 + 3\varepsilon$

Note that for the (preliminary) cells that are T^0 -pentagons we cannot guarantee that $1 + \varepsilon$ is a lower bound for their cell density, so we do not have an overall nontrivial bound for the lower density yet.

5.2 The First Modification of Cell Design and Area Distribution

Recall that a triple of pentagons, each congruent to P_ε and whose λ -cores do not overlap, is called *semiregular* if their intersection contains a point c (a central pin) closer to each of their central vertices than 3λ . Moreover, if the assumptions of Lemma 2 are satisfied for a pentagon in the triple, then at each of the six outer obtuse vertices of the triple, three pentagons are pinned by a pin that is outer in each of them (see Definition 4).

Lemma 5 *Each T^0 -pentagon is part of a semiregular triple consisting of T -pentagons. Moreover, at each outer obtuse vertex of every pentagon in the triple, three T -pentagons are pinned at their outer obtuse vertices by an outer pin.*

Proof Since the 3rd surrounding of a T^0 -pentagon cannot contain a D -pentagon (otherwise the surrounded pentagon would be either a T^+ -pentagon or a D -pentagon), the λ -cores of the pentagons in this surrounding do not overlap. Therefore, by Lemma 2, the conclusion of our lemma is satisfied. $\square$

We now consider all semiregular triples of T -pentagons containing at least one T^0 -pentagon and assign them to a *hexagonal cell* whose vertices are the selected outer pins, and we modify the preliminary cell system by creating the hexagonal cells. The T -pentagons assigned to their hexagonal cells are no longer considered as cells. Thus, by Lemma 5, every T -pentagon that is still called a cell must be a T^+ -pentagon. To distinguish the T^+ -pentagons that now are treated as cells from the other T^+ -pentagons, we refer to these cells as the T^+ -cells. The clipped D -pentagons and the areas assigned to them remain unchanged.

Table 2 summarizes the cell types and area distribution after the 1-st refinement.

Recall that by Lemma 3, the area of the hexagon formed by the outer pins of a semiregular triple cannot exceed the area of the regular hexagon H , hence the entry “At most 3” in the table (third row, middle column) is correct.

Note that now the density in each cell is at least $1 + \varepsilon$. However, the cells may not cover the plane. Indeed, part of a pentagon assigned to its hexagonal cell may protrude outside the cell and not be covered by any cell. The next modification will correct this deficiency.

Fig. 8 A cluster of hexagonal cells and a boundary triangle

5.3 The Second, Final Modification

Observe that the hexagonal cells form mutually disjoint, connected *clusters* of edge-to-edge hexagons. A cluster can be defined as a connected component of the union of the hexagonal cells, but we prefer to treat it as a collection of cells. These clusters are the basis for our next modification. The hexagonal cells adjacent to the boundary of a cluster will be called *boundary hexagons*, and all other cells in a cluster will be referred to as *interior hexagons*.

Notice first that Lemma 5, our choice of N , and our classification of the pentagons imply that *each cluster is surrounded by semiregular triples of T^+ -pentagons*. In the second part of Lemma 5 we already established that in the clusters, all boundary hexagons must be adjacent to T -pentagons outside the clusters. None of these pentagons can be a T^0 -pentagon, since otherwise it would, along with the semiregular triple to which it belongs, lie within the cluster. Thus all of them are T^+ -pentagons. None of them can be intersected by a D -pentagon, because that would contradict the classification of the T^0 -pentagon in the adjacent triple of the cluster. Thus every pentagon outside the cluster and adjacent to the cluster is part of a semiregular triple outside the cluster, as shown in Fig. 8. (For simplicity, the cluster shown in the figure is small, so that each of its hexagonal cells is a boundary hexagon, but in general this is not the case.)

Because of the problem with covering of the area near the boundary of the clusters, the boundary hexagons will be treated differently from the interior ones. A boundary hexagon will not be assigned the entire three pentagons of its triple, but only the sum of the areas of its intersections with these three pentagons. According to Lemma 4, this sum is at least as large as the area of the hexagonal cell. The remaining pieces of such pentagons are temporarily kept waiting for reassignment. The density of each of

the boundary hexagons is thereby temporarily reduced, but not below 1. The reduction will soon be compensated by a surplus area reassignment.

We now create triangular cells, called *boundary triangles*, making one triangular cell adjacent to the cluster along each boundary edge of the cluster (see Fig. 8) as follows. Consider a single boundary edge pq of a cluster of hexagons, where p and q are two end pins of an edge of a hexagonal boundary cell in the cluster. Let r be the central pin of the semiregular triple adjacent to the edge pq and outside the cluster. The triangle pqr becomes a new cell, and the areas of the pieces of pentagons that lie in this new cell will be assigned to it. What is left from the T^+ -cell (or from either of the two T^+ -cells) will be called a *reduced T^+ -cell*.

Of the total surplus area of 30ε on every T^+ -pentagon (given to it by some D -pentagon), consider the portion of 27ε as transferable quantity but imagine the surplus area of 3ε spread evenly over the pentagon, like butter on a slice of bread. The pins p and q can come from a single T^+ -pentagon or from two T^+ -pentagons as shown in Fig. 8. In each case we distribute the surplus area in the same way. We cut this (or these two) pentagon(s) by the infinite angular sector $\angle prq$. The reduced T^+ cell (or each of the two reduced T^+ -cells), say of area Φ , keeps a surplus area of $\frac{\Phi}{1+\varepsilon}3\varepsilon$. The part cut off, say of area Ψ , from the T^+ -cell(s) is assigned to the triangular cell with a surplus area of $\frac{3\Psi}{1+\varepsilon}\varepsilon$. Thus far, the surplus area from the original T^+ -cell has been distributed among the pieces in the reduced T^+ cell (or each of the two reduced T^+ -cells) and in the new triangular cell proportionally to their areas like butter on a slice of bread that has been cut with a knife. From the remaining surplus of 27ε , the boundary hexagon that shares the edge pq with the triangle receives 9ε , to assign 3ε to each of the pentagons forming the triple that defines the hexagon. (It is possible that one T^+ -pentagon is adjacent to three boundary hexagons of the cluster, which requires surplus area of 27ε to give to each of the triples an area of 9ε .) The 3ε surplus received by each of these pentagons is spread on it like butter. Any remaining surplus can be divided arbitrarily between the donors and recipients.

The pieces of the pentagons of the boundary hexagon's semiregular triple that lie outside the hexagon and in the boundary triangle are given their share of the surplus area just received from the T^+ -pentagon via the boundary triangle, proportional to their areas. Since the triangular cell is covered by its assigned "buttered" pieces, its density is greater than $1 + 2\varepsilon$. For the same reason, the density in each boundary hexagon has been raised above $1 + 2\varepsilon$ as well. And while the density in each reduced T^+ -cell has dropped, it still remains above $1 + 2\varepsilon$.

We now have the following types of cells:

- D -cells (the clipped D -pentagons);
- cluster-forming hexagonal cells, which are of two kinds:
 - (1) boundary hexagons;
 - (2) interior hexagons;
- T^+ -cells (T^+ -pentagons not assigned to any hexagonal cells);
- boundary triangles (adjacent to the clusters of hexagons); and
- reduced T^+ -cells (T^+ -cells reduced by the creation of triangular cells).

The cells cover the plane, and the required conditions for area assignment are satisfied. The density in each cell is at least $1 + \varepsilon$. This proves that $\vartheta(P_\varepsilon) = 1 + \varepsilon$ if ε

Table 3 The cell types and the area distribution after the second, final modification

Type of cell	Cell's area	Area assigned to cell
D -cell	$< 1 + \varepsilon$	$\geq \text{cell's area} + 3\varepsilon$
T^+ -cell	$1 + \varepsilon$	$\geq 1 + 31\varepsilon$
Interior hexagon	≤ 3	$3 + 3\varepsilon$
Boundary hexagon	H_{bd}	$> (1 + 2\varepsilon)H_{bd}$
Boundary triangle	Δ_{bd}	$> (1 + 2\varepsilon)\Delta_{bd}$
Reduced T^+ -cell	T_{bd}	$> (1 + 2\varepsilon)T_{bd}$

is sufficiently small. Table 3 summarizes the cell types and area distribution after the 2-nd (and final) refinement. The area of the generic boundary hexagon is denoted by H_{bd} , the area of the generic boundary triangle is denoted by Δ_{bd} , and the area of the generic reduced T^+ -cell is denoted by T_{bd} (see Table 3).

Finally, notice that with the exception of interior hexagons, we have a lower bound $1 + 2\varepsilon$ for the local density in every cell. Assuming that the covering is thinnest, that is, of density $1 + \varepsilon$, this local bound implies that there are infinitely many interior hexagons and there is a sequence of such hexagons whose area converges to 3. By Proposition 4, the configurations of the semiregular triples assigned to these hexagonal cells converge to the regular configuration. Therefore the covering must contain crossing pentagons. Moreover, the collection of pentagons that do not cross another pentagon is *residual* among all pentagons of the covering. More precisely, with probability 1, a randomly chosen pentagon belongs to a semiregular triple so close to being regular that each of the three pentagons crosses the other two.

Remark 5 Although in our construction of P_ε we changed the directions of the sides of P_0 by the same small angle, this restriction can be relaxed. Similar examples are obtained by using two different small angles, one at the central vertex of P_0 and another one at the two outer obtuse vertices. As long as these two angles are (independently) sufficiently small, the proof of the theorem with so obtained pentagon in place of the original P_ε is virtually the same. These pentagons form an infinite, two-parameter family of convex disks whose covering density has thus been established. Moreover, for each of these pentagons, in a most economical (least dense) plane covering with the pentagon's congruent replicas crossings are unavoidable.

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