Piercing Families of Convex Sets with the d -Intersection Property in $\mathbb{R}^d$

R.N. Karasev

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Abstract In this paper, we consider finite families of convex sets in $\mathbb{R}^d$ such that every d or fewer sets of the family have a common point. For some families of this type, we give upper bounds on the size of a finite set intersecting all sets of the family.

Keywords Piercing · Helly's theorem · Transversal

1 Introduction

We consider finite families of convex sets in $\mathbb{R}^d$ such that every d or fewer sets have a common point. We are interested in the minimal size of a finite set in $\mathbb{R}^d$ having a common point with every member of the family.

We begin with several definitions:

Definition A t -transversal for a family of sets $\mathcal{F}$ is a set T of cardinality t , such that $\forall S \in \mathcal{F}, S \cap T \neq \emptyset$.

Definition The minimal t such that a t -transversal of the family $\mathcal{F}$ exists is called the *transversal number* or *piercing number* of $\mathcal{F}$, and is denoted $\tau(\mathcal{F})$.

Definition A family of sets $\mathcal{F}$ has *property Π_k* if, for any nonempty $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| \leq k$, the intersection $\bigcap \mathcal{G}$ is not empty.

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R.N. Karasev (✉)

Dept. of Mathematics, Moscow Institute of Physics and Technology, Institutskiy per. 9,
Dolgoprudny, Russia 141700

e-mail: r_n_karasev@mail.ru

Helly's theorem states that the property Π_{d+1} implies $\tau(\mathcal{F}) = 1$ for any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$. We can see that the property Π_d holding in $\mathbb{R}^d$ is closest to Π_{d+1} , and we may expect reasonable upper bounds on $\tau(\mathcal{F})$ in this case. Of course, the example of hyperplanes in general position shows that the property Π_d alone cannot guarantee any bound on $\tau(\mathcal{F})$. Hence, we have to impose limitations on the family $\mathcal{F}$ in order to obtain bounds for the piercing number. In this paper, we mostly consider families of homothets or translates of some compact convex set.

Hereafter, we consider finite families of closed convex sets in $\mathbb{R}^d$ ($d \geq 2$) having the property Π_d . We show that for certain families, the property Π_d implies $\tau(\mathcal{F}) \leq d + 1$. We also give an upper bound which is linear in d for the piercing number of families of Euclidean balls in $\mathbb{R}^d$ with the property Π_d .

The simplest result of this type (Grünbaum's conjecture) is the following, proven by the author in [4].

Theorem 1 *For a family $\mathcal{F}$ of translates of a two-dimensional convex compact set, Π_2 implies $\tau(\mathcal{F}) \leq 3$.*

In this paper, we use the same main idea as in [4], but with improved reasoning which leads to several new results.

Theorem 2 *Let $\mathcal{F}$ be a family of homothets of a centrally symmetric convex compact set in $\mathbb{R}^2$ such that the sizes of sets in $\mathcal{F}$ are no more than two times different. The property $\Pi_2(\mathcal{F})$ then implies $\tau(\mathcal{F}) \leq 3$.*

In [3], Grünbaum proved the upper bound $\tau(\mathcal{F}) \leq 7$ without any size constraint. Theorem 2 gives a lower piercing number with size constraint. The bound $\tau(\mathcal{F}) \leq 3$ in this theorem is tight and is well known to be tight even for families of equal unit disks.

Using the same technique, we prove another result of this kind for Euclidean balls in $\mathbb{R}^d$.

Theorem 3 *Let $\mathcal{F}$ be a family of Euclidean balls in $\mathbb{R}^d$ with radii no more than d times different. The property $\Pi_d(\mathcal{F})$ then implies $\tau(\mathcal{F}) \leq d + 1$.*

The existence of a $(d + 1)$ -transversal for a family of equal balls with the property Π_d is proved in [1]. Theorem 3 generalizes this result to families of balls with size constraint, although the author is not sure that the bound $\tau(\mathcal{F}) \leq d + 1$ in Theorem 3 is tight. Using this result, we can give an upper bound on the piercing number of a family of balls in $\mathbb{R}^d$ with the property Π_d and without any size constraint as follows.

Theorem 4 *Let $\mathcal{F}$ be a family of Euclidean balls in $\mathbb{R}^d$. The property $\Pi_d(\mathcal{F})$ then implies $\tau(\mathcal{F}) \leq 3(d + 1)$ when $d \geq 5$, and $\tau(\mathcal{F}) \leq 4(d + 1)$ when $d \leq 4$.*

It seems that the bound in this theorem can be improved, especially for small d . In the case $d = 2$, Theorem 4 gives $\tau(\mathcal{F}) \leq 12$, while Danzer's result (see [1, 2]) gives $\tau(\mathcal{F}) \leq 4$. For a family of positive homothets of a simplex in $\mathbb{R}^d$, we have the bound $d + 1$ without size constraints.

Theorem 5 *Let $\mathcal{F}$ be a family of positive homothets of a simplex in $\mathbb{R}^d$. The property $\Pi_d(\mathcal{F})$ then implies $\tau(\mathcal{F}) \leq d + 1$.*

The author does not know whether Theorem 5 gives a tight upper bound on the piercing number for $d > 2$. For $d = 2$, the bound is tight, even for families of equal triangles. In this case, the piercing problem is equivalent to the following covering problem: If every two points of a closed set $S \subset \mathbb{R}^2$ can be covered by a translate of triangle T , then can S be covered by three translates of T ? Taking $S = -T$, we can see that three translates are necessary. Here, the set S is infinite, but by standard compactness reasoning, some of its finite subsets still needs three translates of T to be covered.

Theorem 3 can be generalized to the case where the sets in the family do not have to be homothets of each other. We need some definitions to formulate the result.

Definition A convex compact set in $\mathbb{R}^d$ is called *R-upper convex* if it is an intersection of balls of radius R .

Definition A convex compact set in $\mathbb{R}^d$ is called *R-lower convex* if it is a union of balls of radius R .

Theorem 6 *Let $\mathcal{F}$ be a family of convex compact sets in $\mathbb{R}^d$. Let every set in $\mathcal{F}$ be R-lower convex and dR-upper convex for some constant $R > 0$. The property $\Pi_d(\mathcal{F})$ then implies $\tau(\mathcal{F}) \leq d + 1$.*

The following question remains open: Does the upper bound $\tau(\mathcal{F}) \leq d + 1$ (or some other bound linear in d) hold for families of translates of a convex compact set?

2 Some Consequences of the Property Π_d

To prove theorems in this paper, we explicitly construct a set with $d + 1$ elements and prove that it is a transversal for $\mathcal{F}$. This construction only uses the property Π_d of some family $\mathcal{F}$ of convex closed sets in $\mathbb{R}^d$, but it does not provide a transversal of $\mathcal{F}$ in the general case. First, we define a family of half-spaces that can test the nonexistence of a common point of a family $\mathcal{F}$ of compact convex sets.

Definition Let $\{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F}$. A family of half-spaces

$$\{H_1, H_2, \dots, H_{d+1}\},$$

where, for any $i = 1, \dots, d + 1$, $K_i \subseteq H_i$, is called a *test family of half-spaces* for the family $\mathcal{F}$.

Definition If a test family of half-spaces for $\mathcal{F}$ has an empty intersection, we call it a *nonintersecting* test family of half-spaces.

We need some lemmas.

Lemma 1 Let $\mathcal{F}$ have the property Π_d . Then, for every nonintersecting test family of half-spaces $\{H_1, H_2, \dots, H_{d+1}\}$ for $\mathcal{F}$, the set

$$S(H_1, H_2, \dots, H_{d+1}) = \text{cl}\left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} H_i\right)$$

is a simplex.

Proof The family $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$ has an empty intersection, while all of its proper subfamilies have a common point. For $i = 1, \dots, d+1$, let

$$x_i \in \bigcap_{j \neq i} H_j.$$

If the points $\{x_i\}_{i=1}^{d+1}$ do not form a simplex, then they lie in a $(d-1)$ -dimensional hyperplane and by Radon's theorem, the set of indices $[d+1]$ can be partitioned into I_1 and I_2 so that there exists

$$x \in \text{conv}\{x_i\}_{i \in I_1} \cap \text{conv}\{x_i\}_{i \in I_2}.$$

However, in this case, $x \in \bigcap \mathcal{H}$.

Hence, the points $\{x_i\}_{i=1}^{d+1}$ form a simplex S and the half-spaces H_i contain its respective facets. If the family $\mathcal{H}$ covers S , then by Sperner's lemma, $\bigcap \mathcal{H} \cap S \neq \emptyset$, which is not true. Thus, we have a point $x \in \text{int } S$ such that for all $i = 1, \dots, d+1$, $x \notin H_i$.

Consider some $y \in \mathbb{R}^d \setminus S$. If y is not contained in any half-space of $\mathcal{H}$, then the segment $[x, y]$ does not intersect any half-space of $\mathcal{H}$ (this is true because the complement of a half-space is convex). Hence, $z = [x, y] \cap \text{bd } S$ is not contained in any of the half-spaces in $\mathcal{H}$. This contradicts with the fact that $\text{bd } S$ is covered by $\mathcal{H}$. Thus, we have $\mathbb{R}^d \setminus S \subseteq \bigcup \mathcal{H}$, and $S(H_1, H_2, \dots, H_{d+1}) = \text{cl}(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} H_i)$ is a polytope with nonempty interior and $d+1$ facets; therefore, it must be a d -dimensional simplex. $\square$

Lemma 2 If a family $\mathcal{F}$ with the property Π_d does not have a nonintersecting test family of half-spaces, then $\bigcap \mathcal{F} \neq \emptyset$.

Proof If $|\mathcal{F}| \leq d$, the statement is obvious. Otherwise, by Helly's theorem, we find $\mathcal{K} = \{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F}$ such that $\bigcap \mathcal{K} = \emptyset$. For $i = 1, \dots, d+1$, let L_i denote the set of linear functions $l(x)$ such that $l(x) \leq 0$ for all $x \in K_i$. By the standard properties of convex duality, if $\bigcap \mathcal{K} = \emptyset$, then the constant function $l(x) \equiv 1 \in \text{conv} \bigcup_{i=1}^{d+1} L_i$. Equivalently, there are nonnegative numbers $\{a_i\}_{i=1}^{d+1}$ and functions $\{l_i\}_{i=1}^{d+1}$ ($l_i \in L_i \forall i = 1, \dots, d+1$) such that

$$1 = \sum_{i=1}^{d+1} a_i l_i(x).$$

Let $H_i = \{x : l_i(x) \leq 0\}$. These sets have no common point because the above equality cannot hold at a common point. For all $i = 1, \dots, d+1$, $K_i \subseteq H_i$, and each H_i either is a half-space or coincides with the whole of $\mathbb{R}^d$. If we replace those H_i 's that are equal to $\mathbb{R}^d$ with arbitrary half-spaces containing the respective K_i 's, we obtain a nonintersecting test family of half-spaces. $\square$

Lemma 3 *Let the family $\mathcal{F}$ have the property Π_d . Then, either $\tau(\mathcal{F}) = 1$, or there exists a nonintersecting test family of half-spaces $\{H_1, H_2, \dots, H_{d+1}\}$ such that the volume of $S(H_1, H_2, \dots, H_{d+1})$ is maximal over all nonintersecting test families of half-spaces.*

Proof If there is no nonintersecting test family of half-spaces, then by Lemma 2, $\tau(\mathcal{F}) = 1$.

Otherwise, it is sufficient to show that for every fixed subfamily

$$\mathcal{K} = \{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F}$$

with empty intersection, the maximum of $\text{vol } S(H_1, H_2, \dots, H_{d+1})$ over all nonintersecting test families $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$ corresponding to $\mathcal{K}$ is attained. This is sufficient to prove the lemma since the number of subfamilies $\mathcal{K}$ is finite.

Any d sets from the family $\mathcal{K}$ have a nonempty intersection, so we find that for all $i = 1, \dots, d+1$,

$$x_i \in \bigcap_{j \neq i} K_j.$$

As in the proof of Lemma 1,

$$S(H_1, H_2, \dots, H_{d+1}) \subseteq \text{conv}\{x_1, x_2, \dots, x_{d+1}\};$$

hence, $\text{vol}(S(H_1, H_2, \dots, H_{d+1}))$ is bounded.

It is clear that if we take, instead of H_i , the unique inner support half-space for K_i contained in H_i , the volume of $S(H_1, H_2, \dots, H_{d+1})$ increases. Every support half-space of K_i is determined by its unit normal vector n_i , so the variety of the families of support half-spaces for the sets of $\mathcal{K}$ has the topology of a Cartesian product of $d+1$ unit spheres. We consider the subset X of this space that corresponds to nonintersecting test families for $\mathcal{K}$. The vertices of $S(H_1, H_2, \dots, H_{d+1})$ and its volume are continuous functions on X .

Let $\sup_X \text{vol}(S(H_1, H_2, \dots, H_{d+1})) = 2\varepsilon > 0$. We can then easily check that the variety of nonintersecting test families $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$ with

$$\text{vol}(S(H_1, H_2, \dots, H_{d+1})) \geq \varepsilon$$

is closed; hence, the volume attains its maximum value 2ε at some point of X . $\square$

The above lemmas allow the following definition.

Definition Let $\mathcal{F}$ be a finite family of convex closed sets in $\mathbb{R}^d$ with the property Π_d , and let $\bigcap \mathcal{F} = \emptyset$. Denote by $S(\mathcal{F})$ the simplex $S(H_1, H_2, \dots, H_{d+1})$ with maximal volume for some nonintersecting test family of half-spaces $\{H_1, H_2, \dots, H_{d+1}\}$. Denote by $s(\mathcal{F})$ the convex hull of the mass centers of the facets of $S(\mathcal{F})$.

The importance of these simplices is explained by the following lemma.

Lemma 4 *Let $\mathcal{F}$ be a finite family of convex closed sets in $\mathbb{R}^d$ with the property Π_d and having no common point. In this case, for any $S \in \mathcal{F}$, $S \cap s(\mathcal{F}) \neq \emptyset$.*

To prove Lemma 4, we need another lemma.

Lemma 5 *Let S be a simplex in $\mathbb{R}^d$ with vertices $\{v_1, v_2, \dots, v_{d+1}\}$, and let S' be a simplex with vertices $\{v'_1, v'_2, \dots, v'_d, v_{d+1}\}$. Let the points v'_i ($i = 1, \dots, d$) lie on the rays $v_{d+1}v_i$, respectively. If $\text{int } S'$ contains the mass center of $\{v_1, \dots, v_d\}$, then $\text{vol } S' > \text{vol } S$.*

Proof Take a coordinate frame with origin v_{d+1} and base

$$\{v'_1 - v_{d+1}, v'_2 - v_{d+1}, \dots, v'_d - v_{d+1}\}.$$

Let the i th coordinate of v_i be x_i . If $\text{int } S'$ contains the mass center of $\{v_1, \dots, v_d\}$, then

$$\sum_{i=1}^d x_i < d.$$

Then, by the mean inequality, $\prod_{i=1}^d x_i < 1$, which implies $\text{vol } S' > \text{vol } S$. $\square$

Proof of Lemma 4 Consider the subfamily

$$\mathcal{K} = \{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F}$$

corresponding to the nonintersecting test family of half-spaces

$$\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\},$$

where $S(\mathcal{F}) = S(H_1, H_2, \dots, H_{d+1})$. Denote the vertices of $S(\mathcal{F})$ and $s(\mathcal{F})$ by $\{v_1, v_2, \dots, v_{d+1}\}$ and $\{w_1, w_2, \dots, w_{d+1}\}$, respectively.

The volume of $S(\mathcal{F})$ is maximal, so each K_i has to touch its respective facet of $S(\mathcal{F})$.

Assume the contrary—that K_i does not intersect $s(\mathcal{F})$.

Let C_i be a cone with vertex w_i spanned by rays $w_i w_j$ ($j \neq i$). K_i cannot intersect C_i because these sets are separated by $\text{bd } H_i$ and can only intersect in $w_i \in s(\mathcal{F})$.

There then exists a half-space $H'_i \supseteq K_i$ such that $H'_i \cap C_i = \emptyset$. The family of half-spaces $(\mathcal{H} \setminus \{H_i\}) \cup \{H'_i\}$ is nonintersecting ($\bigcap_{j \neq i} H_j \subseteq C_i$) and is a test family. By Lemma 5,

$$\text{vol } S(H_1, \dots, H'_i, \dots, H_{d+1}) > \text{vol } S(\mathcal{F}),$$

and we have a contradiction.

Now, take $K \in \mathcal{F} \setminus \mathcal{K}$ and assume that $K \cap s(\mathcal{F}) = \emptyset$. There exists a half-space $H \supseteq K$ that does not intersect $s(\mathcal{F})$. $H \not\supseteq S(\mathcal{F})$, and we find the vertex of $S(\mathcal{F})$ farthest from H and denote it v_i . $\bigcap_{j \neq i} H_j$ is then a cone spanned by the rays that have

their origins at v_i and do not intersect H . Hence, $H \cap \bigcap_{j \neq i} H_j = \emptyset$, and the family $(\mathcal{H} \setminus \{H_i\}) \cup \{H\}$ has no common point. Considering simplices $S((\mathcal{H} \setminus \{H_i\}) \cup \{H\})$ and $S(\mathcal{F})$, we again arrive at a contradiction via Lemma 5. $\square$

We need another lemma to prove Theorem 4.

Lemma 6 *Suppose we have a simplex S in $\mathbb{R}^d$ ($d \geq 5$), the outer half-spaces of which correspond to facets being $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$. Let a ball B of radius R and the family $\mathcal{H}$ together form a family with the property Π_d . The radius r of the ball inscribed in S then satisfies the inequality*

$$r \leq \frac{1}{\lfloor \frac{d+1}{2} \rfloor - 1} R \leq \frac{1}{2} R.$$

Proof Let the vertices of S be $\{v_1, v_2, \dots, v_{d+1}\}$. Consider the orthogonal projection $p: \mathbb{R}^d \rightarrow V$ taking pairs $\{v_1, v_2\}, \{v_3, v_4\}$, etc. to single points. If $d+1$ is odd, let p take v_{d-1}, v_d, v_{d+1} to one point; otherwise, p takes pairs to single points. We have a simplex $S' = p(S)$ with $d' + 1 = \lfloor \frac{d+1}{2} \rfloor$ vertices. Let its outer half-spaces of facets be $\mathcal{H}' = \{H'_1, H'_2, \dots, H'_{d'+1}\}$. We also have a ball $B' = p(B)$.

In the barycentric coordinates $\{x_1, x_2, \dots, x_{d+1}\}$ of S and $\{y_1, y_2, \dots, y_{d'+1}\}$ of S' , the map p is given by

$$\begin{aligned} y_1 &= x_1 + x_2 \\ y_2 &= x_2 + x_3 \\ &\vdots \\ y_{d'+1} &= x_d + x_{d+1} \quad \text{or} \quad y_{d'+1} = x_{d-1} + x_d + x_{d+1}. \end{aligned}$$

It is easy to see that B' has a common point with every family $\mathcal{H}' \setminus \{H'_i\}$. These common points form a simplex containing S' , hence, B' contains S' . We then construct a homothety with its center at the mass center of S' and with scale ratio $-\frac{1}{d'}$. This takes B' to the ball B'' of radius $\frac{1}{d'} R$ that intersects all facets of S' . So, the radius of the inscribed ball of S' cannot be larger than $\frac{1}{d'} R$ since the inscribed ball is the ball of minimal radius intersecting all facets of the simplex. Hence, the radius of the inscribed ball of S also cannot be larger than $\frac{1}{d'} R$. $\square$

3 Concave Simplices of Curvature Radius R

We require the following definitions.

Definition Let a family $\{B_1, B_2, \dots, B_{d+1}\}$ of Euclidean balls of radius R and with centers $\{o_1, o_2, \dots, o_{d+1}\}$ in $\mathbb{R}^d$ have the following property: All d -balls have a common point, while the whole family has no common point. In this case, we call the set

$$\text{cl}\left(\text{conv}\{o_1, o_2, \dots, o_{d+1}\} \setminus \bigcup_{i=1}^{d+1} B_i\right)$$

a concave simplex of curvature radius R .

In the plane, we consider an arbitrary norm $\|\cdot\|$ and introduce the following non-Euclidean version of the above definition.

Definition Let a family $\{B_1, B_2, B_3\}$ of balls of radius R and with centers $\{o_1, o_2, o_3\}$ with respect to the norm $\|\cdot\|$ in the plane have the following property: Every two-balls intersect, while the whole family has no common point. The set

$$\text{cl}\left(\text{conv}\{o_1, o_2, o_3\} \setminus \bigcup_{i=1}^3 B_i\right)$$

is called a concave simplex of curvature radius R in norm $\|\cdot\|$.

By Sperner's lemma, the simplices defined above are nonempty; otherwise, the family would have a common point. We could define a concave simplex for an arbitrary norm in any dimension, but in general, these do not have the properties we want because the bounded connected component of $\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B_i$ does not have to lie in $\text{conv}\{o_1, o_2, \dots, o_{d+1}\}$ for $d > 2$.

We now prove several lemmas about concave simplices of curvature radius R . Unless specially noted, our reasoning is valid for the cases of both an arbitrary two-dimensional norm and the Euclidean norm in $\mathbb{R}^d$.

Definition Let T be a concave simplex in $\mathbb{R}^d$. A vertex of T is a point on its boundary that belongs to the boundaries of exactly d balls that determine T .

Lemma 7 Let T be a concave simplex of curvature radius R . Then,

- (1) $\text{int } T$ does not intersect the boundary of $S = \text{conv}\{o_1, o_2, \dots, o_{d+1}\}$;
- (2) T is contained in the convex hull of its vertices;
- (3) the number of vertices of T is $d + 1$.

Proof The first statement can be deduced from the fact that each facet of S is contained in a union of balls of radius R with centers in the vertices of the facet.

Each extremal point of T has to be its vertex; hence, T is contained in the convex hull of its vertices.

Each vertex of T corresponds to the intersection of d spheres of radius R . The centers of these spheres form a facet F of S . If the spheres have two intersection points, then their intersection points lie on different sides of the hyperplane of F , so only one of the points can be a vertex of T . In the case of the plane, when the balls are not strictly convex, the intersection of the spheres can be a line segment. But, in this case, a vertex of T is necessarily an endpoint of the segment, and one of the endpoints is on the other side of F .

Finally, we have no more than $d + 1$ vertices of T ; their number cannot be less because $\text{int } T \neq \emptyset$. $\square$

Lemma 8 *A concave simplex T of curvature radius R in the two-dimensional case has a diameter no more than R in the norm $\|\cdot\|$; in the d -dimensional case, it has a diameter no more than $\frac{2}{\sqrt{d^2-1}}R < R$.*

Proof In the two-dimensional case, T lies in the triangle formed by midpoints of the sides of the triangle $o_1o_2o_3$; therefore, its diameter does not exceed R .

In the case of Euclidean balls, consider the point o equidistant from $o_1, o_2, \dots, o_{d+1}$. It is not contained in any of the balls $B_{o_i}(R)$ and is contained in $\text{conv}\{o_1, o_2, \dots, o_{d+1}\}$, hence, $o \in T$. The result of Danzer (see Theorem 6.8 in [1]) implies that $|oo_i| \leq \frac{d}{\sqrt{d^2-1}}R$. The length of any tangent from o to the ball $B_{o_i}(R)$ does not exceed $\frac{1}{\sqrt{d^2-1}}R$ and any line segment from o to a point on $\text{bd } B_{o_i}(R)$ that does not intersect $\text{int } B_{o_i}(R)$ has length no more than $\frac{1}{\sqrt{d^2-1}}R$.

Every vertex of T lies on the radical axis of d (all but one) of the balls $B_{o_i}(R)$, and this radical axis passes through o . Thus, all segments ov lie in T and do not intersect $\text{int } B_{o_i}(R)$. Hence, all segments ov have length no more than $\frac{1}{\sqrt{d^2-1}}R$, which implies the statement of the lemma. $\square$

Lemma 9 *A ball of radius $R' \geq R$ intersects a concave simplex T of curvature radius R if and only if it contains some vertex of T .*

Proof Consider all balls of radius R' that intersect T . Denote the set of their centers by X . Consider all balls of radius R' that contain some vertex of T . Denote the set of their centers by Y . It is clear that

$$Y = \bigcup_{i=1}^{d+1} B_{o_i}(R').$$

By Lemma 8, $\text{diam } T \leq R$ and each ball $B_{o_i}(R')$ contains T . Thus, Y is star-shaped with any center in T . Similarly, X is star-shaped with any center in T . To prove the inclusion $X \subseteq Y$, we only have to prove that $\text{bd } X \subseteq Y$.

Let $x \in \text{bd } X$ and $B = B_x(R')$. Clearly, $B \cap \text{int } T = \emptyset$ and $B \cap \text{bd } T \neq \emptyset$. First, consider the case of the Euclidean norm.

Let $y \in \text{bd } X \cap B$. If y is not a vertex of T , then it lies on at most $d-1$ spheres of radius R that determine T . Let their centers be $o_1, \dots, o_k$, $k \leq d-1$. Considering the linear approximation of T and B in some neighborhood of y , we see that $\text{int } X \cap \text{int } B = \emptyset$ implies that

$$x - y = \sum_{i=1}^k \alpha_i(o_i - y) \quad \forall i = 1, \dots, k, \quad \alpha_i \geq 0.$$

If there is only one index i such that $\alpha_i \neq 0$, then for such i , we have $B \supseteq B_{o_i}(R)$, and in this case, B contains at least d vertices of T that lie on $\text{bd } B_{o_i}(R)$. Otherwise, consider the affine subspace

$$L = \{p \in \mathbb{R}^d : |p - o_1| = |p - o_2| = \dots = |p - o_k|\}.$$

We have $\dim L \geq 2$ and $y \in L$. All intersections $L \cap B_{o_i}(R)$, $i = 1, \dots, k$, are the same ball, $B' \subset L$, and $L \cap B$ is another ball, $B'' \subset L$.

The direction of the vector $x - y$ does not coincide with any of the $o_i - y$, but lies in their cone hull. All the angles between $o_i - y$ and L are equal to some α , so the angle between $x - y$ and L is less than α ; hence the radius of B' is less than the radius of B'' . The intersection $\text{int } T \cap L$ in some neighborhood of y coincides with $L \setminus B'$, and B'' intersects $L \setminus B'$ in any neighborhood of y . This contradicts the fact that $B \cap \text{int } T = \emptyset$.

The case of $d = 2$ and arbitrary norm with smooth and strictly convex ball is treated in the same manner. In this case, $k = 1$, and we do not have to use an equidistant set (which is not an affine subspace in general). We can then approximate any norm by a norm with a smooth and strictly convex ball and proceed by going to a limit. $\square$

Lemma 10 *An R -lower convex set intersects a concave simplex T of curvature radius R if and only if it contains a vertex of T .*

Proof This can be trivially deduced from Lemma 9. $\square$

4 Proofs of the Theorems

Proof of Theorems 2 and 3 If $\tau(\mathcal{F}) = 1$, the statement is true. Otherwise, by Lemma 3, we have $S = S(\mathcal{F})$ and $s = s(\mathcal{F})$. Elements of $\mathcal{F}$ are called *balls*. Denote the maximal radius of the balls in $\mathcal{F}$ by R . Take the balls $K_1, K_2, \dots, K_{d+1}$ corresponding to S ; these touch the facets of S in the respective vertices of s . Consider the balls $K'_1, K'_2, \dots, K'_{d+1}$ with radius R touching the respective facets of S in vertices of s .

It can be easily seen that $K'_i \supseteq K_i$ for all $i = 1, \dots, d + 1$. Every d -ball of the family K'_i has a common point, while all $d + 1$ balls of K'_i have no common point. Hence, the balls of K'_i define a concave simplex T of radius R . Obviously, $T \supseteq S$. Let the homothety with scale ratio $-1/d$ that takes S to s take T to T' . The latter concave simplex has curvature radius R/d .

By Lemma 4, any $K \in \mathcal{F}$ intersects s and, therefore, intersects T' ; its radius is at least R/d . Hence, by Lemma 9, K contains one of the $d + 1$ vertices of T' . $\square$

Proof of Theorem 4 If $\mathcal{F}$ has a common point, there is nothing to prove. Otherwise, consider the ball $B(\mathcal{F})$ of minimal radius r intersecting every ball in $\mathcal{F}$. By a well-known consequence of Helly's theorem, this ball is the ball of minimal radius intersecting $d + 1$ balls of some subfamily $\mathcal{F}' = \{B_1, B_2, \dots, B_{d+1}\} \subseteq \mathcal{F}$ (remember that $\mathcal{F}$ has the property Π_d) and each intersection $B(\mathcal{F}) \cap B_i$ is exactly one point v_i . Let H_i be the outer support half-space for $B(\mathcal{F})$ in the point v_i . $S = \mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} \text{int } H_i$ is then a simplex with inscribed ball $B(\mathcal{F})$.

Let the ball $B \in \mathcal{F}$ have minimal radius R . First, consider the case $d \geq 5$. Obviously, B and S satisfy the conditions of Lemma 6, and we have $r \leq 1/2R$. Note that

a ball of radius r can be put into a concave simplex of curvature radius

$$R' = \frac{1}{\frac{d}{\sqrt{d^2-1}} - 1} r = (d\sqrt{d^2-1} + d^2 - 1)r \leq 2d^2r \leq d^2R.$$

Thus, the subfamily of balls in $\mathcal{F}$ with radii $\geq d^2R$ has a transversal of cardinality $d+1$, which is the set of vertices of this concave simplex. Other balls in $\mathcal{F}$ can be partitioned into two subfamilies with radii from R to dR and from dR to d^2R . By Theorem 3, these subfamilies have $(d+1)$ -transversals. In total, we obtain a $3(d+1)$ -transversal for $\mathcal{F}$.

In the case $d \leq 4$, we note that all balls in $\mathcal{F}$ intersect the smallest ball in $\mathcal{F}$ of radius R . The latter can be put into a concave simplex of curvature radius

$$(d\sqrt{d^2-1} + d^2 - 1)R \leq d^3R.$$

The vertices of this concave simplex make a transversal for balls in $\mathcal{F}$ with radii $\geq d^3R$. Other balls can be partitioned into three families with $(d+1)$ -transversals to obtain the transversal for $\mathcal{F}$ of size $4(d+1)$. $\square$

Proof of Theorem 5 We omit the obvious case $\tau(\mathcal{F}) = 1$ and consider $S = S(\mathcal{F})$ and $s = s(\mathcal{F})$. Note that any simplex $K \in \mathcal{F}$ is a negative homothet of S and a positive homothet of s . This can be deduced by considering the sets of outer and inner normals to facets of S , K and s . Assume that K does not contain any vertex of s . If K intersects d facets of s , it has to contain their intersection, which is a vertex of s . Hence, there are two facets of s that K does not intersect. Let these facets correspond to the half-spaces H_1 and H_2 from the family $\{H_1, H_2, \dots, H_{d+1}\}$ of outer support half-spaces for facets of S .

Denote by G_i the translate of the half-space H_i containing the respective facet of s in its boundary.

K is a homothet of s , intersects s , and does not intersect the facet $\text{bd } G_1 \cap s$. Hence, $K \subseteq G_1$. Similarly, $K \subseteq G_2$. We show that the intersection

$$G_1 \cap G_2 \cap H_3 \cap \dots \cap H_{d+1}$$

is either empty or consists of one vertex of s , the latter being possible only when $d=2$. For all $i=1, \dots, d+1$, $H_i \subseteq G_i$, and, therefore,

$$G_1 \cap G_2 \cap H_3 \cap \dots \cap H_{d+1} \subseteq G_1 \cap G_2 \cap G_3 \cap \dots \cap G_{d+1} = s.$$

The simplex s intersects half-spaces $H_3, \dots, H_{d+1}$ at one point each; hence,

$$G_1 \cap G_2 \cap H_3 \cap \dots \cap H_{d+1}$$

can be nonempty only when $d=2$, in which case it consists of one vertex of s .

Note that K does not contain any vertex of s and that

$$K \cap \bigcap_{i=3}^{d+1} K_i \subseteq K \cap \bigcap_{i=3}^{d+1} H_i = \emptyset,$$

which contradicts the property Π_d . $\square$

We need another lemma to prove Theorem 6.

Lemma 11 *If a half-space H contains an R -upper convex set K , then there exists a ball B of radius R such that $K \subseteq B \subseteq H$.*

The author is unable to provide the exact reference to the first proof of this fact, but a proof can be found in [5].

Proof of Theorem 6 We omit the case $\tau(\mathcal{F}) = 1$ and consider $S = S(\mathcal{F})$ and $s = s(\mathcal{F})$. Take the sets $K_1, K_2, \dots, K_{d+1}$ in $\mathcal{F}$ that correspond to S and consider the balls $K'_1, K'_2, \dots, K'_{d+1}$ with the following properties: the radius of K'_i is dR , $K'_i \supseteq K_i$, and $K'_i \cap S = K_i \cap S$. Such K'_i can be found by Lemma 11 if we consider, for each K_i , its respective outer support half-space H_i for S . The balls $K'_1, K'_2, \dots, K'_{d+1}$ have the property Π_d and have no common point, so they form a concave simplex T of curvature radius dR and $T \supseteq S$. Let the homothety that takes S to s take T to T' ; the latter has curvature radius R . By Lemma 4, any $K \in \mathcal{F}$ intersects s and, therefore, intersects T' . By Lemma 10, K contains one of the $d + 1$ vertices of T' . $\square$

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