On a Formula Relating Volumes of Medial Sections of Simplices

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Abstract Talata's medial sections of simplices can be classified into families according to how many vertices of the simplex lie on one side of the hyperplane determining the section. Defining a (k, l) -medial section to be one in which k vertices lie on one side and $l = d + 1 - k$ on the other, we show that for any d -simplex the ratio of the sum of the squared volumes of the (k, l) -medial sections to the sum of the squared volumes of the (κ, λ) -medial sections is independent of the geometry of the simplex, depending only on d , k , and κ . We give a formula for the ratio.

Keywords Simplex · Medial section · Volume

In a previous paper [2], the author recalled the classical theorem that the midpoints of any four edges of an Euclidean tetrahedron no three of which lie on a face are the vertices of a planar parallelogram, and derived a simple formula for the area of this *medial parallelogram* of a tetrahedron in terms of the lengths of the six edges. This result was subsequently generalized by Talata [1] to give a formula for the volumes of *medial sections* of d -simplices, that is, of polytopes obtained from d -simplex by intersecting it with a hyperplane equidistant from all the vertices.

The purpose of this note is to sketch the proof of the following theorem and to prove in detail its generalization to medial sections of simplices of arbitrary dimension.

Theorem 1 *The sum of the squared areas of the three medial parallelograms of any Euclidean tetrahedron is one quarter the sum of the squared areas of the four faces.*

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Fig. 1 A generic tetrahedron

Sketch of proof As in [2], we consider a tetrahedron with edge-lengths as in Fig. 1. Fixing a pair of non-incident edges, say those of lengths e and f , the midpoints of the remaining four edges lie in a plane parallel to both of the chosen edges and equidistant from the planes containing each chosen edge and parallel to both, and are the vertices of a parallelogram in this plane.

In [2], it was shown that the area of the medial parallelogram with vertices at the midpoints of the edges other than those of lengths e and f in the tetrahedron of Fig. 1 is

$$\frac{1}{8}\sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2}.$$

Applying this formula to all three medial parallelograms and giving the areas of the faces in terms of Heron's formula, a trivial calculation gives the desired result. $\square$

This result turns out to be an instance of Corollary 8 below.

The generalization to arbitrary dimensions requires slightly less elementary techniques. First recall the following:

Definition 2 (Talata) Given a d -simplex S_d , a *medial hyperplane* is an affine hyperplane H equidistant from all vertices of S_d .

A *medial section* of S_d is the intersection $S_d \cap H$ for some medial hyperplane H .

Observe that medial hyperplanes may be classified according to how many vertices lie on each side of the hyperplane.

Definition 3 A (k, l) -*medial hyperplane* is a medial hyperplane with k vertices lying on one side and $l = d + 1 - k$ vertices lying on the other.

A (k, l) -*medial section* of a d -simplex S_d is its intersection with a (k, l) -medial hyperplane.

In terms of Talata's general notion of medial sections, it is easy to see in the three-dimensional case that a tetrahedron admits three $(2, 2)$ -medial sections and four $(1, 3)$ -medial sections and that Theorem 1 is equivalent to the following:

Corollary 4 *The sum of the squared areas of the $(2, 2)$ -medial sections of a tetrahedron is equal to four times the sum of the squared areas of the $(1, 3)$ -medial sections.*

Proof It suffices to observe that a $(1, 3)$ -medial section is parallel to a face and is similar to that face with linear dimensions halved. $\square$

In general, we have the following:

Proposition 5 *Any choice of k vertices of a d -simplex S_d determines a (k, l) -medial hyperplane separating the chosen k vertices from the remaining $l = d + 1 - k$ vertices.*

Proof Denote the chosen k vertices by $v_1, \dots, v_k$ and the remaining vertices by $w_1, \dots, w_l$.

We begin by constructing a hyperplane through the origin which will be parallel to the desired hyperplane. It follows from the affine independence of the vertices that the translates $v_2 - v_1, \dots, v_k - v_1, w_1 - v_1, \dots, w_l - v_1$ form a basis for d -space, from which it is easily seen that

$$\{v_2 - v_1, \dots, v_k - v_1, w_2 - w_1, \dots, w_l - w_1\}$$

spans a hyperplane of codimension one.

The translate of this plane through v_1 (resp. w_1) is easily seen to contain $v_1, \dots, v_k$ (resp. $w_1, \dots, w_l$). The parallel hyperplane equidistant from the two translates is equidistant from any two points lying in the translates and is thus a medial hyperplane. $\square$

Talata's generalization [1] of the author's formula is given by the following:

Theorem 6 (Talata) *If H is the (k, l) -medial hyperplane separating the first k -vertices $\{v_1, \dots, v_k\}$ of S_d from the remaining l -vertices $\{v_{k+1}, \dots, v_{d+1}\}$, the square of its volume is given by*

$$\frac{1}{(-8)^{d-1}[(k-1)!]^2[(l-1)!]^2} \det M^{(k)},$$

where $M^{(k)}$ is the $(d+3) \times (d+3)$ -matrix with block structure

$$\begin{bmatrix} O & I^{(k)} \\ I^{(k)T} & A \end{bmatrix},$$

where O is a 2×2 -zero matrix, $I^{(k)}$ is the $2 \times (d+1)$ matrix whose first row is k 1's followed by l 0's and whose second row is k 0's followed by l 1's, and A is the (necessarily symmetric) $(d+1) \times (d+1)$ -matrix whose (i, j) -entry is the square of the distance from v_i to v_j .

In these terms, we state the general result:

Theorem 7 For $k + l = \kappa + \lambda = d + 1$, for any d -simplex S_d , the ratio of the sum of the squares of the volumes of the (k, l) -medial sections of S_d to the sum of the squares of the volumes of the (κ, λ) -medial sections of S_d is given by

$$\frac{2^{\delta_{\kappa\lambda}}[(\kappa - 1)!]^3[(\lambda - 1)!]^3}{2^{\delta_{kl}}[(k - 1)!]^3[(l - 1)!]^3},$$

where δ is the Kronecker delta.

From the observation that the $(1, d)$ -medial sections are parallel to $(d - 1)$ -dimensional faces and similar to their parallel face with all linear dimensions halved we have the following:

Corollary 8 The sum of the squares of the volumes of $(d - 1)$ -dimensional faces of any d -simplex S_d is

$$\frac{4^{d-1} 2^{\delta_{k,l}}}{\binom{d-1}{k-1}^3}$$

times the sum of the squares of the volumes of the (k, l) -medial sections of S_d .

Proof of Theorem 7 Now, observe that the (k, l) -medial sections of S_d are in one-to-one correspondence with the k -element subsets of the vertices, unless $k = l$, in which case, they are in one-to-two correspondence with each section corresponding to the unordered pair of complementary k -element subsets.

We may regard this as establishing a one-to-one (resp. one-to-two) correspondence between the medial sections and (k, l) -unshuffles in the case where $k \neq l$ (resp. $k = l$).

Recall that a (k, l) unshuffle is a permutation σ which maps the set $\{1, \dots, k + l\}$ to itself in such a way that $\sigma^{-1}(1) < \dots < \sigma^{-1}(k)$ and $\sigma^{-1}(k + 1) < \dots < \sigma^{-1}(k + l)$. Plainly these are in one-to-one correspondence with k -element subsets: the permutation lists the elements of the subset in increasing order, followed by the elements of the complementary l -element subset in increasing order.

From this and from the main result of Talata [1] it follows that the sum of the squared volumes of the (k, l) -medial sections is given by

$$\frac{2^{-\delta_{kl}}}{(-8)^{d-1}[(k-1)!]^2[(l-1)!]^2} \sum_{\sigma \in \text{Unshuf}(k,l)} \det \begin{bmatrix} O & I^{(k)} \\ I^{(k)T} & \sigma^T A \sigma \end{bmatrix},$$

where $\text{Unshuf}(k, l)$ denotes the set of (k, l) -unshuffles, and by abuse of notation, σ denotes the permutation matrix as well as the unshuffle itself, and δ is the Kronecker delta.

The method of the proof is now quite simple: we expand each determinant in minors to obtain a sum of minors of the various $\sigma^T A \sigma$'s obtained by deleting one of the first k rows, one of the first k columns, one of the last l rows, and one of the last l

columns, then shuffle the rows and columns to obtain a linear combination of minors obtained from A by omitting two rows and two columns. The key is keeping track of the signs and the number of summands.

In general, if M is a matrix, let $M_{u,v}^{s,t}$ denote the minor obtained by deleting the s th and t th columns and the u th and v th rows.

Successively expanding the determinants in the sum above by minors along the first row, the second row, the first column, and the second column gives

$$\begin{aligned} & \sum_{\sigma \in \text{Unshuf}(k,l)} \det \begin{bmatrix} O & I^{(k)} \\ I^{(k)T} & \sigma^T A \sigma \end{bmatrix} \\ &= \sum_{\sigma \in \text{Unshuf}(k,l)} \sum_{s,u=1}^k \sum_{t,v=1}^l (-1)^{s+t-1+u+v-1} \det[\sigma^T A \sigma]_{u,v}^{s,t}. \end{aligned}$$

We now need to make several observations about unshuffles: a (k, l) -unshuffle σ can be encoded by $\langle \{a_1 < \dots < a_k\}, \{b_1 > \dots > b_l\} \rangle$, where $\{a_1, \dots, a_k\} \cup \{b_1, \dots, b_l\} = \{1, \dots, k+l\}$.

Observe now that if $\sigma = \langle \{a_1 < \dots < a_k\}, \{b_1 > \dots > b_l\} \rangle$, then

$$\text{sgn}(\sigma) = (-1)^{\sum_{i=1}^k a_i - i} = (-1)^{\sum_{j=1}^l b_j - d - 2 + j},$$

since the exponents are the number of transpositions used when implementing the unshuffle by moving each a_i in turn to its final position and respectively moving each b_j in turn to its final position.

We want to use this observation to determine the sign of the permutations induced by the shuffle σ^{-1} on the rows and columns of $\sigma^T A \sigma$ when rewriting the minor $\det[\sigma^T A \sigma]_{u,v}^{s,t}$ as $\pm \det A_{\sigma^{-1}(u), \sigma^{-1}(v)}^{\sigma^{-1}(s), \sigma^{-1}(t)}$.

Now since the sign of a permutation and its inverse are equal, it suffices to determine the sign of the permutation induced on $\{1, \dots, l+k\} \setminus \{a_s, b_{d+2-t}\}$, since a_s and b_{d+2-t} are the elements which the unshuffle σ maps to position s and t , respectively.

Now observe that if a_s is omitted, the number of transpositions is decreased by $a_s - s$, since moving earlier a_i 's into place takes the same number of transpositions, while moving later a_i 's into place takes the same number as when a_s is not removed. Dually omitting b_{d+2-t} reduces the number of transpositions by $b_{d+2-t} - d - 2 + t$. Of course, these observations apply to the particular implementation of the unshuffle used to give the formulas for $\text{sgn}(\sigma)$ given above, but mod 2, they apply to all presentations of σ .

If $a_s < b_{d+2-t}$, then the number of permutations is reduced by $a_s - s + b_{d+2-t} - d - 2 + t$, since the removal of either element does not change the number of transpositions needed to move the other into place. However, if $a_s > b_{d+2-t}$, then we have over counted the reduction of the number of transpositions by 1, the transposition moving a_s past b_{d+2-t} , so the number of transpositions is reduced by $a_s - s + b_{d+2-t} - d - 2 + t - 1 = a_s - s + b_{d+2-t} - d - 3 + t$ instead.

Thus the sign of the permutation induced by σ on $\{1, \dots, l+k\} \setminus \{a_s, b_{d+2-t}\}$ is given by

$$(-1)^{[\sum_{i=1}^k a_i - i] - a_s + s - b_{d+2-t} - d - 2 - t - [a_s > b_{d+2-t}]},$$

where

$$[x > y] := \begin{cases} 1, & \text{if } x > y, \\ 0, & \text{otherwise.} \end{cases}$$

It thus follows that

$$\begin{aligned} & \det[\sigma^T A \sigma]_{u,v}^{s,t} \\ &= (-1)^{a_s + s + b_{d+2-t} - t - [a_s > b_{d+2-t}] + a_u + u + b_{d+2-v} - v - [a_u > b_{d+2-v}]} \det A_{\sigma^{-1}(u), \sigma^{-1}(v)}^{\sigma^{-1}(s), \sigma^{-1}(t)}, \end{aligned}$$

since all other summands in the exponent are duplicated, and we may change the signs of terms in the exponent without changing the value of the expression. Here, of course, $\sigma(a_s) = s$, $\sigma(b_{d+2-t}) = t$, $\sigma(a_u) = u$, and $\sigma(b_{d+2-v}) = v$.

Applying this to the expansion by minors given above yields

$$\begin{aligned} & \sum_{\sigma \in \text{Unshuf}(k,l)} \det \begin{bmatrix} O & I^{(k)} \\ I^{(k)T} & \sigma^T A \sigma \end{bmatrix} \\ &= \sum_{\sigma \in \text{Unshuf}(k,l)} \sum_{s,u=1}^k \sum_{t,v=1}^l (-1)^{e(\sigma,s,t,u,v)} \det A_{\sigma^{-1}(u), \sigma^{-1}(v)}^{\sigma^{-1}(s), \sigma^{-1}(t)}, \end{aligned}$$

where $e(\sigma, s, t, u, v) = \sigma^{-1}(s) + \sigma^{-1}(t) - [\sigma^{-1}(s) > \sigma^{-1}(t)] + \sigma^{-1}(u) + \sigma^{-1}(v) - [\sigma^{-1}(u) > \sigma^{-1}(v)]$.

Reindexing by the position of the rows and columns omitted from A , rather than their positions after the unshuffle is applied, we have

$$\begin{aligned} & \sum_{\sigma \in \text{Unshuf}(k,l)} \det \begin{bmatrix} O & I^{(k)} \\ I^{(k)T} & \sigma^T A \sigma \end{bmatrix} \\ &= \sum_{\sigma \in \text{Unshuf}(k,l)} \sum_{s,u \in \{a_1, \dots, a_k\}} \sum_{t,v \in \{b_1, \dots, b_l\}} (-1)^{s+t-[s>t]+u+v-[u>v]} \det A_{uv}^{st}. \end{aligned}$$

We now wish to simplify this sum by collecting all terms in which the same minor $\det A_{uv}^{st}$ of A occurs.

There are three cases:

Case 1 s, t, u , and v are pairwise different. Then $s \neq u$ and $t \neq v$.

Case 2 There are three different values among s, t, u , and v . We may assume w.l.o.g. that $s = u$ and $t \neq v$.

Case 3 There are two different values among s, t, u , and v . Thus $s = u$ and $t = v$.

Case 1. Here, given an unshuffle represented by $\langle \{a_1, \dots, a_k\}, \{b_1, \dots, b_l\} \rangle$, with $s, u \in \{a_1, \dots, a_k\}$ and $t, v \in \{b_1, \dots, b_l\}$, there is another unshuffle represented by $\langle \{a_1, \dots, a_k\} \cup \{t\} \setminus \{s\}, \{b_1, \dots, b_l\} \cup \{s\} \setminus \{t\} \rangle$ with t and u in the first block and s and v in the second block.

The first suffle contributes a summand of $(-1)^{s+t-[s>t]+u+v-[u>v]} \det A_{uv}^{st}$, while the second contributes a summand of $(-1)^{t+s-[t>s]+u+v-[u>v]} \det A_{uv}^{ts}$. The two minors are the same, while the signs are opposite, and thus the summands cancel each other in pairs in this case.

Case 2. In this case, all summands will have the same sign, since the sign depends only on $s = u, t$, and v .

Case 3. As in Case 2, all summands have the same sign, that is, positive now.

Now, consider the number of summands contributing the same minor in each of Cases 2 and 3:

In Case 2, we have two types of summands contributing the minor $\det A_{sv}^{st}$, those in which $s \in \{a_1, \dots, a_k\} \not\ni t, v$ and those in which $t, v \in \{a_1, \dots, a_k\} \not\ni s$. To specify either type of such a summand, it suffices to choose the first block by specifying the other a_i 's. Thus there are

$$\binom{d-2}{k-1} + \binom{d-2}{k-2} = \binom{d-1}{k-1}$$

summands of $\det A_{sv}^{st}$.

In Case 3, we need to count summands in which $s = u \in \{a_1, \dots, a_k\} \not\ni t = v$ or $s = u \notin \{a_1, \dots, a_k\} \ni t = v$. There are plainly $\binom{d-1}{k-1}$ of each type and thus a total of

$$2 \binom{d-1}{k-1}.$$

From this it follows that the sum of the squared volumes of the (k, l) -medial sections of a d -simplex is given by

$$\frac{2^{-\delta_{kl}}}{(-8)^{d-1}[(k-1)!]^2[(l-1)!]^2} \binom{d-1}{k-1} \times \left\{ \left[\sum_{s \neq t \neq v; s \neq v; s, t, v=1}^{d+1} c(s, t, v) \det A_{sv}^{st} \right] + 2 \left[\sum_{s \neq t; s, t=1}^{d+1} \det A_{st}^{st} \right] \right\},$$

where $c(s, t, v) = (-1)^{t+v-[s>t]-[s>v]}$, and from this Theorem 7 immediately follows. $\square$

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