Set Partition Complexes

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Abstract The Hom complexes were introduced by Lovász to study topological obstructions to graph colorings. The vertices of $\text{Hom}(G, K_n)$ are the n -colorings of the graph G , and a graph coloring is a partition of the vertex set into independent sets. Replacing the independence condition with any hereditary condition defines a set partition complex. We show how coloring questions arising from, for example, Ramsey theory can be formulated with set partition complexes.

It was conjectured by Babson and Kozlov, and proved by Čukić and Kozlov, that $\text{Hom}(G, K_n)$ is $(n - d - 2)$ -connected, where d is the maximal degree of a vertex of G . We generalize this to set partition complexes.

Keywords Set partition complex · Hom complex · Topological combinatorics · Ramsey theory

1 Introduction

The last years have seen a fast development of topological tools in combinatorics. The Lovász conjecture regarding Hom complexes was proved by Babson and Kozlov [2], and several related questions have been resolved [1, 4–8, 14, 16, 17].

The Hom complexes encode graph colorings, which we view as partitions into independent sets. Our first goal is to introduce the new concept of set partition complexes and then explain in an independent section how they are related to Hom complexes. Then we prove a connectivity theorem for set partition complexes, which, when specialized to Hom complexes, recovers the theorem of Čukić and Kozlov [4].

In the last section we give an example of connections to Ramsey theory and hints on future research.

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2 Notation and Basics

For any positive integer n , we let $[n]$ denote the set $\{1, 2, \dots, n\}$.

The simplex with vertex set V is denoted by Δ^V , and the vertex set of a simplicial complex Σ is denoted by $V(\Sigma)$. If $V \subseteq V(\Sigma)$, then the induced subcomplex $\Sigma[V]$ is the set of simplices from Σ with vertices in V , and $\Sigma \setminus V = \Sigma[V(\Sigma) \setminus V]$.

A topological space T is n -connected if for all $0 \leq i \leq n$, any continuous map from the i -sphere to T can be extended to a continuous map from the $(i+1)$ -ball to T . Hence arcwise connected and 0-connected are the same. We define all nonempty spaces to be (-1) -connected and all spaces to be n -connected for $n \leq -2$. There are several good surveys on combinatorial topology, see, for example, [3, 13].

The vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$. An *independent* set is a subset of vertices of a graph with no edges between them. If A and B are sets, then K_A is the complete graph with vertex set A , and $K_{A,B}$ is the complete bipartite graph with vertex set $A \sqcup B$ and all edges between A and B . Usually $K_n = K_{[n]}$ and $K_{m,n} = K_{[m],[m+n] \setminus [m]}$ are used.

3 The Set Partition Complex

In this section we discuss the construction of set partition complexes. The set partition complex is a subcomplex of a direct product of simplices. For any finite simplicial complex Σ and finite set S , let $\Pi(\Sigma, S)$ denote the direct product of simplices Δ^S indexed by the vertices of Σ . A cell of $\Pi(\Sigma, S)$ is then a direct product $\prod_{v \in V(\Sigma)} \sigma_v$ of simplices, and its dimension is $\sum_{v \in V(\Sigma)} \dim \sigma_v$.

Definition 3.1 The *set partition complex* $\text{Part}(\Sigma, S)$ is a subcomplex of $\Pi(\Sigma, S)$ where a cell $\sigma = \prod_{v \in V(\Sigma)} \sigma_v$ is in the subcomplex if $\{v \in V(\Sigma) \mid s \in \sigma_v\} \in \Sigma$ for all $s \in S$.

First note that this really defines a subcomplex. If $\sigma \in \Pi(\Sigma, S)$, $\sigma' \in \text{Part}(\Sigma, S)$, and $\sigma \subseteq \sigma'$ where $\sigma = \prod_{v \in V(\Sigma)} \sigma_v \in \Pi(\Sigma, S)$ and $\sigma' = \prod_{v \in V(\Sigma)} \sigma'_v$, then $\{v \in V(\Sigma) \mid s \in \sigma_v\} \subseteq \{v \in V(\Sigma) \mid s \in \sigma'_v\} \in \Sigma$ for all $s \in S$ and hence $\sigma \in \text{Part}(\Sigma, S)$. The topology of $\text{Part}(\Sigma, S)$ is inherited from the topology of $\Pi(\Sigma, S)$, and it is a regular CW-complex (see [15] for a definition).

We note that the set of functions

$$\eta : V(\Sigma) \rightarrow 2^S \setminus \{\emptyset\} \quad \text{such that} \quad \eta^{-1}(s) \in \Sigma \text{ for all } s \in S$$

indexes the cells of $\text{Part}(\Sigma, S)$. This observation will be frequently used in the rest of the paper.

Example 3.2 A geometrical realization of the set partition complex $\text{Part}(\Delta^{[4]} \setminus \{[4]\}, [2])$ is drawn in Fig. 1. For example, the vertices $(1, 1, 2, 2)$, $(1, 1, 1, 2)$, $(2, 1, 1, 2)$, $(2, 1, 2, 2)$ are on the boundary of the maximal cell indexed by the function η defined by $\eta(1) = \{1, 2\}$, $\eta(2) = \{1\}$, $\eta(3) = \{1, 2\}$, and $\eta(4) = \{2\}$. All the maximal cells are indexed by functions η of the form $\eta(a) = \{1\}$, $\eta(b) = \{2\}$, and $\eta(c) = \eta(d) = \{1, 2\}$, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$, and hence are 2-dimensional.

Fig. 1 $\text{Part}(\Delta^{[4]} \setminus \{[4]\}, [2])$ is realized as the boundary of a rhombic dodecahedron, a two-dimensional sphere with twelve maximal cells

4 The Connection to Hom Complexes

In this section we discuss the connection between Hom complexes and set partition complexes. Though not required in later sections, it provides a motivation for our construction.

The study of set partition complexes originated from the theory of Hom complexes. For any graphs G and H , the polyhedral complex $\text{Hom}(G, H)$ has as its vertex set the graph homomorphisms from G to H . The most general definition of Hom complexes can be found in the survey [14], but we will mainly discuss the case where H is a complete graph. A graph homomorphism from a graph G into K_n corresponds to a proper coloring of the vertices of G with n colors. Among the vertices in G with the same color there are no edges, they form an independent set. A coloring of G is thus a partition of the vertex set into independent sets.

Definition 4.1 For any graph G , the *independence complex* of G , $\text{Ind}(G)$, is a simplicial complex with the same vertex set as G and with simplices the independent sets of G .

The set partition complex $\text{Part}(\text{Ind}(G), [n])$ describes the n -colorings of G and will coincide with a certain Hom complex.

Definition 4.2 The complex $\text{Hom}(G, H)$ is the subcomplex of $\prod_{v \in V(G)} \Delta^{V(H)}$ where a cell $\sigma = \prod_{v \in V(G)} \sigma_v$ is in the subcomplex if, for every $\{u, v\} \in E(G)$, there is a complete bipartite subgraph K_{σ_u, σ_v} in H .

One can index the cells of $\text{Hom}(G, H)$ in almost the same way as for set partition complexes, and in fact this is a standard idea in the theory of Hom complexes. They are indexed by the functions

$$\eta : V(G) \rightarrow 2^{V(H)} \setminus \{\emptyset\} \quad \text{such that } K_{\eta(u), \eta(v)} \subseteq H \text{ for all } \{u, v\} \in E(G).$$

Proposition 4.3 *As subcomplexes of $\prod_{v \in V(G)} \Delta^{V(H)}$, the complexes $\text{Part}(\text{Ind}(G), [n])$ and $\text{Hom}(G, K_n)$ are the same.*

Proof Specializing the indexing of $\text{Hom}(G, H)$ with functions η to $\text{Hom}(G, K_n)$, we get the set

$$\begin{aligned} & \{\eta : V(G) \rightarrow 2^{[n]} \setminus \{\emptyset\} \mid \{u, v\} \in E(G) \Rightarrow K_{\eta(u), \eta(v)} \subseteq K_n\} \\ &= \{\eta : V(G) \rightarrow 2^{[n]} \setminus \{\emptyset\} \mid \{u, v\} \in E(G) \Rightarrow \eta(u) \cap \eta(v) = \emptyset\} \\ &= \{\eta : V(G) \rightarrow 2^{[n]} \setminus \{\emptyset\} \mid \{u, v\} \in E(G) \Rightarrow \{u, v\} \notin \eta^{-1}(w) \text{ for } w \in [n]\} \\ &= \{\eta : V(G) \rightarrow 2^{[n]} \setminus \{\emptyset\} \mid \eta^{-1}(w) \text{ is independent in } G \text{ for } w \in [n]\} \\ &= \{\eta : V(G) \rightarrow 2^{[n]} \setminus \{\emptyset\} \mid \eta^{-1}(w) \in \text{Ind}(G) \text{ for } w \in [n]\}, \end{aligned}$$

which is exactly the functions that index $\text{Part}(\text{Ind}(G), [n])$. $\square$

If d is the maximal degree of a vertex of G , Babson and Kozlov [1] conjectured that $\text{Hom}(G, K_n)$ is $(n - d - 2)$ -connected, and this was proved by Čukić and Kozlov [4]. Let $\check{\chi}(G)$ be the largest number of colors a greedy algorithm might use to color G , so that $\chi(G) \leq \check{\chi}(G) \leq d + 1$. A more formal definition of $\check{\chi}(G)$ as $\text{gr}(\text{Ind}(G))$ will be given later. Engström [9] improved the result by showing that $\text{Hom}(G, K_n)$ is $(n - \check{\chi}(G) - 1)$ -connected, and Theorem 5.2 in the next section is a generalization of the result to set partition complexes. Hence one realizes that the structure of the independent sets and not the maximal degree implies the connectivity bounds for Hom complexes, and this inspired the study of set partition complexes. Other ways to generalize Hom complexes have been pursued, for example, by Živaljević [19].

5 Connectivity of Set Partition Complexes

In this section we prove a connectivity theorem for set partition complexes. One of the parameters in the connectivity bound is related to the performance of a greedy set partition algorithm which we discuss next. Given a finite simplicial complex Σ , a *greedy covering* $S_1, S_2, \dots, S_k$ of Σ is a sequence of disjoint sets $S_i \in \Sigma$ which partition $V(\Sigma)$ so that S_i is a maximal face of $\Sigma[S_i \cup S_{i+1} \cup \dots \cup S_k]$ for all $1 \leq i \leq k$.

What is the connection to greedy algorithms? Suppose that $v_1, v_2, \dots, v_m$ is an ordering of the vertex set of Σ . The output of the greedy algorithm is a partition $S_1, S_2, \dots, S_k$ of $V(\Sigma)$ where all $S_i \in \Sigma$. The algorithm starts with all $S_i = \emptyset$. First, it puts v_1 into S_1 . Then it treats $v_2, v_3, \dots, v_m$ like this: when it gets v_i , it puts into the S_j with $S_j \cup \{v_i\} \in \Sigma$ but $S_l \cup \{v_i\} \notin \Sigma$ for $1 \leq l < j$. Sometimes this kind of greedy algorithm is called a first-fit algorithm.

Definition 5.1 We let $\text{gr}(\Sigma)$ denote the largest number of sets in a greedy covering of Σ .

The rest of this section will be spent proving the following theorem.

Theorem 5.2 *The complex $\text{Part}(\Sigma, [n])$ is $(n - \text{gr}(\Sigma) - 1)$ -connected.*

As mentioned in the previous section, this generalizes Theorem 2.3 in [9], a strengthening of the main Theorem of [4], which was conjectured by Babson and Kozlov [1]. Before turning to the proof, we first need some properties of greedy coverings.

Lemma 5.3 *If $V \subseteq V(\Sigma)$, then $\text{gr}(\Sigma[V]) \leq \text{gr}(\Sigma)$.*

Proof It suffices to prove this when V and $V(\Sigma)$ differ by a vertex v . Let $S_1, S_2, \dots, S_{\text{gr}(\Sigma[V])}$ be a greedy covering of $\Sigma[V]$. If $S_i \cup \{v\} \notin \Sigma$ for $1 \leq i \leq \text{gr}(\Sigma[V])$, then $S_1, S_2, \dots, S_{\text{gr}(\Sigma[V])}, \{v\}$ is a greedy covering of Σ , and hence $\text{gr}(\Sigma[V]) + 1 \leq \text{gr}(\Sigma)$.

Otherwise there is an S_j for which $S_j \cup \{v\} \in \Sigma$ and $S_i \cup \{v\} \notin \Sigma$ for $1 \leq i < j$. Then $S_1, S_2, \dots, S_j \cup \{v\}, \dots, S_{\text{gr}(\Sigma[V])}$ is a greedy covering of Σ , and again $\text{gr}(\Sigma[V]) \leq \text{gr}(\Sigma)$. $\square$

Lemma 5.4 *If σ is a maximal face of Σ , then $\text{gr}(\Sigma) > \text{gr}(\Sigma \setminus \sigma)$.*

Proof Let $S_1, S_2, \dots, S_{\text{gr}(\Sigma \setminus \sigma)}$ be a greedy covering of $\Sigma \setminus \sigma$. Then $\sigma, S_1, S_2, \dots, S_{\text{gr}(\Sigma \setminus \sigma)}$ is a greedy covering of Σ . $\square$

We prove the connectivity bound of Theorem 5.2 with a nerve lemma, and the next step is to gain control of certain subcomplexes of set partition complexes. Recall that we index a cell $\sigma = \prod_{v \in V(\Sigma)} \sigma_v \in \text{Part}(\Sigma, [n])$ with the function $\eta : V(\Sigma) \rightarrow 2^{[n]} \setminus \{\emptyset\}$ defined by $\eta(v) = \sigma_v$. We will sometimes use η to denote the corresponding cell σ . If $\eta(v) \supseteq \eta'(v)$ for all $v \in V(\Sigma)$, then $\eta \geq \eta'$, and if also $\eta \neq \eta'$, then $\eta > \eta'$.

Our proof of the next lemma relies on the notion of elementary collapses. In a regular CW-complex, the removal of a maximal cell τ , together with a cell τ' which is on the boundary of τ and not on the boundary of any other cell, is an elementary collapse step and in particular a deformation retraction. There is a discrete version of Morse theory for regular CW-complexes based on this fact [11].

Lemma 5.5 *If $\sigma' \subset \sigma \in \Sigma$, then the complex $\Psi := \{\eta \in \text{Part}(\Sigma, [n]) \mid n \in \eta(u) \Rightarrow u \in \sigma\}$ collapses onto a complex isomorphic to $\Psi' := \{\eta \in \text{Part}(\Sigma \setminus (\sigma \setminus \sigma'), [n]) \mid n \in \eta(u) \Rightarrow u \in \sigma'\}$. In particular, Ψ and Ψ' are homotopy equivalent.*

Proof It suffices to prove this when $\sigma \setminus \sigma' = \{v\}$. We prove the lemma by finding pairs of cells whose removal are elementary collapse steps. The elementary collapse steps are deformation retractions, and the homotopy type is preserved.

Let $\eta_1, \eta_2, \dots, \eta_k$ be an ordering of $\{\eta \in \Psi \mid n \notin \eta(v)\}$ such that if $\eta_i > \eta_j$, then $i < j$. Since $v \in \sigma$ and $n \in \eta(u) \Rightarrow u \in \sigma$ for all $\eta \in \Psi$, one can extend η so that

$n \in \eta(v)$ without falling out of Ψ . For $1 \leq i \leq k$, define $\eta_i^* : V(\Sigma) \rightarrow 2^{[n]} \setminus \{\emptyset\}$ as $\eta_i^*(w) = \eta_i(w)$ for $w \neq v$ and $\eta_i^*(v) = \eta_i(v) \cup \{n\}$.

By construction there is no $\eta \in \Psi \setminus \{\eta_1^*, \eta_1, \eta_2^*, \eta_2, \dots, \eta_{i-1}^*, \eta_{i-1}\}$ for which $\eta > \eta_i^*$ and for which $\eta > \eta_i$ only when $\eta = \eta_i^*$. The successive pairwise removal of η_i^*, η_i in the order $i = 1, 2, \dots, k$ collapses Ψ onto $\Psi'' := \{\eta \in \Psi \mid \eta(v) = \{n\}\}$. There is an isomorphism $i : \Psi'' \rightarrow \Psi'$ by $\eta(u) = (i(\eta))(u)$ for $u \in V(\Sigma) \setminus \{v\}$. $\square$

The only use of Lemma 5.5 in this paper is incorporated in Lemma 5.6.

Lemma 5.6 *If $\sigma \in \Sigma$, then the complex $\Psi := \{\eta \in \text{Part}(\Sigma, [n]) \mid n \in \eta(u) \Rightarrow u \in \sigma\}$ is homotopy equivalent to $\text{Part}(\Sigma \setminus \sigma, [n-1])$.*

Proof Set $\sigma' = \emptyset$ and apply Lemma 5.5 to conclude that Ψ is homotopy equivalent to $\{\eta \in \text{Part}(\Sigma \setminus \sigma, [n]) \mid n \in \eta(u) \Rightarrow u \in \emptyset\} = \text{Part}(\Sigma \setminus \sigma, [n-1])$. $\square$

If Ψ_1 and Ψ_2 are two k -connected complexes and $\Psi_1 \cap \Psi_2$ is $(k-1)$ -connected, then $\Psi_1 \cup \Psi_2$ is k -connected. More generally, if $\Psi_1, \Psi_2, \dots, \Psi_m$ are k -connected and every t -fold intersection of them is $(k-t+1)$ -connected, then $\Psi_1 \cup \Psi_2 \cup \dots \cup \Psi_m$ is k -connected by the connectivity nerve lemma [3] or by a standard spectral sequence argument.

Now we are ready to prove Theorem 5.2, namely that $\text{Part}(\Sigma, [n])$ is $(n - \text{gr}(\Sigma) - 1)$ -connected.

Proof The idea for our proof is to cover $\text{Part}(\Sigma, [n])$ with subcomplexes of $\text{Part}(\cdot, \cdot)$ -type and then use the nerve lemma.

We do induction on $\text{gr}(\Sigma)$ and on $n - \text{gr}(\Sigma)$. If $\text{gr}(\Sigma) = 1$, then Σ is a simplex, so $\text{Part}(\Sigma, [n])$ is a product of simplices, hence contractible, and in particular $(n - \text{gr}(\Sigma) - 1)$ -connected. If $n - \text{gr}(\Sigma) = 0$, then $\text{Part}(\Sigma, [n])$ is nonempty and $(n - \text{gr}(\Sigma) - 1)$ -connected, since $n - \text{gr}(\Sigma) - 1 = -1$.

Now suppose that $\text{gr}(\Sigma) > 1$ and $n > \text{gr}(\Sigma)$, and let us define a cover of $\text{Part}(\Sigma, [n])$. Let $\mathcal{S}$ be the set of maximal faces of Σ , and let $\Psi_\sigma = \{\eta \in \text{Part}(\Sigma, [n]) \mid n \in \eta(u) \Rightarrow u \in \sigma\}$ for all $\sigma \in \mathcal{S}$. If $\eta \in \text{Part}(\Sigma, [n])$, then $\eta^{-1}(n)$ is a subset of an element of $\mathcal{S}$, and thus $\text{Part}(\Sigma, [n]) = \bigcup_{\sigma \in \mathcal{S}} \Psi_\sigma$.

To apply the nerve lemma, we need to calculate the connectivity of intersections of subcomplexes in the cover. The complex Ψ_σ is homotopy equivalent to $\text{Part}(\Sigma \setminus \sigma, [n-1])$ by Lemma 5.6 and $((n-1) - (\text{gr}(\Sigma) - 1) - 1)$ -connected by induction and Lemma 5.4. For any nonempty subset $\mathcal{S}'$ of $\mathcal{S}$, the intersection of Ψ_σ for $\sigma \in \mathcal{S}'$ is

$$\left\{ \eta \in \text{Part}(\Sigma, [n]) \mid n \in \eta(u) \Rightarrow u \in \bigcap_{\sigma \in \mathcal{S}'} \sigma \right\},$$

which by Lemma 5.6 is homotopy equivalent to

$$\text{Part}\left(\Sigma \setminus \bigcap_{\sigma \in \mathcal{S}'} \sigma, [n-1]\right),$$

and the latter is $((n-1) - \text{gr}(\Sigma) - 1)$ -connected by induction and Lemma 5.3. All the required connectivities are checked, and we apply the nerve lemma to conclude that $\text{Part}(\Sigma, [n])$ is $(n - \text{gr}(\Sigma) - 1)$ -connected. $\square$

In this section we discuss a particular example, which illustrates a connection between set partition complexes and Ramsey theory, and use this to suggest future directions for research.

In this paper we introduced set partition complexes, a generalization of $\text{Hom}(G, K_n)$ complexes. A natural question is if there is a common generalization of set partition complexes and $\text{Hom}(G, H)$ complexes. An application of such a construction would be to answer coloring questions other than for graph vertices, for example, from Ramsey theory. In [10], for any two simplicial complexes Σ_1 and Σ_2 , we will define a polyhedral complex whose vertices are indexed by the maps φ from $V(\Sigma_1)$ to $V(\Sigma_2)$ such that $\varphi^{-1}(\sigma) \in \Sigma_1$ for all $\sigma \in \Sigma_2$. The faces are the polyhedral cliques as for the Hom complexes and the set partition complexes. If Σ_2 is of dimension zero, we get a set partition complex, and if Σ_1 and Σ_2 are the independence complexes of G and H , we get $\text{Hom}(G, H)$.

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