Weighted Derivations and the cd-Index

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Abstract Weighted derivations W_1 and W_2 allowed R. Ehrenborg and M. Readdy (Discrete Comput. Geom. 21:389–403, 1999) to give a recursive description of the **cd**-indices of the lattices of the regions of the arrangements $\mathcal{A}_n$ and $\mathcal{B}_n$. In part motivated by this, we describe a new basis for the subspace spanned by **ab**-indices of all simplicial graded posets and determine the action of certain linear maps (associated with weighted derivations W_k) on this basis. Extending the “pyramid” and “prism” operations, we define operations Σ_k on graded posets and show that relations between the **ab**-indices (or **cd**-indices, for an Eulerian poset P) of barycentric subdivisions of $\Sigma_k(P)$ and P can be described by using the linear maps associated by weighted derivations W_k . Finally, in response to a question from Ehrenborg and Readdy (Discrete Comput. Geom. 21:389–403, 1999), we obtain the formulae that express the **cd**-index of the lattice $R_{\mathcal{D}_n}$ of regions of the arrangement $\mathcal{D}_n$ in terms of the **cd**-indices of $R_{\mathcal{B}_n}$ and $R_{\mathcal{A}_{n-2}}$.

Keywords **ab**-index · **cd**-index · Eulerian posets · Simplicial poset · h -vector · Barycentric subdivisions

1 Introduction

The f -vector of an n -polytope P is an ordered n -tuple $f(P) = (f_0, \dots, f_{n-1})$, where $f_k = f_k(P)$ denotes the number of k -dimensional faces of P . As usual, we let $f_{-1} = f_n = 1$. It is well known that the f -vectors of all n -dimensional polytopes satisfy only one nontrivial linear equation, namely, the Euler–Poincaré formula

$$f_0 - f_1 + \dots + (-1)^{n-1} f_{n-1} = 1 - (-1)^n.$$

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The h -vector of an n -dimensional simplicial polytope P is an n -tuple $h(P) = (h_0, h_1, \dots, h_n)$, where

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}. \quad (1)$$

This transformation was first noted by Sommerville in [12]. The introduction of the h -vector was of great significance in the proof of the upper-bound theorem and in the characterization of f -vectors of simplicial polytopes; see [6, 11, 15, 17]. It is known that the Dehn–Sommerville equations $h_i = h_{n-i}$ are the only linear equations for f -vectors of simplicial polytopes; see [1].

For any graded poset P of rank $n+1$ the associated f -vector is an n -tuple $f(P) = (f_0, \dots, f_{n-1})$, where f_i counts the number of the elements of P whose rank is $i+1$. For any polytope and, more generally, for any graded poset P of rank $n+1$ one defines its h -vector by the formulae (1). If we encode f and h -vectors as polynomials

$$f(P, x) = \sum_{i=0}^n f_{i-1} x^{n-i} \quad \text{and} \quad h(P, x) = \sum_{i=0}^n h_i x^{n-i},$$

then from (1) it follows that $h(P, x) = f(P, x-1)$.

Let P be a finite, graded poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$ as the minimum and maximum elements, and let r denote the rank function of P . For $S \subseteq [n]$, we denote by $f_S(P)$ the number of chains $x_1 < x_2 < \dots < x_k$ in P , with $S = \{r(x_1), \dots, r(x_k)\}$. The vector $(f_S(P))_{S \subseteq [n]}$ is called the *flag f -vector* of P . For $S \subseteq [n] = \{1, 2, \dots, n\}$ let

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P). \quad (2)$$

The vector $(h_S(P))_{S \subseteq [n]}$ is called the *flag h -vector* of P .

The flag f and h -vectors of a poset P are encoded by the **ab-index** $\Psi(P)$, a homogenous polynomial in the noncommutative variables **a** and **b**:

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \Psi(P) = \sum_S h_S u_S, \quad \text{where}$$

$$u_S = u_1 u_2 \dots u_n, \quad \text{and} \quad u_i = \begin{cases} \mathbf{a}, & \text{if } i \notin S, \\ \mathbf{b}, & \text{if } i \in S. \end{cases}$$

The **ab-index** of P can also be defined in the following way. To every chain $c = \{\hat{0} < x_1 < x_2 < \dots < x_k < \hat{1}\}$ of P we associate a *weight* $wt(c) = w_1 w_2 \dots w_n$, where

$$w_i = \begin{cases} \mathbf{b}, & \text{if } i \in \{r(x_1), r(x_2), \dots, r(x_k)\}; \\ \mathbf{a} - \mathbf{b}, & \text{otherwise.} \end{cases}$$

Now, from the relations (2), it follows that

$$\Psi(P) = \sum_{\text{chains } c \text{ in } P} wt(c). \quad (3)$$

For a graded poset P of rank $n + 1$ we define its barycentric subdivision $Sd(P)$ as the face poset of the reduced order complex $\Delta(P \setminus \{\hat{0}_P, \hat{1}_P\})$ with a maximal element $\hat{1}$ adjoined; see [7]. If P is a face poset of an n -sphere Ω , then $Sd(P)$ is the face poset of the barycentric subdivision of $\partial\Omega$ (with added $\hat{1}$). The flag f and h -vectors of a graded poset P have the following interpretation as a refinement of the f and h -vector of the barycentric subdivision of P (see [1]):

$$f_i(SdP) = \sum_{|S|=i+1} f_S(P), \quad h_i(SdP) = \sum_{|S|=i} h_S(P).$$

By using the **ab**-index of a graded poset P , the previous relation can be written as

$$h(Sd(P), x) = \Psi_P(1, x). \quad (4)$$

A finite, graded poset is *Eulerian* if every interval of rank at least one contains as many elements of even rank as of odd rank, i.e., the Euler–Poincaré formula holds for every interval in P (see [13]). The face lattice of a polytope and, more generally, of all regular CW -decompositions of spheres (see [16]) is always Eulerian. If Γ is a finite regular cell decomposition of an n -disk, then the poset of its cells $P(\Gamma)$ is a semi-Eulerian poset. By adding a new n -cell σ such that $\partial\sigma = \partial\Gamma$, we obtain a regular CW -decomposition of an n -sphere Γ' called the *semisuspension* of Γ . Note that the face poset $P(\Gamma')$ is an Eulerian poset.

The affine span of the flag f -vectors of all polytopes (and more generally, of all Eulerian posets) is described by the generalized Dehn–Sommerville equations; see [2]. J. Fine conjectured and M. Bayer and A. Klapper showed [3] that the **ab**-index of an Eulerian poset P may be written uniquely as a polynomial $\Phi(P)$ in the noncommutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the **cd**-index of P , and this is the most efficient way to encode the flag f -vector of an Eulerian poset.

One of the advantages of the **cd**-index compared to other bases for flag f -vectors of Eulerian posets is its nice behavior under operations over posets (polytopes). If P is the face lattice of a polytope V , then $P \times B_1$ and $P \diamond B_2$ are the face lattices of the pyramid and prism of V , where $\times$ denotes the usual Cartesian product and $\diamond$ denotes the diamond product, given by $P \diamond Q = (P \setminus \{\hat{0}_P\}) \times (Q \setminus \{\hat{0}_Q\}) \cup \{\hat{0}\}$, and B_n denotes the Boolean algebra on the n elements $[n] = \{1, 2, \dots, n\}$. Therefore, for a graded poset P one defines the pyramid and prism of P as

$$\text{Pyr}(P) = P \times B_1, \quad \text{Prism}(P) = P \diamond B_2.$$

Special derivations G and D of the polynomial algebra $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ were introduced by R. Ehrenborg and M. Readdy in [8]. On generators these derivations act as follows:

$$G(\mathbf{a}) = \mathbf{ba}, \quad G(\mathbf{b}) = \mathbf{ab}; \quad D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab} + \mathbf{ba}.$$

Ehrenborg and Readdy used G and D to define the linear maps $\text{Pyr}, \text{Prism}: \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle \rightarrow \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ by

$$\text{Pyr}(u) = u \cdot (\mathbf{a} + \mathbf{b}) + G(u); \quad \text{Prism}(u) = u \cdot (\mathbf{a} + \mathbf{b}) + D(u).$$

Theorem 1 (R. Ehrenborg, M. Readdy [8]) *For a graded poset P the following holds:*

$$\begin{aligned}\Psi(\text{Pyr}(P)) &= \Psi(P) \cdot (\mathbf{a} + \mathbf{b}) + G(\Psi(P)) = \text{Pyr}(\Psi(P)), \\ \Psi(\text{Prism}(P)) &= \Psi(P) \cdot (\mathbf{a} + \mathbf{b}) + D(\Psi(P)) = \text{Prism}(\Psi(P)).\end{aligned}$$

Moreover, the subalgebra $\mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$ is closed under the action of the derivations G and D and for an Eulerian poset P the following holds:

$$\Phi(\text{Pyr}(P)) = \Phi(P) \cdot \mathbf{c} + G(\Phi(P)), \quad \Phi(\text{Prism}(P)) = \Phi(P) \cdot \mathbf{c} + D(\Phi(P)).$$

2 Bases for Simplicial Posets

For $i = 0, 1, \dots, n$ let Λ_i^n be an $(n-1)$ -dimensional simplicial disk (or sphere, for $i = n$) obtained as the union of any $i+1$ facets of an n -simplex Δ_n . In [14], R. Stanley defines **cd**-polynomials

$$\Psi_0^n = \Phi(\Delta_{n-1}) \cdot \mathbf{c}, \quad \text{and} \quad \Psi_i^n = \Phi((\Lambda_i^n)') - \Phi((\Lambda_{i-1}^n)') \quad \text{for } 1 \leq i \leq n-1. \quad (5)$$

Here $(\Lambda_i^n)'$ denotes the semisuspension of the disk Λ_i^n . A poset P is *simplicial* if for all $x < \hat{1}$ the interval $[\hat{0}, x]$ is isomorphic to a Boolean algebra.

Theorem 2 (R. Stanley, [14]) *Let P be a simplicial Eulerian poset of rank $n+1$ whose h -vector is $h(P) = (h_0, h_1, \dots, h_n)$. Then*

$$\Phi(P) = \sum_{i=0}^{n-1} h_i \Psi_i^n. \quad (6)$$

The Dehn–Sommerville equations $h_i(P) = h_{n-i}(P)$ hold for any simplicial Eulerian poset P of rank $n+1$. In light of Theorem 2 and the symmetry of the h -vector the **cd**-polynomials $\Psi_i^n + \Psi_{n-i}^n$ (for $i = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$) form a basis for the subspace of $\mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$ spanned by the **cd**-indices of all simplicial Eulerian posets. Here we assume that $\Psi_n^n = 0$.

A poset P is *quasisimplicial* if for any coatom c of P , the interval $[\hat{0}, c]$ is simplicial, that is, for any corank 2 element x of P , the interval $[\hat{0}, x]$ is a Boolean algebra. We now prove the quasisimplicial analogue of Theorem 2.

Corollary 3 *Let P be a quasisimplicial Eulerian poset of rank $n+1$ whose h -vector is $h(P) = (h_0, h_1, \dots, h_n)$. Then*

$$\Phi(P) = \sum_{i=0}^{n-1} h_i \Psi_i^n. \quad (7)$$

Proof We will follow the proof of Theorem 3.1 from [14]. Note that the face posets of the CW -spheres $(\Lambda_i^n)'$ are quasisimplicial. For any integer $i = 0, 1, \dots, n-1$ the

h -vector of $(\Lambda_i^n)'$ is

$$h((\Lambda_i^n)') = (\overbrace{1, 1, \dots, 1}^{i+1}, 0, \dots, 0, 1)$$

and the flag f -vectors of the spheres $(\Lambda_i^n)'$ for $i = 0, 1, \dots, n-1$ are linearly independent. From the relation (5) it follows that

$$\Phi((\Lambda_i^n)') = \Psi_0^n + \Psi_1^n + \dots + \Psi_i^n,$$

and formulae (7) is true for $(\Lambda_i^n)'$.

G. Kalai in [10] showed that the dimension of the subspace spanned by the flag f -vectors of all quasisimplicial n -polytopes (Eulerian posets) is n and therefore the **ab**-index of the spheres $(\Lambda_i^n)'$ form a basis for the space spanned by flag f -vectors of quasisimplicial posets.

Therefore, we conclude that formula (6) holds for all Eulerian quasisimplicial posets. $\square$

Now we describe a basis for the **ab**-indices of simplicial graded posets. Let P_i^n denote the face poset of Λ_i^n with added maximal element

$$P_i^n = P(\Lambda_i^n) \cup \{\hat{1}\}.$$

Note that P_i^n is a graded poset of rank $n+1$ and its h -vector is $h(P_i^n) = (\underbrace{1, 1, \dots, 1}_{i+1}, 0, \dots, 0)$. Since the flag f -vector of a simplicial graded poset can be read from its f -vector

$$f_{\{s_1, s_2, \dots, s_k\}} = \binom{s_k}{s_1, s_2 - s_1, \dots, s_k - s_{k-1}} f_{s_k-1},$$

we conclude that for $n \in \mathbb{N}_0$ and $i = 0, 1, \dots, n$ the **ab**-index of the posets P_i^n form a basis for the subspace spanned by flag f -vectors of all graded simplicial posets.

Remark 4 Note that Λ_i^{n+1} can be obtained as the pyramid of Λ_i^n . Therefore, when we identify $(\hat{1}_{P_i^n}, 1)$ with $\hat{1}_{P_i^{n+1}}$, we obtain that:

$$P_i^{n+1} \cong (P_i^n \times B_1) \setminus \{(\hat{1}_{P_i^n}, \emptyset)\} = \text{Pyr}(P_i^n) \setminus \{(\hat{1}_{P_i^n}, \emptyset)\}.$$

As we have that $[\hat{0}, (\hat{1}_{P_i^n}, \emptyset)]_{\text{Pyr}(P_i^n)} \cong P_i^n$, from the relation (3) and Theorem 1 it follows that the **ab**-indices of those posets satisfy

$$\Psi(P_i^{n+1}) = \text{Pyr}(\Psi(P_i^n)) - \Psi(P_i^n) \cdot \mathbf{b} = G(\Psi(P_i^n)) + \Psi(P_i^n) \cdot \mathbf{a}. \quad (8)$$

Further, the face poset of $(\Lambda_i^n)'$ and P_i^n differ only in the coatom σ (a new n -cell in the semisuspension) and we can note that $[\hat{0}, \sigma]_{P((\Lambda_i^n)')} \cong P(\partial \Lambda_i^n)$.

If T_k^n denotes the polytope obtained as the stellar subdivision of an n -simplex over a k -face, then it is not hard to see that $\partial \Lambda_i^n$ is combinatorially equivalent to the boundary of T_i^{n-1} . Therefore, we obtain that

$$\Psi((\Lambda_i^n)') = \Psi(P_i^n) + \Phi(T_i^{n-1}) \cdot \mathbf{b}. \quad (9)$$

Definition 5 For all $n \in \mathbb{N}_0$ and $0 \leq i \leq n$ we define **ab**-polynomials X_i^n :

$$\begin{aligned} X_0^0 &= 1, & X_0^n &= \Psi(P_0^n), & \text{and} \\ \text{for all } i &= 1, 2, \dots, n & \text{ let } X_i^n &= \Psi(P_i^n) - \Psi(P_{i-1}^n). \end{aligned}$$

Note that for $n > 0$ we have $X_0^n = \Psi(B_{n-1}) \cdot \mathbf{a}$. Using the same arguments as in Theorem 3.1 in [14] and Corollary 3, we obtain the following result.

Proposition 6 Let P be a simplicial graded poset of rank $n + 1$ whose h -vector is $h(P) = (h_0, h_1, \dots, h_n)$. Then

$$\Psi(P) = \sum_{i=0}^n h_i X_i^n.$$

Here, in contrast with Theorem 2, the summation ends at n because $X_n^n \neq 0$.

Proposition 7 For all $0 \leq i \leq n$ the polynomials X_i^n satisfy the following relations:

- (i) $X_i^{n+1} = G(X_i^n) + X_i^n \cdot \mathbf{a}$.
- (ii) $X_{i+1}^{n+1} - X_i^{n+1} = X_i^n \cdot (\mathbf{b} - \mathbf{a})$.
- (iii) $X_{i+1}^{n+1} = G(X_i^n) + X_i^n \cdot \mathbf{b}$.
- (iv) If ω denotes the involution on the $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ which interchanges the variables $\mathbf{a}$ and $\mathbf{b}$ in **ab**-polynomials, we have that $X_i^n = \omega(X_{n-i}^n)$.
- (v) $X_i^n + X_{n-i}^n = \Psi_i^n + \Psi_{n-i}^n$.

Proof (i) For $i = 0$ we obtain the formulae (i) from Definition 5 and relation (8). From the linearity of G , for all $i > 0$ we have that:

$$X_i^{n+1} = G(\Psi(P_i^n)) + \Psi(P_i^n) \cdot \mathbf{a} - G(\Psi(P_{i-1}^n)) - \Psi(P_{i-1}^n) \cdot \mathbf{a} = G(X_i^n) + X_i^n \cdot \mathbf{a}.$$

(ii) We shall prove the formulae by induction. Note that the formulae holds for $n = 0$ from Definition 5. Assume that the statement holds for all integers less than $n + 1$. From statement (i) we obtain that

$$X_{i+1}^{n+1} - X_i^{n+1} = G(X_{i+1}^n - X_i^n) + (X_{i+1}^n - X_i^n) \cdot \mathbf{a}.$$

Under the inductive assumption, we have that

$$X_{i+1}^{n+1} - X_i^{n+1} = G(X_i^{n-1} \cdot (\mathbf{b} - \mathbf{a})) + X_i^{n-1} \cdot (\mathbf{b} - \mathbf{a}) \cdot \mathbf{a}. \quad (10)$$

As we obtain that

$$G(X_i^{n-1} \cdot (\mathbf{b} - \mathbf{a})) = G(X_i^{n-1}) \cdot (\mathbf{b} - \mathbf{a}) + X_i^{n-1} \cdot (\mathbf{ab} - \mathbf{ba}),$$

applying (i) on $G(X_i^{n-1})$ it follows that

$$G(X_i^{n-1} \cdot (\mathbf{b} - \mathbf{a})) = X_i^n \cdot (\mathbf{b} - \mathbf{a}) - X_i^{n-1} \cdot \mathbf{a}(\mathbf{b} - \mathbf{a}) + X_i^{n-1} \cdot (\mathbf{ab} - \mathbf{ba}).$$

Substituting the previous equations in (10) we obtain the statement (ii).

(iii) Follows from adding relations (i) and (ii).

(iv) Again, we use induction on n . Assume that relation (iv) holds for all integers less than n . From the relation (i) and inductive assumption, we have that

$$\omega(X_i^n) = \omega(G(X_i^{n-1}) + X_i^{n-1} \cdot \mathbf{a}) = \omega(G(X_i^{n-1})) + X_{n-i-1}^{n-1} \cdot \mathbf{b}.$$

From the definition of the derivation G , it is easily seen that $\omega \circ G = G \circ \omega$. Now, from this, the inductive assumption and formulae (iii), we obtain that

$$\omega(X_i^n) = G(X_{n-1-i}^{n-1}) + X_{n-i-1}^{n-1} \cdot \mathbf{b} = X_{n-i}^n.$$

(v) From Definition 5 it follows that $X_0^n = \Psi(\Delta_{n-1}) \cdot \mathbf{a}$, and from (iv) and the symmetry of the $\mathbf{ab}$ -index of simplices it follows that $X_n^n = \Psi(\Delta_{n-1}) \cdot \mathbf{b}$. Therefore, from the relation (5), we obtain that $\Psi_0^n = X_0^n + X_n^n$.

For $i > 0$, from the definition of polynomials Ψ_i^n and formulae (9) we obtain:

$$\Psi_i^n = X_i^n + (\Phi(T_i^{n-1}) - \Phi(T_{i-1}^{n-1})) \cdot \mathbf{b}, \quad \Psi_{n-i}^n = X_{n-i}^n + (\Phi(T_{n-i}^{n-1}) - \Phi(T_{n-i-1}^{n-1})) \cdot \mathbf{b}$$

As the polytopes T_i^{n-1} and T_{n-i}^{n-1} are combinatorially equivalent, their $\mathbf{cd}$ -indices are the same and therefore it follows that formulae (v) is true. $\square$

3 Weighted Derivations

A left weighted derivation on a graded algebra R is a function $f : R \times \mathbb{N} \rightarrow R$ such that

$$\begin{aligned} f(1, p) &= 0, & f(\alpha \cdot u + \beta \cdot v, p) &= \alpha \cdot f(u, p) + \beta \cdot f(v, p), \\ f(u \cdot v, p) &= f(u, p) \cdot v + u \cdot f(v, |u| + p), \end{aligned}$$

where α and β are scalars and $|u|$ denotes the degree of u .

R. Ehrenborg and M. Readdy in [9] defined the following weighted derivations on the algebra $\mathbb{Q}(\mathbf{c}, \mathbf{d})$:

$$W_k(\mathbf{c}, p) = (1 + kp)2\mathbf{d}; \quad W_k(\mathbf{d}, p) = (1 + kp)\mathbf{dc} + (1 + k(p + 1))\mathbf{cd}. \quad (11)$$

Note that W_0 is just the derivation D .

These weighted derivations were used in order to describe the following recursive relations for the **cd**-indices of the lattices of the regions of arrangements $\mathcal{A}_n$ and $\mathcal{B}_n$ (Theorem 5.2. in [9]):

$$\Phi(R_{\mathcal{A}_{n+1}}) = \mathbf{c} \cdot \Phi(R_{\mathcal{A}_n}) + W_1(\Phi(R_{\mathcal{A}_n}), 1), \quad (12)$$

$$\Phi(R_{\mathcal{B}_{n+1}}) = \mathbf{c} \cdot \Phi(R_{\mathcal{B}_n}) + W_2(\Phi(R_{\mathcal{B}_n}), 1). \quad (13)$$

The main tool for the proof of the above relations is Theorem 3.1 from [4], and an appropriate lexicographically shelling for the intersection lattices of these arrangements.

We extend the definition of the weighted derivations W_k on the algebra $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$. On the generators, these derivations are defined by:

$$W_k(\mathbf{a}, n) = W_k(\mathbf{b}, n) = (1 + kn)(\mathbf{ab} + \mathbf{ba}), \quad W_k(1, n) = 0.$$

Here, we use the same notation because the restriction of these derivations on $\mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$ coincide with (11). Let $\mathcal{W}_k : \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle \rightarrow \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ be a linear map defined by $\mathcal{W}_k(u) = (\mathbf{a} + \mathbf{b}) \cdot u + W_k(u, 1)$.

Theorem 8 For all $n \geq i \geq 0$ we have that

$$\mathcal{W}_k(X_i^n) = (1 + ki) \cdot X_i^{n+1} + (1 + k(n - i))X_{i+1}^{n+1}. \quad (14)$$

Proof We use induction on n and i . We have that

$$\mathcal{W}_k(X_0^0) = \mathcal{W}_k(1) = \mathbf{a} + \mathbf{b} = X_0^1 + X_1^1.$$

Also, we obtain that

$$\mathcal{W}_k(X_0^1) = \mathcal{W}_k(\mathbf{a}) = \mathbf{a}^2 + \mathbf{ba} + (1 + k)(\mathbf{ab} + \mathbf{ba}) = X_0^2 + (1 + k)X_1^2,$$

and

$$\mathcal{W}_k(X_1^1) = \mathcal{W}_k(\mathbf{b}) = \mathbf{ab} + \mathbf{b}^2 + (1 + k)(\mathbf{ab} + \mathbf{ba}) = (1 + k)X_1^2 + X_2^2.$$

Assume that the relation (14) holds for all **ab**-polynomials considered whose degree is less than n . From the definition of the polynomials X_i^n and the inductive assumption it follows that

$$\begin{aligned} \mathcal{W}_k(X_0^n) &= \mathcal{W}_k((X_0^{n-1} + X_1^{n-1} + \cdots + X_{n-1}^{n-1}) \cdot \mathbf{a}) \\ &= \left(X_0^n + (2 + kn) \sum_{i=1}^{n-1} X_i^n + X_n^n \right) \cdot \mathbf{a} + (1 + kn) \sum_{i=0}^{n-1} X_i^{n-1} \cdot (\mathbf{ab} + \mathbf{ba}). \end{aligned}$$

As we know that $X_0^{n+1} = (X_0^n + X_1^n + \cdots + X_n^n) \cdot \mathbf{a}$, in order to prove that relation (14) holds for X_0^n we have to verify that

$$(1 + kn) \left(\sum_{i=1}^{n-1} X_i^n \cdot \mathbf{a} + \sum_{i=0}^{n-1} X_i^{n-1} \cdot (\mathbf{ab} + \mathbf{ba}) \right) = (1 + kn)X_1^{n+1}.$$

But, from the definition of X_0^n and formulae (iv) of Proposition 7 we have that

$$\sum_{i=1}^{n-1} X_i^n \cdot \mathbf{a} = X_0^{n+1} - X_0^n \cdot \mathbf{a} - X_n^n \cdot \mathbf{a}, \quad \sum_{i=0}^{n-1} X_i^{n-1} \cdot (\mathbf{ab} + \mathbf{ba}) = X_0^n \cdot \mathbf{b} + X_n^n \cdot \mathbf{a}.$$

Now, we apply statement (ii) of Proposition 7, and the proof for X_0^n is complete.

Assume that the relation (14) holds for all polynomials X_j^n when $i > j \geq 0$. We shall prove that (14) holds for X_i^n . From the relation (ii) of Proposition 7, the inductive assumption and the fact that $W_k(\mathbf{a}, n) = W_k(\mathbf{b}, n)$, we obtain that

$$\begin{aligned} \mathcal{W}_k(X_i^n) &= \mathcal{W}_k(X_{i-1}^n + X_{i-1}^{n-1} \cdot (\mathbf{b} - \mathbf{a})) = \mathcal{W}_k(X_{i-1}^n) + \mathcal{W}_k(X_{i-1}^{n-1}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= (1 + k(i-1))X_{i-1}^{n+1} + (1 + k(n-i+1))X_i^{n+1} \\ &\quad + ((1 + k(i-1))X_{i-1}^n + (1 + k(n-i))X_i^n) \cdot (\mathbf{b} - \mathbf{a}). \end{aligned}$$

Applying relation (ii) of Proposition 7 again, we conclude that formulae (14) holds for X_i^n . $\square$

For an n -dimensional CW sphere X , we can define the following sequence of $(n+1)$ -spheres: $S_1(X) = \text{Pyr}(X)$, $S_2(X) = \text{Prism}(X)$, and $S_{k+2}(X)$ is obtained by gluing the spheres $S_k(X)$ and $\text{Prism}(X)$ over a common base X . Using a result of R. Stanley (Lemma 6.3 in [8]) it follows that

$$\Phi(S_{k+2}(X)) = \Phi(S_k(X)) + \Phi(\text{Prism}(X)) - \Phi(X) \cdot \mathbf{c}.$$

By successive application of Theorem 1, we obtain the following proposition.

Proposition 9 *If X is a CW -sphere, then*

$$\Phi(S_k(X)) = \Phi(X) \cdot \mathbf{c} + F_k(\Phi(X)),$$

where F_k is a sequence of the derivations defined by $F_{2k} = kD$ and $F_{2k+1} = G + kD$.

If P is the face poset of a CW -sphere X , then the face poset of $S_k(X)$ can be identified with the set

$$\Sigma_k(P) = \{(x, i) : x \in P \setminus \{\hat{0}_P, \hat{1}_P\}, 0 \leq i \leq k\} \cup \{\hat{0}_{\Sigma_k}, \hat{1}_{\Sigma_k}, (\hat{1}_P, 0), (y, k)\},$$

where $y = \hat{1}_P$, if k is even, and $y = \hat{0}_P$, if k is odd. The order relation on $\Sigma_k(P)$ is defined by

$$(x, i) < (x', j) \iff \begin{cases} x < x', i = j, \\ x \leq x', i = j \pm 1, & j \text{ is odd.} \end{cases}$$

We can use the above to define the operation $P \mapsto \Sigma_k(P)$ for an arbitrary poset P . It is easy to see that the $\mathbf{ab}$ -index of the poset $\Sigma_k(P)$ is of the form

$$\Psi(\Sigma_k(P)) = \Psi(P) \cdot (\mathbf{a} + \mathbf{b}) + F_k(\Psi(P)). \quad (15)$$

Also, if P is an Eulerian poset, then $\Sigma_k(P)$ is also Eulerian. Therefore, the operation $P \mapsto \Sigma_k(P)$ may be useful for questions regarding the flag vectors of Eulerian posets. This will be the subject of further research.

Theorem 10 *Let P be a graded poset of rank $n + 1$ and let k be a nonnegative integer. Then we have that*

$$\Psi(Sd(\Sigma_k(P))) = (\mathbf{a} + \mathbf{b}) \cdot \Psi(Sd(P)) + W_k(\Psi(Sd(P)), 1) = \mathcal{W}_k(\Psi(Sd(P))).$$

In particular, for an Eulerian poset P the following holds:

$$\Phi(Sd(\Sigma_k(P))) = \mathbf{c} \cdot \Phi(Sd(P)) + W_k(\Phi(Sd(P)), 1).$$

Proof Note that the posets $Sd(P)$ and $Sd(\Sigma_k(P))$ are both simplicial. Let $h(Sd(P)) = (h_0, h_1, \dots, h_n)$ be the h -vector of $Sd(P)$. Applying the relations (4) and (15) we obtain the h -vector of $Sd(\Sigma_k(P))$:

$$\begin{aligned} h(Sd(\Sigma_k(P))) &= (\bar{h}_0, \bar{h}_1, \dots, \bar{h}_{n+1}) \quad \text{where} \\ \bar{h}_i &= (1 + ki)h_i + (1 + k(n - i + 1))h_{i-1}. \end{aligned}$$

Now, from Proposition 6 and Theorem 8 it follows that

$$\begin{aligned} \mathcal{W}_k(\Psi(Sd(P))) &= \mathcal{W}_k\left(\sum_{i=0}^n h_i X_i^n\right) = \sum_{i=0}^n h_i ((1 + ki)X_i^{n+1} + (1 + k(n - i)X_{i+1}^{n+1})) \\ &= \sum_{i=0}^{n+1} ((1 + ki)h_i + (1 + k(n - i + 1))h_{i-1})X_i^{n+1} \\ &= \Psi(Sd(\Sigma_k(P))). \end{aligned}$$

As the weighted derivations agree on **ab** and **cd**-polynomials, the second identity holds in the case of Eulerian posets. $\square$

It is well known (see in [5]) that the lattices $R_{\mathcal{A}_n}$ and $R_{\mathcal{B}_n}$ are isomorphic to $Sd(B_{n+1})$ and $Sd(C_n)$ (the barycentric subdivisions of the face posets of the n -simplex and n -cube), respectively. So, the relations (12) and (13) represent special cases of the previous formula.

As an interesting question, the authors of [9] asked whether there is a recursive relation for the lattices of the regions of arrangements $\mathcal{D}_n : x_i \pm x_j = 0$ for $0 \leq i < j \leq n$. We determine such a recursion in Corollary 12.

Remark 11 Maximal cells in the decomposition of $\mathbb{R}^n$ with hyperplanes of the arrangement $\mathcal{B}_n$ are in the bijection with the “signed” permutation of the set $[n]$. The region which corresponds to the permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ is supported by the hyperplanes $x_{\sigma_1} = 0$ and $x_{\sigma_i} = \text{sgn}(\sigma_i \cdot \sigma_{i+1})x_{\sigma_{i+1}}$ for $i = 1, 2, \dots, n - 1$.

As the arrangement $\mathcal{D}_n$ does not contain the coordinate hyperplanes, the coatoms in the lattice $R_{\mathcal{D}_n}$ are obtained by “gluing” cells which correspond with the permutations $\sigma_1 \sigma_2 \dots \sigma_n$ and $-\sigma_1 \sigma_2 \dots \sigma_n$. So, $R_{\mathcal{D}_n}$ is combinatorially equivalent to a simplicial poset obtained from $Sd(C_n)$ as follows:

- Chains of C_n that do not contain an edge remain in $R_{\mathcal{D}_n}$;
- Chains of C_n that contain an edge, but do not contain a vertex will disappear; and
- For any edge $e = [u, v]$, two chains of C_n are identified if they contain e and differ only in the vertex they contain.

As the interval $[e, 1]$ in the face lattice of an n -cube is isomorphic to the Boolean lattice B_{n-1} , we have that

$$f_i(R_{\mathcal{D}_n}) = f_i(R_{\mathcal{B}_n}) - 2^{n-1}n(f_{i-1}(R_{\mathcal{A}_{n-2}}) + f_{i-2}(R_{\mathcal{A}_{n-2}})).$$

Now, from the relation (1) it follows that

$$h(R_{\mathcal{D}_n}, x) = h(R_{\mathcal{B}_n}, x) - n2^{n-1}x \cdot h(R_{\mathcal{A}_{n-2}}, x). \quad (16)$$

For a simplicial Eulerian poset P whose h -vector is $h(P) = (h_0, h_1, \dots, h_n)$, $\text{Pyr}(P)$ is a quasisimplicial poset with $h(\text{Pyr}(P)) = (h(P), 1)$; see [1]. Then, from Corollary 3 and Theorem 1 we have that

$$\Phi(\text{Pyr}(P)) = \sum_{i=0}^n h_i \Psi_i^{n+1} = \text{Pyr}(\Phi(P)).$$

As $G(\Psi_i^n) = \Psi_{i+1}^{n+1}$ holds for all $i < n$ (see [8]), from relation (16) we obtain the following corollary.

Corollary 12 *The cd-indices of the lattice of the regions of the arrangements of the type $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{D}_n$ satisfy the following relation*

$$\Phi(R_{\mathcal{D}_n}) = \Phi(R_{\mathcal{B}_n}) - 2^{n-1}nG(\text{Pyr}(\Phi(R_{\mathcal{A}_{n-2}}))).$$

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