Line Transversals in Large $T(3)$ - and $T(4)$ -Families of Congruent Discs

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Abstract A family of closed discs is said to have property $T(k)$ if to every subset of at most k discs there belongs a common line transversal. A family of discs is said to be d -disjoint, $d \geq 1$, if the mutual distance between the centers of the discs is larger than d . It is known that a d -disjoint $T(3)$ -family $\mathcal{F}$ of unit diameter discs has a line transversal if $d = \sqrt{2}$. Similarly, a d -disjoint $T(4)$ -family has a line transversal if $d = 2/\sqrt{3}$. Both results are sharp in d , i.e., they do not hold for smaller values of d . The main result of this paper is that while the above lower bounds on d cannot be relaxed in general, some reduction of d can be compensated by imposing a proper d -dependent lower bound on the size of the family in both cases.

Keywords Disc · d -disjoint · Line transversal

1 Introduction

Helly type problems for line transversals of families of convex domains have been studied by a number of authors. The interested reader is referred to the survey papers [4, 6, 9] or [8]. In this paper finite families of congruent discs (solid circles) are considered in the Euclidean plane.

Definitions Throughout this paper the term *disc* is used for a closed solid circle of diameter 1 if not specified otherwise. Two discs are said to be d -disjoint, $d \geq 1$, if their centers are more than d apart. A *family* is d -disjoint if every pair of its members is d -disjoint. A *line transversal* or simply *transversal* to a family of discs is a straight line having a nonempty intersection with every member of the family. If a family has

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a transversal we also say that the family has property T . A family $\mathcal{F}$ of discs is a $T(k)$ -family if every subset of at most k members of the family has property T . The *diameter* $\text{diam}\{\mathcal{F}\}$ of family $\mathcal{F}$ is the diameter of the union of the discs of the family. The *width* of a finite point set is the width of the narrowest strip covering the set.

While by a theorem of Danzer [3] $T(5) \Rightarrow T$ if the discs are disjoint (1-disjoint), simple examples show that a finite $T(k)$ -family of disjoint discs does not necessarily have a transversal if $k < 5$. However, we have the following theorems:

Theorem 1 For $d = \sqrt{2}$ every d -disjoint $T(3)$ -family of unit diameter discs has a transversal.

Theorem 2 For $d = 2/\sqrt{3}$ every d -disjoint $T(4)$ -family of unit diameter discs has a transversal.

(Theorem 3 was first proved by Grünbaum [7] as an application of the topological Helly theorem. An elementary proof can be found for both theorems in [2].)

Case $k = 3$ The family consisting of four discs set about the vertices of a square of side length $\sqrt{2}$ (a family, d -disjoint for any $d < \sqrt{2}$), shows that Theorem 1 is sharp in d (Fig. 1a). It is natural to ask whether in families consisting of *more than four* discs even a smaller distance requirement is enough to guarantee property T . This question is answered in the affirmative: a reduction in d can be compensated by a lower bound on the cardinality $|\mathcal{F}|$ (or on the diameter $\text{diam}\{\mathcal{F}\}$) of the family—to some extent.

It is well known that the set of centers of a $T(3)$ -family of unit diameter discs can be covered by a strip of width 2 (see, e.g., [5]). This universal upper bound on the width implies that for any d a d -disjoint family of *many* discs has *large* diameter. Thus, for a given d , a lower bound on the diameter can be replaced by a proper lower bound on the number of the discs. The theorems below will be formulated for the weaker assumption of the two. We prove

Fig. 1 (a) $k = 3$, $d < \sqrt{2}$. (b) $k = 4$, $d < 2/\sqrt{3} - \varepsilon$

Theorem 3 *For any $d > 2/\sqrt{3}$ there exists $D_3(d)$ such that any d -disjoint $T(3)$ -family $\mathcal{F}$ of unit diameter discs has a transversal if $\text{diam}\{\mathcal{F}\} > D_3(d)$.*

It will also be shown that the lower bound $d > 2/\sqrt{3}$ is sharp.

Theorem 4 *For $d < 2/\sqrt{3}$ and any integer $N > 4$ a d -disjoint $T(3)$ -family consisting of N unit diameter discs can be given such that the family has no transversal.*

Case $k = 4$ Like in the case $k = 3$, a construction consisting of five discs [2] shows that Theorem 2 is sharp in d (see Fig. 1b). (For $k = 4$, however, we do not have a single configuration for every $d < 2/\sqrt{3}$.) The question, whether in families of *more than five* discs even a smaller distance requirement is enough to guarantee property T , is answered again in the affirmative: a reduction in d can be compensated by an increase in the diameter (or cardinality) of the family—to some extent. We prove

Theorem 5 *For any $d > 1$ there exists $D_4(d)$ such that any d -disjoint $T(4)$ -family $\mathcal{F}$ of unit diameter discs has a transversal if $\text{diam}\{\mathcal{F}\} > D_4(d)$.*

The fact that a further decrease in d cannot be compensated by a lower bound on the size of the family follows from a result of Aronov et al. [1]:

Theorem 6 *For $d = 1$ and any integer $N > 5$ a d -disjoint $T(4)$ -family consisting of at least N unit diameter discs can be given such that the family has no transversal.*

The above results can be summarized in two diagrams (see Fig. 2).

2 Proofs

In the following proofs the centers of the families will be considered as points of a Cartesian coordinate system. $D(X)$ denotes the unit diameter disc centered at point X .

2.1 Proof of Theorem 3

In contrary to the claim of Theorem 3, we assume that the family has no transversal, i.e., the width of the set of centers is larger than 1. We are going to show that this assumption is in contradiction with the $T(3)$ property of the family.

Let $d > 2/\sqrt{3}$, $\mathcal{F}$ a $T(3)$ -family and S a strip of minimal width covering all centers in $\mathcal{F}$. S is supposed to be horizontal bounded by the straight lines $y = 0$ and $y = y_a$. $A(x_a, y_a)$, $B(0, 0)$ and $C(x_c, 0)$ are centers of three discs of the family stretching S . As the family is d -disjoint $x_c > d$. The angles of the triangle at vertices A , B and C will be denoted by α , β and γ , respectively. Since S is a narrowest covering strip, it can be supposed that neither of the angles β and γ is obtuse and $x_a > 0$. It is assumed that $\beta \geq \gamma$.

The heights of the triangle belonging to the base opposite to vertex A and B are h_a and h_b , respectively (see Fig. 3). By our assumption, $h_a > 1$; on the other hand, by

Fig. 2 (a) $T(3) \Rightarrow T$. (b) $T(4) \Rightarrow T$

property $T(3)$, the smallest height of the triangle must not exceed 1, $h_b \leq 1$. Then, side AC must intersect the unit circle about B . Consequently, A must be an inner point of the domain bounded from the left by the y -axis, from above by line L connecting points $C'(d, 0)$ and $A'(0, h)$, where $h = d/\sqrt{d^2 - 1}$, and from below by the circle of radius d about B (see Fig. 4). The lowest boundary point of this domain is denoted by $M(f, g)$. Since $g = \lambda h$, where

$$\lambda = \frac{|MC'|}{|A'C'|} = \frac{2\sqrt{d^2 - 1}}{\sqrt{d^2 + h^2}} = 2 - 2/d^2,$$

for any $d > 2/\sqrt{3}$

$$g = 2\sqrt{1 - 1/d^2} > 1. \quad (2.1)$$

Clearly, if a line is intersecting both discs $D(A)$ and $D(B)$, then this line must be at least as steep as the line tangent to $D(B)$ from above and to $D(M)$ from below which, by (2.1), cannot be horizontal. Obviously, if the diameter of the family is sufficiently large, then the disc most distant from $D(B)$ cannot have a common transversal with $D(A)$ and $D(B)$.

Fig. 3 The narrowest strip covering all centers

Fig. 4 Transversal of smallest possible slope to $D(A)$ and $D(B)$

Fig. 5 Large $T(3)$ family without transversal

2.2 Proof of Theorem 4

Let $N \geq 5$ be given. We are to show that there exists a $T(3)$ -family consisting of at least N discs such that the family has no transversal.

Consider centers $A(v, 1)$, $B(0, 0)$ and $C(2v, 0)$, where $v = 1/\sqrt{3}$ and denote by s_A , s_B and s_C the straight lines parallel to sides BC , CA and AB and at equal distance from the vertices of triangle ABC , respectively. Let the family $\mathcal{F}$ consist of the discs about points A , B and C , $N - 4$ discs centered at points $P_k((2k - 6)v, 0)$, $k = 5, \dots, N$, and the disc centered at $Q(-v, 1 + \varepsilon)$, where $\varepsilon > 0$ is specified as follows. Let s' be the straight line tangent to $D(B)$ from above and $D(P_N)$ from below and the value of ε is chosen so that $D(Q)$ is tangent to s' from above. Finally, s'' is the straight line tangent to $D(Q)$ and $D(A)$ from below (see Fig. 5).

Clearly, $\mathcal{F}$ is d -disjoint for any $d < 2/\sqrt{3}$. It is easy to check that $\mathcal{F}$ is a $T(3)$ -family as well: s_A is intersecting all discs but $D(Q)$, s' is intersecting all discs but $D(A)$ and s'' is intersecting all discs but $D(B)$; finally, s_B is common transversal to $D(A)$, $D(B)$ and $D(Q)$.

Lines s_A , s_B and s_C are the only lines meeting discs $D(A)$, $D(B)$ and $D(C)$ simultaneously, therefore, if a transversal to $\mathcal{F}$ exists it is necessarily one of these three lines. However, s_A does not intersect $D(Q)$, s_B does not meet $D(P_5)$ and s_C is missing both. This proves our claim.

2.3 Proof of Theorem 5

Contrary to the claim of Theorem 5, we assume that the family has no transversal, i.e., the width of the set of centers is larger than 1. We are going to show that this assumption is in contradiction with the $T(4)$ property of the family.

Let $d > 1$. In a d -disjoint family of discs let $\delta(d) = \arcsin(1/d)$ denote the least upper bound for the angle by which a common transversal of two discs can deviate from the line connecting the two centers (Fig. 6).

Let $\mathcal{F}$ be a $T(4)$ -family and S be a narrowest strip covering the centers of $\mathcal{F}$. Without loss of the generality we can suppose that S is horizontal and $A(x_a, y_a)$, $B(x_b, 0)$ and $C(x_c, 0)$ are centers of three discs of the family stretching S . The angles of triangle ABC at vertices A , B and C will be denoted by α , β and γ , respectively. Since S is a narrowest covering strip, it can be supposed that neither of the angles β and γ is obtuse. According to our assumption property T does not hold, thus the width y_a of the covering strip S is larger than 1 and disc $D(A)$ is separated from the other two discs $D(B)$ and $D(C)$ by a horizontal line (Fig. 7).

Let t be an assumed transversal of discs $D(A)$, $D(B)$ and $D(C)$ and let τ denote the angle of the horizontal line and t —in this order. (The existence of a transversal line to the three discs follows even from the weaker property $T(3)$.) Our goal is to show that—if the diameter of the family is larger than a number depending *only* on d —then there exists a further disc $D(Z)$ in the family which cannot be intersected by t . This would yield the intended contradiction.

As the family is d -disjoint we have $|AB| > d$ implying $\tau \in (\beta - \delta(d), \beta + \delta(d))$, where $\delta(d) = \arcsin(1/d) < \pi/2$. This interval must *not* contain the horizontal di-

Fig. 6 Two discs and their common transversals

Fig. 7 The narrowest strip covering all centers

rection as the discs are separated by the horizontal line $y = y_a/2$. On the other hand $\beta \leq \pi/2$, hence

$$\tau \in (0, \pi/2 + \delta(d)). \quad (2.2)$$

Similarly, $|AC| > d$ implies $\tau \in (\pi - \gamma - \delta(d), \pi - \gamma + \delta(d))$. Again, since the horizontal direction is excluded and $\gamma \leq \pi/2$ we have

$$\tau \in (\pi/2 - \delta(d), \pi). \quad (2.3)$$

Consequently, from (2.2) and (2.3) we have

$$\tau \in (\pi/2 - \delta(d), \pi/2 + \delta(d)). \quad (2.4)$$

If the diameter of the family is sufficiently large—which can be forced out by a large lower bound on the number of the discs as well—there exists a disc $D(Z)$ in the family such that its distance from $D(A)$ is larger than any prescribed value. This value can, however, be set so that the direction of any line intersecting $D(A)$ and $D(Z)$ simultaneously is necessarily so close to the horizontal direction that it cannot fulfill the d -dependent inequality (2.4).

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