Distributions of Points in d Dimensions and Large k -Point Simplices

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Abstract We consider a variant of Heilbronn’s triangle problem by investigating for a fixed dimension $d \geq 2$ and for integers $k \geq 2$ with $k \leq d$ distributions of n points in the d -dimensional unit cube $[0, 1]^d$, such that the minimum volume of the simplices, which are determined by $(k + 1)$ of these n points is as large as possible. Denoting by $\Delta_{k,d}(n)$, the supremum of this minimum volume over all distributions of n points in $[0, 1]^d$, we show that $c_{k,d} \cdot (\log n)^{1/(d-k+1)} / n^{k/(d-k+1)} \leq \Delta_{k,d}(n) \leq c'_{k,d} / n^{k/d}$ for fixed $2 \leq k \leq d$, and, moreover, for odd integers $k \geq 1$, we show the upper bound $\Delta_{k,d}(n) \leq c''_{k,d} / n^{k/d + (k-1)/(2d(d-1))}$, where $c_{k,d}, c'_{k,d}, c''_{k,d} > 0$ are constants.

Keywords Heilbronn’s triangle problem · Hypergraphs · Independence number

1 Introduction

For integers, $n \geq 3$, Heilbronn’s problem asks for the supremum $\Delta_2(n)$ of the minimum area of a triangle formed by three of n points over all distributions of n points in the unit square $[0, 1]^2$. For primes n , no three of the points $P_k = (1/n) \cdot (l \bmod n, l^2 \bmod n)$, $l = 0, 1, \dots, n - 1$, are collinear, which gives the lower bound $\Delta_2(n) = \Omega(1/n^2)$, as has been observed by Erdős, see [17]. Komlós, Pintz and Szemerédi [12] improved this to the currently best known lower bound $\Delta_2(n) = \Omega(\log n / n^2)$, and in [7], a deterministic polynomial in n time algorithm was given, which achieves this lower bound on $\Delta_2(n)$. Upper bounds on $\Delta_2(n)$ have been given in a series of papers by Roth [17–20] and Schmidt [22]. The currently best known upper bound has been obtained by Komlós, Pintz and Szemerédi [11], who proved for some constant $c > 0$ that $\Delta_2(n) = O(2^{c\sqrt{\log n}} / n^{8/7})$. If n points are chosen uniformly at random and independently of each other in $[0, 1]^2$, the expected value of

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the minimum area of a triangle formed by three of these n points is $\Theta(1/n^3)$, as has been shown by Jiang, Li and Vitany [10].

A variant of Heilbronn's problem, which has been considered by Barequet in [2], asks for dimension $d \geq 2$, for the supremum $\Delta_d(n)$ of the minimum volume of a simplex determined by $(d+1)$ of n points in the d -dimensional unit cube $[0, 1]^d$, where the supremum is taken over all distributions of n points in $[0, 1]^d$. For fixed $d \geq 1$, it is shown in [2] that the lower bound $\Delta_d(n) = \Omega(1/n^d)$. This has been improved in [13] by a logarithmic factor to $\Delta_d(n) = \Omega(\log n/n^d)$ for fixed $d \geq 2$, and in [16], for the case of dimension $d = 3$, a deterministic polynomial in n time algorithm has been given, which achieves the lower bound $\Delta_3(n) = \Omega(\log n/n^3)$. An upper bound of $\Delta_d(n) = O(1/n)$ follows from the pigeonhole-principle. Recently, by considering the angles between lines, which are determined by pairs of points, Brass [8] improved this upper bound to $\Delta_d(n) = O(1/n^{1+1/(2d)})$ for fixed odd integers $d \geq 3$.

Here we consider the following generalization of Heilbronn's problem: given fixed integers d, k with $1 \leq k \leq d$, find for every integer $n \geq k$ a distribution of n points in the d -dimensional unit cube $[0, 1]^d$, such that the minimum k -dimensional volume of a $(k+1)$ -point simplex arising from these n points is as large as possible. Let $\Delta_{k,d}(n)$ denote the supremum—over all distributions of n points in $[0, 1]^d$ —of the minimum volume of a $(k+1)$ -point simplex among n points in $[0, 1]^d$, i.e., $\Delta_d(n) = \Delta_{d,d}(n)$.

It is easy to see that $\Delta_{1,d} = \Theta(1/n^{1/d})$ for the fixed dimension $d \geq 1$: the lower bound follows by considering the points of the standard $n^{1/d} \times \dots \times n^{1/d}$ -grid in $[0, 1]^d$, and the upper bound follows by an argument of packing balls in $[0, 1]^d$. Lower and upper bounds on $\Delta_{2,d}(n)$ for fixed $d \geq 2$, i.e., areas of triangles in $[0, 1]^d$, were given by this author in [14], where it has been shown that $c_{2,d} \cdot (\log n)^{1/(d-1)} / n^{2/(d-1)} \leq \Delta_{2,d}(n) \leq c'_{2,d} / n^{2/d}$ for constants $c_{2,d}, c'_{2,d} > 0$.

Here we prove the following lower and upper bounds on $\Delta_{k,d}(n)$:

Theorem 1 *Let d, k be fixed integers with $2 \leq k \leq d$. Then for constants $c_{k,d}, c'_{k,d}, c''_{k,d} > 0$, for every integer $n \geq k$, it is*

$$c_{k,d} \cdot \frac{(\log n)^{\frac{1}{d-k+1}}}{n^{\frac{k}{d-k+1}}} \leq \Delta_{k,d}(n) \leq \frac{c'_{k,d}}{n^{\frac{k}{d}}} \quad \text{for every } k, \quad (1)$$

$$\Delta_{k,d}(n) \leq \frac{c''_{k,d}}{n^{\frac{k}{d} + \frac{k-1}{2d(d-1)}}} \quad \text{for } k \text{ odd}. \quad (2)$$

For $d = 2$ and $k = 2$, the lower bound in (1) is just the result from [12]. For $k = d$, the upper bound in (2) yields the bound from [8], and the lower bound in (1) gives the result from [13]. For $k = 2$ and any fixed dimension $d \geq 2$, the bounds in (1) yield the above mentioned result from [14]. Indeed, our arguments for proving Theorem 1 give a randomized polynomial in n time algorithm, which finds a distribution of n points in $[0, 1]^d$ that achieves the lower bound in (1).

Independently from this work, in [4] Barequet and Naor showed—taking also attention to the involved parameters—the bounds $f(k, d)/n^{k/(d-k+1)} \leq \Delta_{k,d}(n) \leq (k^{k/d} \cdot d^{k/2})/(k! \cdot n^{k/d})$ for arbitrary integers $1 \leq k \leq d$, where the function $f(k, d)$ only depends on d, k . Note that our bounds on $\Delta_{k,d}(n)$ are better for fixed $2 \leq k \leq d$:

the lower bound in (1) by a factor of $\Theta((\log n)^{1/(d-k+1)})$, and for odd k , the upper bound (2) by a factor of $\Theta(n^{(k-1)/(2d(d-1))})$.

We remark that the on-line situation—the points are positioned one after the other in $[0, 1]^d$ and suddenly this process stops—of the variant of Heilbronn's problem for $(d+1)$ -point simplices in $[0, 1]^d$ has been investigated by Barequet [3], where he proved the existence of distributions of n points in $[0, 1]^d$ for the cases $d=3$ and $d=4$, such that the volume of every $(d+1)$ -simplex is $\Omega(1/n^{10/3})$ and $\Omega(1/n^{127/24})$, respectively. Recently, in extending these results, Barequet and Shaikhhet [5, 21] showed by packing arguments for the on-line situation the existence of configurations of n points in $[0, 1]^d$, where for fixed $k \leq d$ the volume of each $(k+1)$ -point simplex among these n points is $\Omega(1/n^{(d+1)\ln \frac{d-2}{d-k+1} + 0.735d - k + 2.8881})$ for fixed $d \geq 5$ and $3 \leq k \leq d$. Thus, with respect to the off-line situation as discussed above, where the number n of points is known in advance, so far there is a large gap between the lower bounds.

2 Notation

We introduce some notation, which is used throughout this paper.

For points $P = (p_1, \dots, p_d) \in [0, 1]^d$ and $Q = (q_1, \dots, q_d) \in [0, 1]^d$ denote by $\text{dist}(P, Q) := ((p_1 - q_1)^2 + \dots + (p_d - q_d)^2)^{1/2}$ the Euclidean distance between P and Q . A $(k+1)$ -point simplex is given by $(k+1)$ points $P_1, \dots, P_{k+1} \in [0, 1]^d$ and is defined as the convex hull of $P_1, \dots, P_{k+1}$, i.e., it is the set of all points $P_1 + \sum_{i=2}^{k+1} \lambda_i \cdot (P_i - P_1)$ with $\lambda_i \geq 0$, $i = 2, \dots, k+1$, and $\sum_{i=2}^{k+1} \lambda_i \leq 1$. The $(k$ -dimensional) volume of a $(k+1)$ -point simplex, which is determined by the points $P_1, \dots, P_{k+1} \in [0, 1]^d$, $1 \leq k \leq d$, is given by $\text{vol}(P_1, \dots, P_{k+1}) := (1/k!) \cdot \prod_{j=2}^{k+1} \text{dist}(P_j; \langle P_1, \dots, P_{j-1} \rangle)$, where $\text{dist}(P_j; \langle P_1, \dots, P_{j-1} \rangle)$ denotes the Euclidean distance of the point P_j from the affine space $\langle P_1, \dots, P_{j-1} \rangle$, which is generated by the points $P_1, \dots, P_{j-1}$. Hence, if $(k+1)$ points are contained in a $(k-1)$ -dimensional affine space, then $\text{vol}(P_1, \dots, P_{k+1}) = 0$.

In our arguments, we use hypergraphs. A hypergraph $\mathcal{G} = (V, \mathcal{E})$ with vertex-set V and edge-set $\mathcal{E}$ is called k -uniform if $|E| = k$ for each edge $E \in \mathcal{E}$. If the hypergraph $\mathcal{G}$ contains edges of different cardinalities, then $\mathcal{G}$ is called non-uniform. For a hypergraph $\mathcal{G}$, we indicate by $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_k)$ that $\mathcal{E}_i$ is the set of all i -element edges in $\mathcal{G}$, $i = 2, \dots, k$. A subset $I \subseteq V$ of the vertex-set V is called independent if no edge from $\mathcal{E}$ is contained in I . The largest size $|I|$ of an independent set in $\mathcal{G}$ is the independence number $\alpha(\mathcal{G})$ of $\mathcal{G}$. A hypergraph $\mathcal{G} = (V, \mathcal{E})$ is called linear if $|E \cap E'| \leq 1$ for all distinct edges $E, E' \in \mathcal{E}$.

3 A Lower Bound on $\Delta_{k,d}(n)$

In this section, we prove the lower bound (1) in Theorem 1, namely, that for fixed integers d, k with $2 \leq k \leq d$ there are constants $c_{k,d} > 0$, such that for every integer $n \geq k$, it is $\Delta_{k,d}(n) \geq c_{k,d} \cdot (\log n)^{\frac{1}{d-k+1}} / n^{\frac{k}{d-k+1}}$.

Proof Let d, k be fixed integers with $2 \leq k \leq d$. For arbitrary integers $n \geq k$ and a suitable constant $\beta > 0$, we select uniformly at random and independently of each other $N := n^{1+\beta}$ points $P_1, P_2, \dots, P_N$ from the d -dimensional unit cube $[0, 1]^d$.

For suitable constants $\gamma_j > 0$, $j = 2, \dots, k$, and a number $V_0 > 0$, which are fixed later in connection with Lemmas 3 and 4, we form a random non-uniform hypergraph $\mathcal{G} = \mathcal{G}(N^{-\gamma_2}, \dots, N^{-\gamma_k}, V_0) = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_{k+1})$ with vertex-set $V = \{1, 2, \dots, N\}$, where vertex i corresponds to the random point $P_i \in [0, 1]^d$, $i = 1, \dots, N$. For $j = 2, \dots, k$, let $\{i_1, \dots, i_j\} \in \mathcal{E}_j$ be a j -element edge, if and only if, the $(j-1)$ -dimensional volume of the simplex determined by the points $P_{i_1}, \dots, P_{i_j}$ is at most $N^{-\gamma_j}$, i.e., $\text{vol}(P_{i_1}, \dots, P_{i_j}) \leq N^{-\gamma_j}$. Moreover, let $\{i_1, \dots, i_{k+1}\} \in \mathcal{E}_{k+1}$ be a $(k+1)$ -element edge, if and only if, $\text{vol}(P_{i_1}, \dots, P_{i_{k+1}}) \leq V_0$ and $\{i_1, \dots, i_{k+1}\}$ does not contain any j -element edges $E \in \mathcal{E}_j$ for $j = 2, \dots, k$.

Let $I \subseteq V$ be an independent set in this hypergraph $\mathcal{G}$. Then by definition of the edge-set of $\mathcal{G}$, for distinct vertices $i_1, \dots, i_{k+1} \in I$, we infer that the volume of the simplex, which is determined by the corresponding points $P_{i_1}, \dots, P_{i_{k+1}} \in [0, 1]^d$, satisfies $\text{vol}(P_{i_1}, \dots, P_{i_{k+1}}) > V_0$. Thus, an independent set $I \subseteq V$ in $\mathcal{G}$ yields $|I|$ many points in $[0, 1]^d$, such that the volume of each simplex determined by k of these $|I|$ points is bigger than V_0 .

Our aim is to show the existence of a large independent set $I \subseteq V$ in $\mathcal{G}$. For doing so, we use an extension by Duke, Rödl and this author [9] of a result by Ajtai, Komlós, Pintz, Spencer, and Szemerédi [1] on the independence number of linear uniform hypergraphs.

Theorem 2 ([1, 9]) *Let $k \geq 2$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E}_{k+1})$ be a linear, $(k+1)$ -uniform hypergraph on $|V| = N$ vertices with average degree $t^k := (k+1) \cdot |\mathcal{E}_{k+1}|/N$.*

Then for some constant $C_{k+1}^ > 0$, the independence number $\alpha(\mathcal{G})$ of $\mathcal{G}$ satisfies*

$$\alpha(\mathcal{G}) \geq C_{k+1}^* \cdot \frac{N}{t} \cdot (\log t)^{\frac{1}{k}}. \quad (3)$$

We remark that for arbitrary $(k+1)$ -uniform hypergraphs $\mathcal{G}$ on N vertices with average degree t^k one can prove only the lower bound $\alpha(\mathcal{G}) = \Omega(N/t)$ (Turán bound) and there exist $(k+1)$ -uniform hypergraphs with an upper bound on the independence number of $O(N/t)$. However, as Theorem 2 shows, if the hypergraph $\mathcal{G}$ is linear, then there is a bigger lower bound on the independence number $\alpha(\mathcal{G})$.

The difficulty in our arguments is to find a suitable induced uniform subhypergraph $\mathcal{G}^*$ of the random nonuniform hypergraph $\mathcal{G}$, to which we can apply Theorem 2. For doing so, we estimate the numbers of edges and certain pairs of edges in $\mathcal{G}$. For $j = 2, \dots, k$, let $BP_j(\mathcal{G})$ be the set of all “bad j -pairs of $(k+1)$ -point simplices” in $\mathcal{G}$, which are those unordered pairs $\{E, E'\}$ of distinct $(k+1)$ -element edges $E, E' \in \mathcal{E}_{k+1}$ in $\mathcal{G}$, which share j vertices, i.e., $|E \cap E'| = j$. We show that in the random nonuniform hypergraph $\mathcal{G}$, the expected numbers $E(|\mathcal{E}_i|)$ and $E(|BP_j(\mathcal{G})|)$ of i -element edges and of bad j -pairs of $(k+1)$ -point simplices, respectively, $i, j = 2, \dots, k$, are not too big, i.e., and are of the order $o(N)$. Then we discard one vertex from each i -element edge $E \in \mathcal{E}_i$, $i = 2, \dots, k$, which yields a $(k+1)$ -uniform subhypergraph of $\mathcal{G}$. Moreover, we discard one vertex from each bad

j -pair of $(k+1)$ -point simplices, $j = 2, \dots, k$. This yields a *linear, $(k+1)$ -uniform* subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_{k+1}^*)$ of $\mathcal{G}$, and $\mathcal{G}^*$ fulfills the assumptions in Theorem 2, and then we apply it.

To obtain upper bounds on the expected numbers $E(|\mathcal{E}_i|)$ of i -element edges in $\mathcal{G}$, $i = 2, \dots, k+1$, we estimate for a given number $v > 0$, the probability that i points, which are chosen uniformly at random and independently of each other in $[0, 1]^d$, determine a simplex of volume at most v .

Lemma 1 *Let d, k be fixed integers with $1 \leq k \leq d$. For $i = 2, \dots, k+1$, and random points $P_1, \dots, P_i \in [0, 1]^d$ there are constants $c_i^* > 0$, such that for every number $v > 0$ it is*

$$\text{Prob}(\text{vol}(P_1, \dots, P_i) \leq v) \leq c_i^* \cdot v^{d-i+2}. \quad (4)$$

Proof Let $P_1, \dots, P_i$ be i points, which are chosen uniformly at random and independently of each other in $[0, 1]^d$. We assume that these points are numbered such that for $2 \leq g \leq h \leq i$, it is

$$\text{dist}(P_g; \langle P_1, \dots, P_{g-1} \rangle) \geq \text{dist}(P_h; \langle P_1, \dots, P_{g-1} \rangle). \quad (5)$$

The point P_1 may be anywhere in $[0, 1]^d$. Given the point $P_1 \in [0, 1]^d$, the probability that the Euclidean distance of the point $P_2 \in [0, 1]^d$ from P_1 is in the infinitesimal interval $[r_1, r_1 + dr_1]$, is at most the difference of the volumes of the d -dimensional balls with center P_1 and with radii $(r_1 + dr_1)$ and r_1 , respectively, hence

$$\text{Prob}(r_1 \leq \text{dist}(P_1, P_2) \leq r_1 + dr_1) \leq d \cdot C_d \cdot r_1^{d-1} dr_1,$$

where $C_l := \pi^{l/2} / \Gamma(l/2 + 1)$ denotes the volume of the l -dimensional unit ball in $\mathbb{R}^l$, $l = 1, \dots, d$.

Given the points $P_1, P_2 \in [0, 1]^d$ with $\text{dist}(P_1, P_2) = r_1$, the probability that the distance $\text{dist}(P_3; \langle P_1, P_2 \rangle)$ of the point $P_3 \in [0, 1]^d$ from the affine line $\langle P_1, P_2 \rangle$ is in the interval $[r_2, r_2 + dr_2]$, is at most the difference of the volumes of two cylinders, which are centered at the affine line $\langle P_1, P_2 \rangle$ and have radii $(r_2 + dr_2)$ and r_2 , respectively. By assumption (5) and the triangle inequality, both cylinders have height $2 \cdot r_1 = 2 \cdot \text{dist}(P_1, P_2)$. Thus, we infer

$$\text{Prob}(r_2 \leq \text{dist}(P_3; \langle P_1, P_2 \rangle) \leq r_2 + dr_2) \leq 2 \cdot r_1 \cdot (d-1) \cdot C_{d-1} \cdot r_2^{d-2} dr_2.$$

In general, let $P_1, \dots, P_g \in [0, 1]^d$, $g < i$, with $\text{dist}(P_f; \langle P_1, \dots, P_{f-1} \rangle) = r_{f-1}$, $f = 2, \dots, g$, be given. By (5) and the triangle inequality, the projection of the point P_{g+1} onto the affine space $\langle P_1, \dots, P_g \rangle$ is contained in a $(g-1)$ -dimensional parallelepiped of volume $2^{g-1} \cdot r_1 \cdots r_{g-1}$. If $\text{dist}(P_{g+1}; \langle P_1, \dots, P_g \rangle) \leq r$, then the point P_{g+1} is contained in the Cartesian product of a $(g-1)$ -dimensional parallelepiped of volume $2^{g-1} \cdot r_1 \cdots r_{g-1}$ and a $(d-g+1)$ -dimensional ball of radius r . Hence, for $g < i-1$, we obtain

$$\begin{aligned} \text{Prob}(r_g \leq \text{dist}(P_{g+1}; \langle P_1, \dots, P_g \rangle) \leq r_g + dr_g) \\ \leq 2^{g-1} \cdot r_1 \cdots r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g} dr_g. \end{aligned} \quad (6)$$

Moreover, for $g = i - 1$, in order to satisfy $\text{vol}(P_1, \dots, P_i) \leq v$, we must have

$$\frac{1}{(i-1)!} \cdot \text{dist}(P_i; \langle P_1, \dots, P_{i-1} \rangle) \cdot \prod_{f=2}^{i-1} r_{f-1} \leq v,$$

hence,

$$\text{dist}(P_i; \langle P_1, \dots, P_{i-1} \rangle) \leq \frac{(i-1)! \cdot v}{r_1 \cdots r_{i-2}}. \quad (7)$$

With (5) the projection of the point P_i onto the affine space $\langle P_1, \dots, P_{i-1} \rangle$ is contained in a $(i-2)$ -dimensional parallelepiped of volume $2^{i-2} \cdot r_1 \cdots r_{i-2}$. Thus, by (7) the point P_i is contained in the Cartesian product of an $(i-2)$ -dimensional parallelepiped of volume $2^{i-2} \cdot r_1 \cdots r_{i-2}$ and a $(d-i+2)$ -dimensional ball of radius $((i-1)! \cdot v)/(r_1 \cdots r_{i-2})$, which happens with probability at most

$$2^{i-2} \cdot r_1 \cdots r_{i-2} \cdot C_{d-i+2} \cdot \left(\frac{(i-1)! \cdot v}{r_1 \cdots r_{i-2}} \right)^{d-i+2}. \quad (8)$$

Summarizing the estimates (6) and (8), we obtain for constants $c_i^*, c_i^{**} > 0$:

$$\begin{aligned} & \text{Prob}(\text{vol}(P_1, \dots, P_i) \leq v) \\ & \leq \int_{r_{i-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} 2^{i-2} \cdot C_{d-i+2} \cdot \frac{((i-1)! \cdot v)^{d-i+2}}{(r_1 \cdots r_{i-2})^{d-i+1}} \\ & \quad \cdot \prod_{g=1}^{i-2} (2^{g-1} \cdot r_1 \cdots r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g}) dr_{i-2} \cdots dr_1 \\ & \leq c_i^{**} \cdot v^{d-i+2} \cdot \int_{r_{i-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} \left(\prod_{g=1}^{i-2} r_g^{2i-2g-3} \right) dr_{i-2} \cdots dr_1 \\ & \leq c_i^* \cdot v^{d-i+2} \quad \text{as } 2 \cdot i - 2 \cdot g - 3 > 0, \end{aligned}$$

which proves inequality (4). $\square$

Corollary 1 *Let d, k be fixed integers with $2 \leq k \leq d$. For $i = 2, \dots, k$, there exist constants $c_i, c_{k+1} > 0$, such that*

$$E(|\mathcal{E}_i|) \leq c_i \cdot N^{i-\gamma_i(d-i+2)} \quad \text{and} \quad E(|\mathcal{E}_{k+1}|) \leq c_{k+1} \cdot V_0^{d-k+1} \cdot N^{k+1}. \quad (9)$$

Proof There are $\binom{N}{i}$ possibilities to choose i out of the N random points $P_1, \dots, P_N \in [0, 1]^d$, and using the definition of the edge-set of $\mathcal{G}$, by (4) with $v := N^{-\gamma_i}$, $i = 2, \dots, k$, and $v := V_0$ for $i = k+1$ the inequalities (9) follow. $\square$

Next we give upper bounds on the expected numbers $E(|BP_j(\mathcal{G})|)$ of bad j -pairs of $(k+1)$ -point simplices in $\mathcal{G}$, $j = 2, \dots, k$.

Lemma 2 Let d, k be fixed integers with $2 \leq k \leq d$. For $j = 2, \dots, k$, there exist constants $c_{p,j} > 0$, such that

$$E(|BP_j(\mathcal{G})|) \leq c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)}. \quad (10)$$

Proof For $j = 2, \dots, k$, we show an upper bound of $O(V_0^{2(d-k+1)} \cdot N^{\gamma_j(d-k+1)})$ on the probability that $(2k+2-j)$ points, which are chosen uniformly at random and independently of each other in $[0, 1]^d$, form a bad j -pair of $(k+1)$ -point simplices.

Note that $\{i_1, \dots, i_{k+1}\} \in \mathcal{E}_{k+1}$, if and only if, $\text{vol}(P_{i_1}, \dots, P_{i_{k+1}}) \leq V_0$ and $\{i_1, \dots, i_{k+1}\}$ does not contain any i -element edges $E \in \mathcal{E}_i$, $i = 2, \dots, k$. Let the two $(k+1)$ -point simplices, which form a bad j -pair are given by the points $P_1, \dots, P_{k+1} \in [0, 1]^d$ and $P_1, \dots, P_j, Q_{j+1}, \dots, Q_{k+1} \in [0, 1]^d$, and both sets of points determine an edge in $\mathcal{E}_{k+1}$, hence

$$\text{vol}(P_1, \dots, P_{k+1}) \leq V_0 \quad \text{and} \quad \text{vol}(P_1, \dots, P_j, Q_{j+1}, \dots, Q_{k+1}) \leq V_0.$$

By (4) with $v := V_0$, we know that for some constant $c_{k+1}^* > 0$:

$$\text{Prob}(\text{vol}(P_1, \dots, P_{k+1}) \leq V_0) \leq c_{k+1}^* \cdot V_0^{d-k+1}. \quad (11)$$

By construction of the hypergraph $\mathcal{G}$, we have $\text{vol}(P_1, \dots, P_j) > N^{-\gamma_j}$, and we condition on this in the following. Given the points $P_1, \dots, P_j, Q_{j+1}, \dots, Q_g \in [0, 1]^d$, we infer for $g = j, \dots, k-1$:

$$\begin{aligned} \text{Prob}(r_g \leq \text{dist}(Q_{g+1}; \langle P_1, \dots, P_j, Q_{j+1}, \dots, Q_g \rangle)) &\leq r_g + \text{dr}_g \\ &\leq (\sqrt{d})^{g-1} \cdot (d+1-g) \cdot C_{d+1-g} \cdot r_g^{d-g} \text{dr}_g, \end{aligned} \quad (12)$$

since all points Q_{g+1} with $\text{dist}(Q_{g+1}; \langle P_1, \dots, P_j, Q_{j+1}, \dots, Q_g \rangle) \leq r$, are contained in the Cartesian product of a $(g-1)$ -dimensional parallelepiped of volume at most $(\sqrt{d})^{g-1}$ and a $(d+1-g)$ -dimensional ball of radius r .

Finally, given the points $P_1, \dots, P_j, Q_{j+1}, \dots, Q_k$ in the unit cube $[0, 1]^d$, such that $\text{dist}(Q_g; \langle P_1, \dots, P_j, Q_{j+1}, \dots, Q_{g-1} \rangle) = r_{g-1}$, $g = j+1, \dots, k$, to fulfill $\text{vol}(P_1, \dots, P_j, Q_{j+1}, \dots, Q_{k+1}) \leq V_0$, we must have

$$\frac{(j-1)!}{k!} \cdot \text{dist}(Q_{k+1}; \langle P_1, \dots, P_j, Q_{j+1}, \dots, Q_k \rangle) \cdot \text{vol}(P_1, \dots, P_j) \cdot \prod_{g=j}^{k-1} r_g \leq V_0,$$

thus, with $\text{vol}(P_1, \dots, P_j) > N^{-\gamma_j}$, we conclude

$$\text{dist}(Q_{k+1}; \langle P_1, \dots, P_j, Q_{j+1}, \dots, Q_k \rangle) < \frac{k!}{(j-1)!} \cdot \frac{V_0 \cdot N^{\gamma_j}}{\prod_{g=j}^{k-1} r_g}.$$

Hence, the point Q_{k+1} is contained in the Cartesian product of a $(k-1)$ -dimensional parallelepiped of volume $(\sqrt{d})^{k-1}$ and a $(d-k+1)$ -dimensional ball of radius

$(k! \cdot V_0 \cdot N^{\gamma_j}) / ((j-1)! \cdot \prod_{g=j}^{k-1} r_g)$, which happens with probability at most

$$(\sqrt{d})^{k-1} \cdot C_{d-k+1} \cdot \left(\frac{k!}{(j-1)!} \cdot \frac{V_0 \cdot N^{\gamma_j}}{\prod_{g=j}^{k-1} r_g} \right)^{d-k+1}. \quad (13)$$

Putting (11–13) together, we obtain for constants $c_{k+1}^*, c_{p,j}, c_{p,j}^* > 0$, $j = 2, \dots, k$, the following upper bound

$$\begin{aligned} & \text{Prob}(\{P_1, \dots, P_{k+1}\}, \{P_1, \dots, P_j, Q_{j+1}, \dots, Q_{k+1}\} \text{ is a bad } j\text{-pair}) \\ & \leq c_{k+1}^* \cdot V_0^{d-k+1} \cdot \int_{r_{k-1}=0}^{\sqrt{d}} \dots \int_{r_j=0}^{\sqrt{d}} d^{\frac{k-1}{2}} \cdot C_{d-k+1} \cdot \frac{(k! \cdot V_0 \cdot N^{\gamma_j})^{d-k+1}}{((j-1)! \cdot \prod_{g=j}^{k-1} r_g)^{d-k+1}} \\ & \quad \cdot \prod_{g=j}^{k-1} (d^{\frac{g-1}{2}} \cdot (d+1-g) \cdot C_{d+1-g} \cdot r_g^{d-g}) dr_{k-1} \dots dr_j \\ & \leq c_{p,j}^* \cdot V_0^{2(d-k+1)} \cdot N^{\gamma_j(d-k+1)} \cdot \int_{r_{k-1}=0}^{\sqrt{d}} \dots \int_{r_j=0}^{\sqrt{d}} \prod_{g=j}^{k-1} r_g^{k-g-1} dr_{k-1} \dots dr_j \\ & \leq c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{\gamma_j(d-k+1)} \quad \text{as } k-g-1 \geq 0. \end{aligned} \quad (14)$$

Concerning the expected numbers $E(|BP_j(\mathcal{G})|)$ of bad j -pairs of $(k+1)$ -point simplices, there are $\binom{N}{j}$ possibilities to choose the joint j points out of N random points, and less than $\binom{N}{k+1-j}$ to extend these j points to $(k+1)$ points. By (14), we infer for constants $c_{p,j} > 0$, $j = 2, \dots, k$:

$$\begin{aligned} E(|BP_j(\mathcal{G})|) & \leq \binom{N}{j} \cdot \binom{N}{k+1-j}^2 \cdot c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{\gamma_j(d-k+1)} \\ & \leq c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)}, \end{aligned}$$

which proves (10) and finishes the proof of Lemma 2. $\square$

By (9) and (10) and Markov's inequality, there exist $N = n^{1+\beta}$ points $P_1, \dots, P_N$ in $[0, 1]^d$, such that the corresponding hypergraph $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_{k+1})$ on $|V| = N$ vertices, which we consider in the following, satisfies for $i, j = 2, \dots, k$:

$$|V| = N, \quad (15)$$

$$|\mathcal{E}_i| \leq 2 \cdot k \cdot c_i \cdot N^{i-\gamma_i(d-i+2)}, \quad (16)$$

$$|\mathcal{E}_{k+1}| \leq 2 \cdot k \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^{k+1}, \quad (17)$$

$$|BP_j(\mathcal{G})| \leq 2 \cdot k \cdot c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)}. \quad (18)$$

Set for some suitable constant $c^* > 0$, which will be fixed later in connection with inequality (23):

$$V_0 := (c^*)^{\frac{k}{d-k+1}} \cdot (\log n)^{\frac{1}{d-k+1}} / n^{\frac{k}{d-k+1}}. \quad (19)$$

Lemma 3 Let d, k be fixed integers with $2 \leq k \leq d$. For $j = 2, \dots, k$, and fixed $\gamma_j > 0$ with $\gamma_j < (2 \cdot k)/((1 + \beta) \cdot (d - k + 1)) - (2 \cdot k + 1 - j)/(d - k + 1)$, it is

$$|BP_j(\mathcal{G})| = o(|V|). \quad (20)$$

Proof Using (15), (18), (19), and $N = n^{1+\beta}$, we infer

$$\begin{aligned} |BP_j(\mathcal{G})| &= o(|V|) \\ &\Leftarrow V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)} = o(N) \\ &\Leftrightarrow \frac{(\log n)^2}{n^{2k}} \cdot N^{2k+1-j+\gamma_j(d-k+1)} = o(1) \\ &\Leftrightarrow (\log n)^2 \cdot n^{(1+\beta)((2k+1-j)+\gamma_j(d-k+1))-2k} = o(1) \\ &\Leftrightarrow \gamma_j < \frac{2 \cdot k}{(1 + \beta) \cdot (d - k + 1)} - \frac{2 \cdot k + 1 - j}{d - k + 1}, \end{aligned}$$

as claimed. $\square$

Lemma 4 Let d, k be fixed integers with $2 \leq k \leq d$. For $i = 2, \dots, k$, and fixed γ_i with $\gamma_i > (i - 1)/(d - i + 2)$, it is

$$|\mathcal{E}_i| = o(|V|). \quad (21)$$

Proof By (15), (16) and using $N = n^{1+\beta}$, we infer

$$\begin{aligned} |\mathcal{E}_i| &= o(|V|) \\ &\Leftarrow N^{i-\gamma_i(d-i+2)} = o(N) \\ &\Leftrightarrow \gamma_i > \frac{i - 1}{d - i + 2}, \end{aligned}$$

as desired. $\square$

Now we fix $\gamma_i := (i - 1)/(d - i + 3/2)$, $i = 2, \dots, k$, and $\beta := 1/(8 \cdot k \cdot d)$. Certainly, for these choices of γ_i , $i = 2, \dots, k$, the assumptions of Lemma 4 are satisfied. To see that also the assumptions of Lemma 3 are fulfilled, notice that for $0 < \beta \leq 1/(8 \cdot k \cdot d)$ it is $(2 \cdot k)/(1 + \beta) \geq 2 \cdot k - 1/(4 \cdot d)$ and that

$$\begin{aligned} \frac{i - 1}{d - i + 3/2} &< \frac{2 \cdot k}{(1 + \beta) \cdot (d - k + 1)} - \frac{2 \cdot k + 1 - i}{d - k + 1} \\ &\Leftarrow \frac{i - 1}{d - i + 3/2} < \frac{i - 1 - 1/(4 \cdot d)}{d - k + 1} \\ &\Leftrightarrow i^2 - i \cdot (k + 3/2) - i/(4 \cdot d) < -k - 3/4 - 3/(8 \cdot d). \end{aligned} \quad (22)$$

For $2 \leq i \leq k$, the left-hand side of (22) achieves its maximum for $i = 2$ or $i = k$. Namely, if $i := 2$ or $i := k$, then inequality (22) is equivalent to $k > 7/4 - 1/(8 \cdot d)$

or $k > 3/2 - (2 \cdot k - 3)/(4 \cdot d)$, respectively. Both inequalities hold for $k \geq 2$. Hence, by choice of the constants γ_i and β , $i = 2, \dots, k$, the assumptions in both Lemmas 3 and 4 are fulfilled.

In the hypergraph $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_{k+1})$, we discard one vertex from each i -element edge and from each bad j -pair of $(k+1)$ -point simplices, $i, j = 2, \dots, k$. Let $V^* \subseteq V$ be the set of remaining vertices. On the vertex-set V^* , the induced subhypergraph $\mathcal{G}^*$ of $\mathcal{G}$ is $(k+1)$ -uniform and linear, hence $\mathcal{G}^* = (V^*, \mathcal{E}_{k+1}^*)$ with $\mathcal{E}_{k+1}^* := \mathcal{E}_{k+1} \cap [V^*]^{k+1}$, and fulfills $|V^*| = (1 - o(1)) \cdot |V|$ by (20) and (21). Thus, we have $|V^*| \geq N/2$ and $|\mathcal{E}_{k+1}^*| \leq |\mathcal{E}_{k+1}| \leq 2 \cdot k \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^{k+1}$ by (15) and (17), and $\mathcal{G}^*$ has average degree $t^k = (k+1) \cdot |\mathcal{E}_{k+1}^*|/|V^*| \leq 4 \cdot k^2 \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^k =: t_1^k$, hence, $t_1 = \omega(n^\beta)$ by (19) and with $N = n^{1+\beta}$. Now the assumptions in Theorem 2 are fulfilled by this subhypergraph $\mathcal{G}^*$, i.e., $\mathcal{G}^*$ is $(k+1)$ -uniform and linear, and with (3) and (19), by using that the function $f(t) = (\log t)^{1/k}/t$ is decreasing for $t \geq e^{1/k}$, we obtain with $N = n^{1+\beta}$ for constants $C_{k+1}^*, C_{k+1}^{**}, c', c_{k+1}, c^* > 0$:

$$\begin{aligned} \alpha(\mathcal{G}) &\geq \alpha(\mathcal{G}^*) \geq C_{k+1}^* \cdot \frac{|V^*|}{t} \cdot (\log t)^{\frac{1}{k}} \geq C_{k+1}^* \cdot \frac{|V^*|}{t_1} \cdot (\log t_1)^{\frac{1}{k}} \\ &\geq C_{k+1}^* \cdot \frac{N/2}{(4 \cdot k^2 \cdot c_{k+1})^{\frac{1}{k}} \cdot V_0^{\frac{d-k+1}{k}} \cdot N} \cdot (\log(4 \cdot k^2 \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^k))^{\frac{1}{k}} \\ &\geq C_{k+1}^{**} \cdot \frac{n}{c^* \cdot (\log n)^{\frac{1}{k}}} \cdot (\log((4 \cdot k^2 \cdot c_{k+1})^{\frac{1}{k}} \cdot c^* \cdot n^\beta \cdot (\log n)^{\frac{1}{k}}))^{\frac{1}{k}} \\ &\geq n, \end{aligned} \quad (23)$$

where the last inequality follows by choosing in (19) a sufficiently small constant $c^* > 0$, i.e., $c^* < C_{k+1}^* \cdot \beta^{1/k}$ for $n \geq n_0$. This choice is possible as the constants β, C_{k+1}^* do not depend on c^* . Thus, the hypergraph $\mathcal{G}$ contains an independent set $I \subseteq V$ with $|I| = n$. These n vertices yield n points among the N points in $[0, 1]^d$, such that the volume of each $(k+1)$ -point simplex among these n points is bigger than V_0 , i.e., $\Delta_{k,d}(n) = \Omega((\log n)^{1/(d-k+1)} / n^{k/(d-k+1)})$, which proves the lower bound (1) in Theorem 1. $\square$

4 Upper Bounds on $\Delta_{k,d}(n)$

Here we show the upper bounds (1) and (2) in Theorem 1, namely, that for fixed $1 \leq k \leq d$ and constants $c'_{k,d}, c''_{k,d} > 0$, the inequalities $\Delta_{k,d}(n) \leq c'_{k,d}/n^{k/d}$, and, moreover, $\Delta_{k,d}(n) \leq c''_{k,d}/n^{k/d+(k-1)/(2d(d-1))}$ for odd $k \geq 1$ hold. The first upper bound can be obtained by the pigeonhole-principle, as has been done in [4]. Our arguments for proving the second upper bound also give a proof for the first upper bound.

Proof First we prove for fixed $1 \leq k \leq d$ the general upper bound $\Delta_{k,d}(n) \leq c'_{k,d}/n^{k/d}$ for constants $c'_{k,d} > 0$. Given any n points $P_1, P_2, \dots, P_n \in [0, 1]^d$, for some number D with $0 < D \leq 1$, we construct a graph $G = G(D) = (V, E)$ with

vertex-set $V = \{1, \dots, n\}$, where vertex i corresponds to the point $P_i \in [0, 1]^d$, and with edge-set E , where $\{i, j\} \in E$, if and only if, $\text{dist}(P_i, P_j) \leq D$.

An independent set $I \subseteq V$ in this graph $G = G(D)$ yields a subset $I' \subseteq \{P_1, P_2, \dots, P_n\}$ of points in $[0, 1]^d$, such that the Euclidean distance between any two distinct points in I' is bigger than D . Each ball $B_r(P)$ with center $P \in [0, 1]^d$ and radius $r \leq 1$ satisfies $\text{vol}(B_r(P) \cap [0, 1]^d) \geq \text{vol}(B_r(P))/2^d$. The balls with radius $D/2$ and centers from an independent set I' have pairwise empty intersection. Each ball $B_{D/2}(P)$ has volume $C_d \cdot (D/2)^d$, hence $(|I'| \cdot C_d \cdot (D/2)^d)/2^d \leq \text{vol}([0, 1]^d) = 1$, and we infer for the independence number $\alpha(G)$:

$$\alpha(G) \leq \frac{4^d}{C_d \cdot D^d}. \quad (24)$$

Set $D := c/n^{1/d}$ with $c \geq ((k \cdot 2 \cdot 4^d)/C_d)^{1/d}$ a constant. Let $t := (2 \cdot |E|)/n$ denote the average degree of the graph G . If $t < 1$, then we have $|E| < n/2$, and by deleting one vertex from each edge in E , we obtain $\alpha(G) > n/2$. But then (24) yields $n/2 < 4^d/(C_d \cdot D^d)$, hence $k < 1$ by the choice of the constant c , which is a contradiction. Thus, we have $t \geq 1$ and Turán's theorem for graphs yields for the independence number the lower bound $\alpha(G) \geq n/(2 \cdot t)$. With (24), this implies

$$\frac{n}{2 \cdot t} \leq \alpha(G) \leq \frac{4^d}{C_d \cdot D^d} \implies t \geq \frac{C_d}{2 \cdot 4^d} \cdot n \cdot D^d = k, \quad (25)$$

hence, the average degree of the graph G is at least k . Then there exists a vertex $i_1 \in V$ and k edges $\{i_1, i_2\}, \dots, \{i_1, i_{k+1}\} \in E$, which are incident to i_1 . By construction, each point $P_{i_j} \in [0, 1]^d$, $j = 2, \dots, k+1$, satisfies $\text{dist}(P_{i_1}, P_{i_j}) \leq D$, in particular, $\text{dist}(P_{i_j}; \langle P_{i_1}, P_{i_2}, \dots, P_{i_{j-1}} \rangle) \leq c/n^{1/d}$, which implies $\text{vol}(P_{i_1}, \dots, P_{i_{k+1}}) \leq ((1/k!) \cdot c^k)/n^{k/d}$, i.e., $\Delta_{k,d}(n) = O(1/n^{k/d})$.

For odd integers, $k \geq 3$, we are able to improve this upper bound by taking into account also the angles between the directions, which are determined by the edges in the graph G . From (25), we obtain $|E| = n \cdot t/2 \geq (C_d \cdot n^2 \cdot D^d)/4^{d+1}$. We use a modification of an argument of Brass [8]. Each edge $\{i, j\} \in E$ determines an affine line $\langle P_i P_j \rangle$ and this line determines a direction $(P_i P_j)$, which is viewed as a vector of length 1. The volume of the surface of the d dimensional unit ball is equal to $d \cdot C_d$. Let ϕ be such that $|E| \cdot (\sin(\phi/2))^{d-1} \cdot C_{d-1} \geq \binom{k+1}{2} \cdot d \cdot C_d$, say, $\sin \phi = c'/n^{1/(d-1)}$ for a constant $c' > 0$. Then there exist $\binom{k+1}{2}$ directions $(P_i P_j)$, $\{i, j\} \in E$, with a pairwise angle between them at most ϕ . The corresponding set $E^* \subseteq E$ of $\binom{k+1}{2}$ edges covers a subset $S \subseteq V$ of at least $(k+1)$ vertices. Consider a minimum subset $E^{**} \subseteq E^*$ of edges, which covers a subset $S^* \subseteq S$ of exactly $(k+1)$ vertices in G . Since $(k+1)$ is even, this set E^{**} exists and contains only isolated edges and stars, thus $|E^{**}| \geq (k+1)/2$. We pick one vertex from each isolated edge $E \in E^{**}$ and the center of each star. Let $S^{**} \subseteq S^*$ be the set of chosen vertices with $|S^{**}| = s \leq (k+1)/2$.

For each vertex, $v \in S^* \setminus S^{**}$, there exists an edge $\{v, w\} \in E^{**}$ for some vertex $w \in S^{**}$, hence, $\text{dist}(P_v, P_w) \leq D$. Having fixed such vertices $v \in S^* \setminus S^{**}$ and $w \in S^{**}$, for each vertex $u \in S^* \setminus (S^{**} \cup \{v\})$, there exists some vertex $t \in S^{**}$, such that $\{u, t\} \in E^{**}$ and the angle between the directions $(P_u P_t)$ and $(P_v P_w)$ is at

most ϕ . Thus, the Euclidean distance of the point P_u from the affine space generated by the points $P_r, r \in S^{**} \cup \{v\}$, is at most $D \cdot \sin \phi$. The points $P_q, q \in S^{**}$, pairwise have Euclidean distance at most $\sqrt{d}$, hence, the $((s-1)$ -dimensional) volume of the corresponding simplex satisfies $\text{vol}(P_q : q \in S^{**}) \leq (\sqrt{d})^{s-1} / ((s-1)!).$

With $D = c/n^{1/d}$, and $\sin \phi = c'/n^{1/(d-1)}$ and $s \leq (k+1)/2$, for a constant $c''_{k,d} > 0$, we obtain for the volume of the simplex, which is determined by the $(k+1)$ points $P_s, s \in S^*$, the following upper bound

$$\begin{aligned} \text{vol}(P_{s^*}; s^* \in S^*) &\leq \frac{1}{k!} \cdot (\sqrt{d})^{s-1} \cdot D \cdot (D \cdot \sin \phi)^{k-s} \\ &\leq \frac{1}{k!} \cdot d^{\frac{k-1}{4}} \cdot D \cdot \left(\frac{c' \cdot D}{n^{\frac{1}{d-1}}} \right)^{\frac{k-1}{2}} = \frac{c''_{k,d}}{n^{\frac{k}{d} + \frac{k-1}{2d(d-1)}}}, \end{aligned}$$

which finishes the proof of Theorem 1. $\square$

5 Concluding Remarks

The arguments, which were presented here, together with an algorithmic version of Theorem 2, see [6], yield a randomized polynomial in n time algorithm for obtaining a distribution of n points in $[0, 1]^d$, which shows the lower bound $\Delta_{k,d}(n) = \Omega((\log n)^{1/k} / n^{k/(d-k+1)})$ for fixed $2 \leq k \leq d$. In view of the recent results of Barequet and Shaikhet [5], it might be of some interest to achieve similar lower bounds, say $\Omega(1/n^{k/(d-k+1)})$ in the on-line situation. Moreover, it might be of interest to get a deterministic polynomial in n time algorithm, which achieves the lower bound $\Delta_{k,d}(n) = \Omega((\log n)^{1/k} / n^{k/(d-k+1)})$, as well as investigating the case $k > d+1$, but so far concerning this case only partial results are known; compare [15] for dimension $d=2$, where the area of the convex hull of $(k+1)$ points among n points in $[0, 1]^2$ has been considered.

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