(m, n) -Equidistant Sets in $\mathbb{R}^k$, $\mathbb{S}^k$, and $\mathbb{P}^k$

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Received: 16 January 2007 / Revised: 11 December 2007 /

Published online: 21 December 2007

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Abstract We call a metric space X (m, n) -equidistant if, when $A \subseteq X$ has exactly m points, there are exactly n points in X each of which is equidistant from (the points of) A . We prove that, for $k \geq 2$, the Euclidean space $\mathbb{R}^k$ contains an $(m, 1)$ -equidistant set if and only if $k \geq m$. Although the sphere $\mathbb{S}^2$ is $(3, 2)$ -equidistant, $\mathbb{S}^3$ and $\mathbb{R}^4$ contain no $(4, 2)$ -equidistant sets. We discuss related results about projective spaces, and state a conjecture about $\mathbb{S}^2$ analogous to the Double Midset Conjecture.

1 Introduction

The bisector (or midset) of two distinct points a, b in a metric space (X, d) is the set $E(a, b) = \{x \in X : d(x, a) = d(x, b)\}$. This is a natural and important notion, and we review some known results in Sect. 2. We define a generalization of this concept when we start with $m \geq 2$ points, and we obtain some geometric and combinatorial results about subsets of Euclidean spaces, projective spaces, and spheres. In Sect. 3 we prove that, for $k \geq 2$, $\mathbb{R}^k$ contains an $(m, 1)$ -equidistant set (Definition 2.1) if and only if $k \geq m$. Although the circle $\mathbb{S}^1$ and the sphere $\mathbb{S}^2$ are $(2, 2)$ - and $(3, 2)$ -equidistant, respectively, Sect. 4 shows that $\mathbb{S}^3$ is not $(4, 2)$ -equidistant, and that $\mathbb{R}^4$ contains no $(4, 2)$ -equidistant set.

2 Definitions, Examples, and Known Results

Definition 2.1 Let (X, d) be a metric space and $A \subseteq X$. If A has at least two points, then the *equidistance set for A* is the set $E(A) = \{x \in X : d(x, a) = d(x, b) \text{ for all } a, b \in A\}$.

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$a, b \in A\}$. Clearly $E(A) \subseteq X \setminus A$. For convenience if A is empty or has only one point we define $E(A) = X \setminus A$. Call X (m, n) -equidistant if X has at least m points and whenever A has exactly m points $E(A)$ has exactly n points. If $x \in E(A)$, we say that x is *equidistant from* A .

For example, if $a, b \in \mathbb{R}^2 = X$ (and $a \neq b$), then $E(\{a, b\})$ is the perpendicular bisector of a, b . If $X = \mathbb{R}$, then $E(\{a, b\})$ is the midpoint of a, b . Thus, the real line is $(2, 1)$ -equidistant. When X is a circle, $E(\{a, b\})$ always consists of exactly two points, and hence the circle is $(2, 2)$ -equidistant. The vertices of a regular tetrahedron form a set that is both $(2, 2)$ - and $(3, 1)$ -equidistant.

The case when $m = 2$ has been extensively studied. $E(A)$ where $A = \{a, b\}$ is known as the midset (or bisector) of a, b . $(2, n)$ -equidistance is known in the literature as the n -MP, i.e. n -points Midset Property; see [7, 8, 14]. 1-MP, 2-MP, and 3-MP are known as the UMP, DMP, and TMP (Unique, Double, and Triple Midset Property), and in our notation correspond to $(2, 1)$ -, $(2, 2)$ -, and $(2, 3)$ -equidistance.

We will always assume that $m \geq 2$, although some of the statements that follow are also true for $m = 1$ or $m = 0$. One may allow m and n to be infinite. For example, let $D(n)$ denote a discrete space with n points with a constant metric, i.e. there is $c > 0$ such that $d(x, y) = c$ iff $x \neq y$. If n is infinite, then $D(n)$ is (m, n) -equidistant for all (finite or infinite) $m < n$. This is trivial since if X has a constant metric, then $E(A) = X \setminus A$ for all A . If k is finite, then $D(k)$ is (m, n) -equidistant iff $m + n = k$. The plane is $(2, c)$ -equidistant where c is the cardinality of the continuum.

The more interesting cases are when both m and n are finite. The case $n = 0$ is quite different from $n > 0$: If X is $(m, 0)$ -equidistant for some m , then X is $(m', 0)$ -equidistant for all $m' \geq m$. The real line is $(3, 0)$ -equidistant but not $(2, 0)$ -equidistant. It is easy to see that if X is $(2, 0)$ -equidistant, then X has a zero-dimensional topology, i.e. X has a base of sets that are both open and closed. (If S is a sphere centered at x , then there is at most one $y \in S$. If B is the open ball corresponding to S , and if U is the set of points closer to x than to y , then $B \cap U$ is closed and open.) It is known that the covering dimension of a separable metric space X does not exceed n iff X admits a totally bounded metric with respect to which the midset of every two points has dimension at most $n - 1$, [7–10].

An interesting topological characterization of the circle due to Loveland is known in terms of midsets: A compact connected $(2, 2)$ -equidistant subspace of $\mathbb{R}^2$ with the usual metric is homeomorphic to a circle, [7, 8, 12, 15]. We comment more on this in Sect. 5.

The results that we present in the following sections belong to distance geometry (see [3]), and to combinatorial geometry (see [18]). We assume familiarity with elementary notions from linear algebra and convex geometry (e.g., affine subspaces, convex hull). The reader may consult [11] or any linear algebra text.

$\mathbb{R}^k$ is the k -dimensional Euclidean space with the usual metric. By Δ^k we denote the k -dimensional regular simplex in $\mathbb{R}^k$. Its $k + 1$ vertices are $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$, and $(a, a, \dots, a)$, where $a = (1 + \sqrt{k + 1})/k$ is the positive solution of the equation $d((1, 0, \dots, 0), (a, a, \dots, a)) = \sqrt{2}$. Let V be the set of vertices of Δ^k . Clearly, for $u, v \in V$, $d(u, v) = \sqrt{2}$ iff $u \neq v$. Thus, V is a copy of $D(k + 1)$, so it is (m, n) -equidistant (for finite k) iff $m + n = k + 1$. (If we allow k to be infinite, then the standard basis in the generalized Hilbert space of weight k is a copy of $D(k)$.)

It is easy to verify by induction that $\mathbb{R}^k$ contains a copy of $D(m)$ iff $k \geq m - 1$. We emphasize that we always assume the Euclidean metric d on $\mathbb{R}^k$ and on its subspaces. With respect to a different metric (although it may induce the same topology) the plane may contain a copy of $D(4)$; e.g., take the vertices of a square with a horizontal side and consider the metric that takes the maximum of the absolute values of the differences of the corresponding coordinates. (Alternatively, take the vertices of a square with a horizontal diagonal, and consider the taxicab metric.) With respect to the metric $\min\{1, d\}$ the integer lattice $\mathbb{Z}^k$ is a copy of $D(\aleph_0)$ in $\mathbb{R}^k$.

3 $(m, 1)$ -Equidistant Sets in $\mathbb{R}^k$

Clearly $\mathbb{R}^1$ is $(2, 1)$ -equidistant. $\mathbb{R}^2$ is not, but contains $D(3)$ which is $(2, 1)$ -equidistant. $\mathbb{R}^k$ is never $(3, 1)$ -equidistant since if a, b, c are collinear then $E(\{a, b, c\})$ is empty. One may attempt to construct a $(3, 1)$ -equidistant subset of $\mathbb{R}^2$ by starting with three suitably chosen noncollinear points a, b, c and then building (by adding points in countably many steps) the smallest set $X \subset \mathbb{R}^2$ that contains a, b, c , and together with all distinct $p, q, r \in X$ contains the circumcenter of the triangle with vertices p, q, r . “Suitably chosen” a, b, c would mean that the above process would never produce three collinear points. It turns out that this approach is bound to fail.

Theorem 3.1 *Assume that $k \geq 2$. Then $\mathbb{R}^k$ contains an $(m, 1)$ -equidistant set if and only if $k \geq m$.*

One direction is easy since if $k \geq m$, $D(m + 1)$ is an $(m, 1)$ -equidistant subset of $\mathbb{R}^k$. For the other direction we prove a lemma.

If $A \subseteq \mathbb{R}^k$, then by $\text{aff}(A)$ we denote the smallest affine subspace of $\mathbb{R}^k$ containing A . It is easy to verify by induction that if $A \subseteq \mathbb{R}^k$ is an affinely independent set and A has exactly $k + 1$ points, then there is a unique $(k - 1)$ -dimensional sphere in $\mathbb{R}^k$ that contains A . In the induction step it is used that if $A \subseteq \mathbb{R}^k$ is affinely independent and has exactly k points, then A determines a unique $(k - 2)$ -dimensional sphere in $\text{aff}(A)$, and $E(A)$ is the line in $\mathbb{R}^k$ through the center of that sphere and orthogonal to $\text{aff}(A)$.

Next we generalize the observation that if A contains three collinear points, then $E(A)$ is empty. This observation may be restated as follows: If $X \subseteq \mathbb{R}^k$ is (m, n) -equidistant for some $m \geq 3$ and $n \geq 1$, and if L is a line, then $L \cap X$ has at most two points and hence is contained in a 0-dimensional sphere S contained in L . (In particular, such an X cannot be contained in a line.)

Lemma 3.2 *Suppose $X \subseteq \mathbb{R}^k$ is (m, n) -equidistant where $n \geq 1$, and L is a j -dimensional affine subspace of $\mathbb{R}^k$ where $1 \leq j \leq m - 2$. Then*

- (a) $L \cap X$ is contained in a $(j - 1)$ -dimensional sphere in L ,
- (b) $L \cap X$ contains at most $j + 1$ points,
- (c) every subset of X of at most m points is affinely independent,
- (d) $L \cap X$ is affinely independent, and
- (e) $m \leq k + 1$ (this bound will be improved later).

Proof If (a) were false, fix the minimal $j \leq m - 2$ for which there is an affine subspace L of dimension j such that $L \cap X$ is not contained in any $(j - 1)$ -dimensional sphere in L . Then $j \geq 2$ by the observation preceding the lemma. By minimality of j it follows that $\text{aff}(L \cap X) = L$ (for otherwise if $i < j$ is the dimension of $L' = \text{aff}(L \cap X)$, then $L \cap X = L' \cap X$ is contained in an $(i - 1)$ -dimensional sphere in L' which could be extended to a $(j - 1)$ -dimensional sphere in L). Hence there is an affinely independent set $B \subseteq L \cap X$ of exactly $j + 1$ points. B is contained in a unique $(j - 1)$ -dimensional sphere S in L . Let K be the equidistance set for S taken in $\mathbb{R}^k$. Then K is the largest affine subspace of $\mathbb{R}^k$ whose orthogonal projection onto L is the center of S . Also, K is the equidistance set for B taken in $\mathbb{R}^k$. If there is an $a \in (L \cap X) \setminus B$, take a (possibly empty) $J \subseteq X \setminus (B \cup \{a\})$ of exactly $m - j - 2$ points. Let $p \in X$ be equidistant from $B \cup \{a\} \cup J$. The orthogonal projection o of p onto L must coincide with the center of S (since p is equidistant from B), and o must be equidistant from $B \cup \{a\}$, and hence $a \in S$. It follows that $L \cap X \subseteq S$.

If (b) were false, the construction in (a) could be repeated (including the definition of B and uniqueness of S), now under the assumption that $j \leq m - 2$ is minimal such that $L \cap X$ contains at least $j + 2$ points for some j -dimensional affine subspace L . By (a), and by uniqueness of S , $L \cap X = S \cap X$. Take any $a \in (S \cap X) \setminus B$ and any $J \subseteq X \setminus (B \cup \{a\})$ of exactly $m - j - 2$ points. Let $E = E(B \cup \{a\} \cup J)$ (taken in X). By assumption E has exactly n points. Take any $e \in E$ and let $E' = E(B \cup \{e\} \cup J)$. Each point of E' belongs to K (defined as in part (a)), and hence is equidistant from $B \cup \{a\}$. It follows that $E' \subseteq E \setminus \{e\}$, and each of E, E' has exactly n points, a contradiction.

For part (c), if $A \subseteq X$ were affinely dependent of exactly $i \leq m$ points then the dimension j of $\text{aff}(A)$ would be at most $i - 2$. By (b) it follows that A contains at most $i - 1$ points, a contradiction.

Part (d) follows from (b) and (c).

For part (e), if $m \geq k + 2$, then, by (b) with $L = \mathbb{R}^k$, X would contain at most $k + 1$ points, but X has at least m points. $\square$

The next lemma completes the proof of Theorem 3.1.

Lemma 3.3 *If $m > k \geq 2$, then $\mathbb{R}^k$ contains no $(m, 1)$ -equidistant set.*

Proof Assume that $m > k \geq 2$ and $X \subseteq \mathbb{R}^k$ is $(m, 1)$ -equidistant. By Lemma 3.2, $m = k + 1$. Take any $A \subseteq X$ of exactly m points. Then A is affinely independent. Write $A = B \cup \{a\}$ where B has exactly k points. Let e be the unique point of X equidistant from $B \cup \{a\}$. Let e' be the unique point of X equidistant from $B \cup \{e\}$. Since B determines a unique $(k - 2)$ -dimensional sphere S in $L = \text{aff}(B)$, both e and e' must belong to the line K orthogonal to L passing through the center of S . (K is the equidistance set for B taken in $\mathbb{R}^k$. The reader may make an illustration when $k = 3$ and $m = 4$.) Since $K \cap X$ has at most two points, and the unique point in X equidistant from $B \cup \{e'\}$ must belong to K , it follows that e is that point. Thus $d(e', b) = d(e', e) = d(b, e)$ for all $b \in B$, and hence L is the equidistance set (midset in this case) for $\{e, e'\}$ taken in $\mathbb{R}^k$. By Lemma 3.2, $L \cap X$ contains at most k points, and hence $L \cap X = B$. For each $b \in B$, the point b is the unique point in X equidistant

from $(B \setminus \{b\}) \cup \{e, e'\}$, since this point must belong to L and hence to B , but not to $B \setminus \{b\}$.

Now $B \cup \{e, e'\}$ is a copy of $D(k+2)$ in $\mathbb{R}^k$, a contradiction. $\square$

Corollary 3.4 *If $X \subseteq \mathbb{R}^k$ is (m, n) -equidistant where $m \geq 3$ and $n \geq 1$, then $k \geq m$.*

Proof Assume $m > k$. By Lemma 3.2.(e), we have $m = k + 1$. But if $A \subseteq X$ has exactly m points, then A is affinely independent and $E(A)$ taken in $\mathbb{R}^k$ has exactly one point (the center of the $(k-1)$ -dimensional sphere determined by A). Thus n must be equal to 1, which contradicts the above theorem. $\square$

Corollary 3.5 *If $m \geq \max\{k, 3\}$ and $n \geq 3$, then $\mathbb{R}^k$ contains no (m, n) -equidistant set.*

Proof Suppose X were such a set. By Corollary 3.4, $m = k$. Take $A \subseteq X$ of exactly m points. By Lemma 3.2.(c), $\text{aff}(A)$ is $(k-1)$ -dimensional, and hence $E(A)$ (taken in $\mathbb{R}^k$) is a line orthogonal to $\text{aff}(A)$. X must contain $n \geq 3$ points on this line, a contradiction. $\square$

By $\mathbb{S}^k$ we denote the k -dimensional unit sphere in $\mathbb{R}^{k+1}$. The metric on $\mathbb{S}^k$ is the metric d induced from $\mathbb{R}^{k+1}$. Thus, for $a, b \in \mathbb{S}^k$, $d(a, b)$ is the length of the chord from a to b . By $\mathbb{P}^k$ we denote the (real) projective space of dimension k , as the image of $\mathbb{S}^k$ under the quotient map π that glues together diametrically opposite points. For $x, y \in \mathbb{P}^k$ let $d(x, y) = d(\pi^{-1}(x), \pi^{-1}(y)) := \min\{d(a, b) : a \in \pi^{-1}(x), b \in \pi^{-1}(y)\}$ (the latter taken in $\mathbb{S}^k$). If $\pi^{-1}(x) = \{a, a'\}$ and $\pi^{-1}(y) = \{b, b'\}$ (where a, a' are diametrically opposite, and so are b, b'), then $d(x, y)$ is the length of the shorter of the two chords: from a to b , and from a' to b . This deviates from the usual definition of the metric on $\mathbb{P}^k$ which is the following. First consider $\mathbb{S}^k$ with the angular metric δ , i.e. $\delta(a, b)$, is the measure of the central angle subtended by rays from the origin to a and b . Since $\mathbb{S}^k$ has radius 1, $\delta(a, b)$ is also the length of the shortest arc on $\mathbb{S}^k$ from a to b . For $x, y \in \mathbb{P}^k$ let $\delta(x, y) = \delta(\pi^{-1}(x), \pi^{-1}(y))$ (the latter taken in $\mathbb{S}^k$). If we think of x, y as two lines through the origin that intersect $\mathbb{S}^k$ at a, a' and at b, b' , respectively, then $\delta(x, y)$ is the measure of the acute or right angle determined by x, y .

Any sentence about (m, n) -equidistance of subsets of $\mathbb{S}^k$ (or $\mathbb{P}^k$) with respect to d is true if and only if it is true with respect to δ . This follows from the next lemma, the proof of which is trivial.

Lemma 3.6 *Suppose X is a space with two different metrics d and δ such that for all $a, b, p, q \in X$ (not necessarily distinct) $d(a, b) = d(p, q)$ if and only if $\delta(a, b) = \delta(p, q)$. Then X is (m, n) -equidistant with respect to d (for some m, n) if and only if X is (m, n) -equidistant with respect to δ .*

Corollary 3.7 *Let $k \geq 1$. Then $\mathbb{S}^k$ contains an $(m, 1)$ -equidistant set if and only if $k+1 \geq m$. If $k+1 \geq m$, then $\mathbb{P}^k$ contains an $(m, 1)$ -equidistant set.*

Proof $\mathbb{S}^k$ contains a copy of $D(k+2)$, and hence a copy of $D(m+1)$ if $k+1 \geq m$, and $D(m+1)$ is $(m, 1)$ -equidistant. $D(k+2)$ also embeds in $\mathbb{P}^k$ since $D(k+2)$ embeds in $\mathbb{S}^k$ in such a way that no diametrically opposite points are contained in $D(k+2)$.

If X is an $(m, 1)$ -equidistant subset of $\mathbb{S}^k \subseteq \mathbb{R}^{k+1}$ it follows from Theorem 3.1 that $k+1 \geq m$. $\square$

It is known that $\mathbb{P}^2$ contains an equilateral sextuple (i.e., a copy of $D(6)$ and thus a $(5, 1)$ -equidistant set). This is the image under the quotient map π of the set of vertices of a regular icosahedron inscribed in $\mathbb{S}^2$. Neither $\mathbb{P}^2$ nor $\mathbb{P}^3$ contain equilateral septuples (copies of $D(7)$); see [3] §80.

There is no congruent (i.e. distance preserving) embedding of $\mathbb{P}^1$ into $\mathbb{R}^k$, and hence of $\mathbb{P}^j$ into $\mathbb{R}^k$, for any j, k . We are grateful to Charles Anderson for the following argument: Take two slightly overlapping arcs A and B in $\mathbb{P}^1$, each congruent to a quarter-circle. Embed each of A and B in $\mathbb{R}^k$ ($k \geq 2$) such that they overlap just as they did in $\mathbb{P}^1$. Then $A \cup B$ is an arc larger than a quarter-circle, and the distance between its endpoints (in $\mathbb{R}^k$) is larger than the distance between the corresponding points in $\mathbb{P}^1$ (where the maximum distance is the one between the endpoints of a quarter-circle).

It is also known that $\mathbb{P}^2$ contains two congruent triples, such that no congruence between them can be extended to a congruence from $\mathbb{P}^2$ onto $\mathbb{P}^2$, while every congruence between subsets of $\mathbb{R}^k$ can be extended to a congruence from $\mathbb{R}^k$ onto $\mathbb{R}^k$ (see [3]). For example, take $a, b, c \in \mathbb{S}^2$ on a great circle with angular distances $\delta(a, b) = \delta(b, c) = 60^\circ$, and $\delta(a, c) = 120^\circ$. Also, take $a', b', c' \in \mathbb{S}^2$ that form an equilateral triangle with side 60° . Then $\pi(\{a, b, c\})$ is congruent to $\pi(\{a', b', c'\})$ in $\mathbb{P}^2$ (since each is congruent to an equilateral triangle with side 60°) but no congruence between them can be extended onto $\mathbb{P}^2$ since $\pi(\{a, b, c\})$ is contained in an elliptic line in $\mathbb{P}^2$ (see [3] §79), but $\pi(\{a', b', c'\})$ is not. (This should not be confused with well-known topological embeddings of $\mathbb{P}^1$ into $\mathbb{R}^2$, and of $\mathbb{P}^2$ into $\mathbb{R}^4$.)

4 $(m, 2)$ -Equidistance of Spheres

In this section we prove that, although $\mathbb{S}^1$ is $(2, 2)$ -equidistant and $\mathbb{S}^2$ is $(3, 2)$ -equidistant, $\mathbb{S}^3$ is not $(4, 2)$ -equidistant. Moreover, neither $\mathbb{S}^3$ nor $\mathbb{R}^4$ contain a $(4, 2)$ -equidistant set.

Theorem 4.1 $\mathbb{S}^2 \subseteq \mathbb{R}^3$ is $(3, 2)$ -equidistant.

Proof Every three points $a, b, c \in \mathbb{S}^2$ determine a unique plane L and a unique circle $C = L \cap \mathbb{S}^2$. Let K be the line through the center of C orthogonal to L , and let e, e' be the two points where K intersects $\mathbb{S}^2$. Then $E(a, b, c)$ (taken in $\mathbb{S}^2$) is $\{e, e'\}$. $\square$

Theorem 4.2 $\mathbb{S}^3 \subseteq \mathbb{R}^4$ contains no $(4, 2)$ -equidistant set.

Proof Suppose $X \subseteq \mathbb{S}^3$ were $(4, 2)$ -equidistant. Take distinct $a, b, p, q \in X$. By Lemma 3.2, $\{a, b, p, q\}$ is affinely independent, hence $E(\{a, b, p, q\})$ taken in $\mathbb{R}^4$

is a line K through the center of $\mathbb{S}^3$ that intersects $\mathbb{S}^3$ in two diametrically opposite points e, e' . Thus, $E(\{a, b, p, q\})$ taken in X is $\{e, e'\}$. Similarly $E(\{a, b, e, e'\})$ taken in X is s, s' where $s, s' \in \mathbb{S}^3$ are diametrically opposite. The line through e, e' intersects the line through s, s' at the center of $\mathbb{S}^3$ and hence the points e, e', s, s' are not affinely independent, a contradiction. $\square$

More generally (and with more work) we have the following.

Theorem 4.3 $\mathbb{R}^4$ contains no $(4, 2)$ -equidistant set.

Proof Suppose $X \subseteq \mathbb{R}^4$ were $(4, 2)$ -equidistant. Take distinct $e, e', p, q \in X$ and let $E(\{e, e', p, q\}) = \{a, b\}$ (taken in X). Let $E(\{e, e', a, b\}) = \{s, t\}$. Let L be the equidistance set for $\{e, e'\}$ taken in $\mathbb{R}^4$. Let K be the line through e, e' and let o be the midpoint of e, e' . Then L is the three-dimensional affine subspace of $\mathbb{R}^4$ orthogonal to K and passing through o . (One may visualize this by thinking of L as a copy of $\mathbb{R}^3$ with all objects in L colored in black, white, and shades of gray, and then place an extra red line K through o that “is in the 4th dimension” and contains e, e' .) Also, $a, b, s, t \in L$ and a, b, s, t are contained in a unique two-dimensional sphere $S \subseteq L$. The center of S must coincide with o for otherwise the two points in $E(\{a, b, s, t\})$ would determine a line parallel to K , and then $E(\{a, b, s, t\}) \cup \{e, e'\}$ would be a four-point affinely independent subset of X contained in a plane. Since o is the center of S it follows that $E(\{a, b, s, t\}) \subseteq K$ and, since $K \cap X$ has only two points, $E(\{a, b, s, t\}) = \{e, e'\}$. Let r be the radius of S , let $v = d(e, o)$ and let $c = d(e, a)$. So far we know $c^2 = r^2 + v^2$ and $d(e, a) = d(e, b) = d(e, s) = d(e, t) = d(e', a) = d(e', b) = d(e', s) = d(e', t) = d(s, a) = d(s, b) = d(t, a) = d(t, b) = c$. This implies $E(\{e, e', s, t\}) = \{a, b\}$.

In order to show that $\{e, e', a, b, s, t\}$ is a copy of $D(6)$ in $\mathbb{R}^4$ we need to show that $d(e, e') = d(a, b) = d(s, t) = c$. Let B be the equidistance set for a, b, s taken in L . Let T be the equidistance set for a, b, s taken in $\mathbb{R}^4$. Then B is the line in L that goes through the circumcenter of a, b, s , and is orthogonal to the plane P determined by a, b, s . Equivalently, B is orthogonal to P in L and contains o . Also, B is the orthogonal projection of T onto L , and hence T is the plane in $\mathbb{R}^4$ determined by the lines B and K (where K goes through e, e' and intersects B at o). The set $E(\{a, b, s, e\})$ (taken in X) is contained in T and must contain e' for otherwise $\{e, e'\} \cup E(\{a, b, s, e\})$ would be four distinct points of X in T . Say $E(\{a, b, s, e\}) = \{e', w\}$. Then $d(e', e) = d(e', a) = c$. Also, $E(\{a, b, s, e'\}) = \{e, w\}$ since $E(\{a, b, s, e'\}) \subseteq T \setminus \{e'\}$ and $T \cap X = \{e, e', w\}$. Thus $w \in E(\{a, b, e, e'\}) = \{s, t\}$, but $w \neq s$, and hence $w = t$. It follows $d(t, s) = d(t, a) = c$.

In a similar way (using that $E(\{e, e', s, t\}) = \{a, b\}$) one obtains that $d(a, b) = c$. This shows that $\mathbb{R}^4$ contains a copy of $D(6)$, a contradiction which completes the proof. $\square$

5 Comments and Open Questions

Generalizing a well-known construction by transfinite recursion of Mazurkiewicz [16] one may build, for any $n \geq 2$, a subset of $\mathbb{R}^2$ that intersects every line in exactly n points (see [1]). Such a set is $(2, n)$ -equidistant. See also [4].

One may construct a countable $(3, 2)$ -equidistant set $X \subseteq \mathbb{S}^2$. Start with any $X_1 = \{a, b, c\}$. For $j \geq 1$, let X_{j+1} be the minimal set that contains X_j and, together with every triple $\{p, q, r\} \subseteq X_j$, we require that X_{j+1} contains the two (diametrically opposite) points of $\mathbb{S}^2$ equidistant from $\{p, q, r\}$. Clearly each X_j is finite, and the set $X = \cup\{X_j : j = 1, 2, \dots\}$ is $(3, 2)$ -equidistant. One may show that any $(3, 2)$ -equidistant subset of $\mathbb{S}^2$ must be dense in $\mathbb{S}^2$ (and hence infinite). The X_1 we start with could be of any infinite cardinality not exceeding continuum, and the set we end up with will have that same cardinality.

$\mathbb{S}^2$ (and hence $\mathbb{R}^3$) also contains infinite $(3, 1)$ -equidistant sets: Start with any $(3, 2)$ -equidistant set $X \subseteq \mathbb{S}^2$ and for each pair of diametrically opposite points remove one of the two points. One may show that no $(3, 1)$ -equidistant subset of $\mathbb{S}^2$ contains a pair of diametrically opposite points (for if X is such a set that contains the north and the south poles, then a relatively short argument would show that X must contain three points on the equator).

$\mathbb{S}^2$ also contains a $(2, 2)$ -equidistant set that satisfies some special properties: no three points belong to a great circle (and in particular no two points are diametrically opposite), no four points lie in a plane, and no three points form an equilateral triangle. We may build $X = \cup\{X_j : j = 1, 2, \dots\}$ where $X_{j+1} \supseteq X_j$ is a minimal set such that if $a, b \in X_j$ then there are two points in X_{j+1} each of which is equidistant from a, b . The main idea is that given $a, b \in X_j$ the equidistance set for a, b in $\mathbb{S}^2$ is a great circle C , and there are only finitely many points in C the addition of which will create problems (e.g. the formation of an equilateral triangle with two points already present in X_j , etc.). Since C is uncountable we may pick points from C to be added to X_{j+1} that will not create problems. See [4] for related results in the plane.

There is a known example of a $(2, 2)$ -equidistant subset of the plane of exactly eight points: $(\pm 1, \pm 1), (\pm a, 0), (0, \pm a)$, where $a = 1 + \sqrt{3}$ is a solution of the equation $d((1, 1), (a, 0)) = 2$. This example is attributed to Kelly in [5] (Problem F10), where it is asked if eight is the largest (finite) number of points of a $(2, 2)$ -equidistant subset of the plane.

Since $\mathbb{P}^2$ contains $D(6)$, it contains an (m, n) -equidistant set whenever $n \geq 1$ and $m + n \leq 6$. We did not attempt to determine if $\mathbb{P}^2$ contains an infinite $(2, 4)$ -equidistant set (which might be possible to build using that the equidistance set for every two points in $\mathbb{P}^2$ is uncountable: it consists of the image of two mutually perpendicular great circles in $\mathbb{S}^2$). It seems unlikely that $\mathbb{P}^2$ would contain a $(2, 5)$ -equidistant set but we did not verify this.

It is interesting to note that if $X = \mathbb{P}^2$ then, most of the time, $E(\{p, q, r\})$ has exactly four points, and some of the time it has exactly one point. To see this we may take $a, b, c \in \mathbb{S}^2$ that map onto p, q, r , respectively, under the quotient map π . First assume that a, b, c do not lie on a great circle. Let a', b', c' be diametrically opposite to a, b, c , respectively. Let e be a circumcenter in $\mathbb{S}^2$ of a, b, c . (Then the diametrically opposite point e' would be the other circumcenter.) Let f, g, h be circumcenters of $\{a, b, c'\}$, $\{a, b', c\}$ and $\{a', b, c\}$, respectively. Then $\pi(\{e, f, g, h\}) = E(\{p, q, r\})$ in $\mathbb{P}^2$. On the other hand if a, b, c lie on a great circle, then $E(\{p, q, r\}) = \pi(e)$. It is helpful to build a model from a tennis ball and nine rubber bands to be used as great circles. Three great circles pass through $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$. Three more are the equidistance sets in $\mathbb{S}^2$ for $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$. The other three are the equidistance sets for $\{a, c'\}$, $\{b, c'\}$, and $\{a, b'\}$ (which of course are the same as the

equidistance sets for $\{a', c\}$, $\{b', c\}$, and $\{a', b\}$). Using this model it is easy to convince oneself that $\mathbb{P}^2$ contains no $(3, 4)$ -equidistant set. Indeed, if we start with a, b, c and add e, f, g, h as described earlier, then we need to also add a circumcenter for $\{a, b, e\}$, and this will produce (together with e and f) three points on a great circle, unless this circumcenter coincides with f . We leave the details to the reader.

We do not know if $\mathbb{P}^2$ contains an infinite $(3, 1)$ equidistant set.

A result of Berard [2] states that a connected metric space with the UMP is homeomorphic to an interval in $\mathbb{R}$. An article by Loveland and Wayment [15] presented (along with positive answers for some special cases) the following Double Midset Conjecture: If a continuum (i.e. a compact connected metric space) X has the DMP, then X is homeomorphic to a circle. The answer is positive if $X \subseteq \mathbb{R}^2$ with the usual metric [12]. Related results about UMP, DMP, TMP, and n -MP are found in [6–8, 13, 14, 17, 19].

The following seems to be a natural analogue of the Double Midset Conjecture. It also leads to the formulation of a more general problem.

Conjecture 5.1 If X is a $(3, 2)$ -equidistant continuum, then X is homeomorphic to a sphere (i.e. to $\mathbb{S}^2 \subseteq \mathbb{R}^3$).

Problem 5.2 (a) Find all (m, n) with $m \geq 2$ and $n \geq 1$ such that there is a metric continuum X which is (m, n) -equidistant. (b) For each (m, n) as in part (a) find all continua (up to a homeomorphism) which are (m, n) -equidistant.

We do not know the answer to the above conjecture even in the special case when $X \subseteq \mathbb{R}^3$ with the usual metric.

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