Geometric Realization of a Triangulation on the Projective Plane with One Face Removed

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Received: 7 September 2006 / Revised: 16 August 2007 /
Published online: 25 September 2007
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Abstract Let M be a map on a surface F^2 . A *geometric realization* of M is an embedding of F^2 into a Euclidean 3-space $\mathbb{R}^3$ such that each face of M is a flat polygon. We shall prove that every triangulation G on the projective plane has a face f such that the triangulation of the Möbius band obtained from G by removing the interior of f has a geometric realization.

Keywords Triangulation · Geometric realization · Möbius band · Projective plane

1 Introduction

Let F^2 be a surface with at most one boundary component and let G be a map on F^2 . If F^2 has a boundary, we suppose that some cycle of G coincides with the boundary of F^2 . Such a cycle of G is called the *boundary* of G and denoted by ∂G . A k -*cycle* means a cycle of length k . A *triangulation* on F^2 is a map on F^2 such that each face is bounded by a 3-cycle. A vertex of G not on ∂G is called an *inner* vertex. (An *inner* vertex of a path means a vertex not on the ends.) Throughout this paper, we suppose that the graph of a map is *simple*, i.e., with no multiple edges and no loops. For a cycle or path C in G , a *chord* of C means an edge xy of G such that $x, y \in V(C)$ but $xy \notin E(C)$. A k -*wheel* is a triangulation on the disk such that its boundary is a k -cycle, called a *rim*, and there is only one inner vertex adjacent to the k vertices on

This research was partially supported by the Ministry of Education, Science, Sports and Culture of Japan, Grant-in-Aid for Young Scientists (B), 18740045, 2005.

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the boundary. We often use symbols with subscripts to express vertices and so on, in which the subscripts are taken by suitable modulus.

A *geometric realization* of a map G on a surface F^2 is an embedding of F^2 into a Euclidean 3-space $\mathbb{R}^3$ such that each face of G is a flat polygon. We denote a geometric realization of G by $\hat{G}$. Steinitz's theorem states that a spherical map has a geometric realization if and only if its graph is 3-connected [10]. Moreover, Archdeacon et al. proved that every toroidal triangulation has a geometric realization [1]. In general, Grünbaum conjectured that every triangulation on any orientable closed surface has a geometric realization [6], but Bokowski et al. recently showed that a triangulation by K_{12} on the orientable closed surface of genus 6 has no geometric realization [3]. (See also a good survey paper [5].)

Let us consider nonorientable surfaces, in particular, the projective plane. Since the projective plane itself is not embeddable in $\mathbb{R}^3$, no map on the projective plane has a geometric realization. However, the surface obtained from the projective plane by removing a disk, which is a Möbius band, is embeddable in $\mathbb{R}^3$, and hence we can expect that a triangulation on the Möbius band has a geometric realization. For simple notations, we call a triangulation on the projective plane and that of the Möbius band a *projective triangulation* and a *Möbius triangulation*, respectively.

In this paper, we consider geometric realizations of Möbius triangulations. However, Brehm [4] has already found a Möbius triangulation with no geometric realization shown in Fig. 1, in which both express the same Möbius triangulation. (In Fig. 1, identify the vertices with the same label. In the right side, the shaded part means the hole.)

Can we get an affirmative result for geometric realizations of Möbius triangulations? In this paper, answering such a question, we prove the following.

Let G be a projective triangulation and let f be a face of G . Let $G - f$ denote the Möbius triangulation obtained from G by removing the interior of f .

Theorem 1 *Every projective triangulation G has a face f such that the Möbius triangulation $G - f$ has a geometric realization.*

Fig. 1 A Möbius triangulation with no geometric realization

Why does Brehm's example have no geometric realization? Brehm essentially proved in [4] that for each of its spatial embedding, the two disjoint 3-cycles 123 and 456 have a linking number at least 2. (See [9] for the definition of the linking number.) However, two 3-cycles with straight segments have linking number at most one, a contradiction. Hence we have the following fact.

Fact 2 *If a Möbius triangulation G has a boundary cycle C of length 3 and a 3-cycle C' disjoint from C which forms an annular region with C' , then G never has a geometric realization.*

A graph G is said to be *cyclically k -connected* if G has no separating set $S \subset V(G)$ of G with $|S| \leq k - 1$ such that each connected component of $G - S$ has a cycle.

Let G be a projective triangulation. By Fact 2, the cyclically 4-connectedness of G is necessary for a geometric realization of $G - f$ for any face f of G . We conjecture that the condition is also sufficient, as follows.

Conjecture 3 *Let G be a projective triangulation. Then G is cyclically 4-connected if and only if $G - f$ has a geometric realization for any face f of G .*

2 Irreducible Projective Triangulations

Let G be a triangulation on a surface F^2 possibly with a boundary ∂F^2 . Let e be an edge of G not contained in ∂F^2 . *Contraction* of e (or *contracting* e) is to remove e , identify the two endpoints of e and replace two pairs of parallel edges by two single edges. (For an edge e contained in ∂F^2 , the contraction of e can be defined similarly except replacing one pair of parallel edges with a single edge.) Let $[e]$ denote the vertex in the resulting graph obtained from e by its contraction. The inverse operation of the contraction of e is a *splitting* of $[e]$. We say that e is *contractible* if the map obtained from G by contracting e has no multiple edges, and that G is *contractible* to T if G can be transformed into T by contractions of edges. An *irreducible* triangulation is one with no contractible edge.

Figure 2 shows two irreducible projective triangulations $\mathcal{P}_1$ and $\mathcal{P}_2$, in which we identify each pair of antipodal points of the hexagon. The graphs of $\mathcal{P}_1$ and $\mathcal{P}_2$ are isomorphic to K_6 and $K_4 + \bar{K}_3$, respectively, where $\bar{K}_n$ denotes a graph with n vertices

Fig. 2 Irreducible projective triangulations $\mathcal{P}_1, \mathcal{P}_2$ and their submaps $\mathcal{M}_1, \mathcal{M}_2$

and no edges. Let $\mathcal{M}_1$ be the Möbius triangulation obtained from $\mathcal{P}_1$ by removing a vertex and five faces incident to it, and let $\mathcal{M}_2$ be the one obtained from $\mathcal{P}_2$ by removing a vertex of degree 4 and four faces incident to it. The graph of $\mathcal{M}_1$ is isomorphic to K_5 , and so $\mathcal{M}_1$ is called a K_5 -triangulation. On the other hand, the graph of $\mathcal{M}_2$ includes a quadrangulation by K_4 with vertex set $\{1, 2, 3, 4\}$, which is called a K_4 -quadrangulation.

Lemma 4 (Barnette [2]) *The projective plane admits exactly two irreducible triangulations, which are $\mathcal{P}_1$ and $\mathcal{P}_2$ shown in Fig. 2.*

Let us construct a geometric realization of an irreducible projective triangulation with one face removed. For a point x and a straight segment e not intersecting x in $\mathbb{R}^3$, let $\Delta(x, e)$ denote the unique triangular disk in $\mathbb{R}^3$ containing x and the endpoints of e as its corners. Let M be a map and let $\hat{M}$ denote a geometric realization of M . Fix a point x in $\mathbb{R}^3$ not contained in $\hat{M}$. We say that an edge e of $\hat{M}$ is (completely) *seen* from x if $\Delta(x, e)$ intersects $\hat{M}$ only at e , and that an edge e of $\hat{M}$ is *half seen* from x if an interval e' of e containing one of the endpoints of e is completely seen from x .

Proposition 5 *Let P be an irreducible projective triangulation. Then $P - f$ has a geometric realization for any face f of P .*

Proof For each $\mathcal{P}_i$, we first construct a geometric realization of the Möbius triangulation $\mathcal{M}_i$ contained in $\mathcal{P}_i$. Figure 3 shows how to construct geometric realizations $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$. In particular, we take the following 3-dimensional coordinates in $\mathbb{R}^3$:

- For $\hat{\mathcal{M}}_1$, $a = (0, 0, 8)$, $b = (-4, 0, 0)$, $c = (0, 4, 4)$, $d = (0, -4, 4)$, $e = (4, 0, 0)$.

Fig. 3 How to construct $\mathcal{M}_1$ and $\mathcal{M}_2$

- For $\hat{\mathcal{M}}_2$, $a = (-4, 0, 0)$, $b = (0, 0, 2)$, $c = (4, 0, 0)$, $d = (0, 0, 4)$, $e = (4, 0, 0)$, $p = (0, -4, 1)$ and $q = (0, 4, 1)$.

In order to complete a geometric realization of $\mathcal{P}_i$ with one face removed, we have only to add four triangular disks incident to $\partial\hat{\mathcal{M}}_1$ and three triangular disks to $\partial\hat{\mathcal{M}}_2$. Observe that for $i = 1, 2$, there is only one edge on $\partial\hat{\mathcal{M}}_i$ which can not be completely seen from us in Fig. 3. Such an edge in $\partial\hat{\mathcal{M}}_1$ is cd , and the edge in $\partial\hat{\mathcal{M}}_2$ is ab . Therefore, for $\hat{\mathcal{M}}_1$, fixing a point x close to our eyes in $\mathbb{R}^3$ (for example, $x = (1, -1, 20)$), we can add four triangular disks $\Delta(x, ab)$, $\Delta(x, bc)$, $\Delta(x, de)$ and $\Delta(x, ea)$ with no internal collision of any two triangular disks. For $\hat{\mathcal{M}}_2$, fixing a fixed point $x = (1, -1, 20)$ in $\mathbb{R}^3$, for example, we can add three triangular disks $\Delta(x, bc)$, $\Delta(x, cd)$ and $\Delta(x, da)$. Thus, each $\mathcal{P}_i$ with some single face removed has a geometric realization.

It is easy to see that for any two faces f and f' of $\mathcal{P}_i$ ($i = 1, 2$), there exists a homeomorphism over the projective plane carrying f into f' , and hence each of $\mathcal{P}_1$ and $\mathcal{P}_2$ with any single face removed has a geometric realization. $\square$

Remark 6 In the geometric realization of $\mathcal{M}_1$ shown in Fig. 3, all faces are contained in the same half-space of $\mathbb{R}^3$ with respect to a plane containing a, d and e .

3 Planar Patches

A plane graph G with boundary walk C of length $m \geq 3$ is said to be a *near triangulation* and denoted by (G, C) if C is a cycle and if each inner face is a triangle. Let $v_0, \dots, v_{k-1}$ be k distinct vertices, called the *nodes*, specified on ∂G in this cyclic order, where v_i and v_{i+1} are not necessarily adjacent in C , for each i . Let P_i be the path, called a *segment*, on C joining v_i and v_{i+1} but not containing v_{i+2} , for $i = 0, \dots, k-1$. We say that (G, C) is *good* with respect to $v_1, \dots, v_m$ (or $(G, C, \{v_1, \dots, v_k\})$ is *good*) if no P_i has a chord.

Lemma 7 *Let (G, C) be a near triangulation. If C has no chord, then $G - V(C)$ is connected or empty. Moreover, each vertex on C is incident to $G - V(C)$.*

Proof We use induction on $|V(G)|$. If $|V(G)| = 3$, then the lemma clearly holds since $G - V(C)$ is empty. Hence we suppose that $|V(G)| \geq 4$. Let $v \in V(C)$ and let $P = v_1 \cdots v_k$ be the path of G consisting of the neighbors of v , where $v_1, v_k \in V(C)$. Since G has no chord, we have $k \geq 3$. Let $G' = G - v$, which is a near triangulation since G is chordless. Observe that G' might have chords incident to $v_2, \dots, v_{k-1}$. Let $B_1, \dots, B_m$ be the chordless near triangulations contained in G' sharing these chords one another, where $m \geq 1$. By induction hypothesis, the interior $\tilde{B}_i$ of each B_i is connected or empty. Moreover, every B_i has some v_j on its boundary for $j \in \{2, \dots, k-1\}$, and hence the vertex v_j is incident to $\tilde{B}_i$. Therefore, $\tilde{G} = G - V(C)$ is connected, since $V(\tilde{G}) = \{v_2, \dots, v_{k-1}\} \cup V(\tilde{B}_1) \cup \dots \cup V(\tilde{B}_m)$. Clearly, every vertex on C is incident to $\tilde{G}$, since C has no chord. $\square$

We often use the following lemma to find a suitable subgraph in a near triangulation or a Möbius triangulation. Let (G, C) be a near triangulation and let $x, y \in V(C)$.

A path joining x and y and intersecting C only at x and y is called an *internal* (x, y) -*path*. The following immediately follows from Lemma 7.

Lemma 8 *Let (G, C) be a near triangulation and let $x, y \in V(C)$ with $xy \notin E(C)$. There is an internal (x, y) -path in (G, C) if and only if there is no chord pq for any $p, q \in V(C) - \{x, y\}$ such that x, p, y, q appear on C in this cyclic order.*

An embedding $f : C \rightarrow \mathbb{R}^3$ is called an *exhibition* if $f(P_i)$ is a straight segment in $\mathbb{R}^3$ for $i = 0, \dots, k-1$, and if $f(C)$ is projected to some hyperplane as a convex k -gon. Note that each vertex of C which is not a node is also fixed by f . This notion was first introduced in [1].

Lemma 9 *Let $(G, C, \{v_0, \dots, v_{k-1}\})$ be a good near triangulation, where $v_0, \dots, v_{k-1}$ are nodes lying on C in this cyclic order, where $k \geq 3$, and let $f : C \rightarrow \mathbb{R}^3$ be an exhibition. Then $f(C)$ extends to a geometric realization of (G, C) contained in the interior of the convex hull of $f(C)$.*

Proof We use induction on $|V(G)|$. If $|V(G)| = |V(C)| = 3$, then we have nothing to do. If $|V(C)| = 3$ and $|V(G)| \geq 4$, then (G, C) is a maximal plane graph, and $f(C)$ is contained in a plane in $\mathbb{R}^3$. It has already been proved in [11] that a maximal plane graph has a straight-line embedding on the plane with a given outer triangle.

So we suppose that $|V(C)| \geq 4$. Then we can take a chordless internal (x, y) -path P for some vertices $x, y \in V(C)$ belonging to distinct segments of C . (Such a P can be taken, by Lemma 8.) Regarding x and y as new nodes, we consider two near triangulations (G_1, C_1) and (G_2, C_2) such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = V(P)$, which are good with respect to their nodes. Now, joining x and y of $f(C)$ by a straight-line segment corresponding to P in $\mathbb{R}^3$, we can naturally obtain the exhibitions of C_1 and C_2 , say $f(C_1)$ and $f(C_2)$. Then the convex hulls of $f(C_1)$ and $f(C_2)$ intersect only at P . By induction hypothesis, G_1 and G_2 have geometric realizations $\hat{G}_1$ and $\hat{G}_2$ extended from $f(C_1)$ and $f(C_2)$, which are contained in the convex hulls of $f(C_1)$ and $f(C_2)$, respectively. So, $\hat{G}_1 \cup \hat{G}_2$ is a geometric realization of G containing $f(C)$ and contained in the convex hull of $f(C)$. $\square$

Lemma 10 *Let $(G, C, \{v_0, v_1, v_2\})$ be a good near triangulation with $v_0v_1 \in E(C)$. Suppose that G has a face bounded by vv_0v_1 , where $v \in V(G - C)$. Let D be any triangular disk with corners u_0, u_1, u_2 . Then, for any inner point p of D , there is a straight-line embedding of G on D with each segment of C straight such that each v_i coincides with u_i , for $i = 0, 1, 2$, and that v coincides with p .*

Proof Observe that G has a chordless path, say P , from v to some $u \in V(C) - V(P_0)$, since G has no chord v_0v_1 . We may suppose that $u \in V(P_1) - \{v_1\}$. Then the three paths vv_0, vv_1, P decompose G into three near triangulations. Introducing u, v as new nodes, we can see that they are good with respect to their nodes. In D , even if we specify any inner point p as the position of v , we can choose a point q on the segment u_1u_2 of D as the position of u so that the three straight segments pu_0, pu_1, pq decompose D into three convex polygons. Since each boundary of the three polygons is an exhibition, we can apply Lemma 9 to them. $\square$

Note that Lemma 10 does not necessarily hold if an embedding of C into ∂D is fixed. However, moving the positions of inner vertices of the segments $v_1 v_2$ and $v_0 v_2$ on C suitably, we can obtain a required embedding of G into D .

4 Projective Triangulations with K_4 -Quadrangulations

We classify the projective triangulations into those with a K_4 -quadrangulation as a subgraph and those not containing it. In this section, we deal with the former.

We first put an important lemma.

Lemma 11 (Menger [7]) *Let G be a graph and let $v, v_1, \dots, v_k$ be distinct vertices of G . Then G has k disjoint paths from v to v_i for $i = 1, \dots, k$, if and only if there is no $S \subset V(G) - \{v, v_1, \dots, v_k\}$ separating v and $\{v_1, \dots, v_k\}$ in G such that $|S| < k$.*

Lemma 12 *If a projective triangulation G has a K_4 -quadrangulation K as a subgraph, then G has a subdivision of $\mathcal{P}_2$ containing K such that each path of G corresponding to an edge of $\mathcal{P}_2$ is chordless.*

Proof Let K be a K_4 -quadrangulation contained in G and let A_1, A_2 and A_3 be the three facial 4-cycles of K . Let $A_1 = abcd$ and let B_1 be the plane subgraph of G with boundary A_1 .

We take an inner vertex v_1 of B_1 not contained in the interior of any separating 3-cycle of B_1 , as follows. Since G is simple, we have $ac, bd \notin E(B_1)$, and hence B_1 has an inner vertex. If B_1 has no separating 3-cycle, then let v_1 be any inner vertex. Otherwise, let C be a separating 3-cycle of B_1 containing a maximal number of vertices in the interior. Since $ac, bd \notin E(B_1)$, at least one vertex of C , say v , is not contained in A_1 . By the maximality of C , v is not surrounded by any separating 3-cycle. Therefore, let $v_1 = v$.

Then, by Lemma 11, there are four disjoint paths P_a, P_b, P_c, P_d from v_1 to a, b, c, d in B_1 . Note that each of the four paths can be taken to be chordless if they are chosen to be shortest. Similarly, we can find a vertex v_i in the interior of A_i which has four disjoint paths to the four vertices of A_i , for $i = 2, 3$. Therefore, G contains a subdivision of $\mathcal{P}_2$ containing K . $\square$

Proposition 13 *If a projective triangulation G has a K_4 -quadrangulation as a subgraph, then $G - f$ has a geometric realization for some face f of G .*

Proof By Lemma 12, G has a subdivision of $\mathcal{P}_2$ containing a K_4 -quadrangulation K . Let $V(K) = \{a, b, c, d\}$ and let $\{v_1, v_2, v_3\}$ be the vertices of G corresponding to $\bar{K}_3$, as in the above lemma. Suppose that v_1 is in the interior of the 4-cycle $abcd$. Let D denote the plane subgraph of G bounded by P_a, P_b, ab , where P_a and P_b are the chordless paths from v_1 to a and b , respectively. Let f be the face of D incident to ab . Let G' be the Möbius triangulation obtained from G by removing the interior of D . By Proposition 5 and Lemma 9, G' has a geometric realization $\hat{G}'$, since each disk corresponding to a face of $\mathcal{P}_2$ is a good near triangulation. Even if D consists of several faces of G , we can put $D - f$ in $\partial \hat{G}'$ avoiding a collision of faces, by Lemma 10, moving the positions of inner vertices of P_a and P_b suitably. $\square$

Lemma 14 (Mukae and Nakamoto [8]) *If a projective triangulation G has no K_4 -quadrangulation as a subgraph, then G is contractible to $\mathcal{P}_1$.*

5 Split- K_5 's in Projective Triangulations

Consider the irreducible projective triangulation $\mathcal{P}_1$ isomorphic to K_6 . Let v be a vertex of $\mathcal{P}_1$ and let L_v be the link of v . Let G be a projective triangulation contractible to $\mathcal{P}_1$. Then G has a cycle, say C , which is contracted to L_v . We call C a *rim* in G . Clearly, cutting G along a rim C , we can obtain a Möbius triangulation G_M and a near triangulation G_D both of whose boundaries are C such that G_M is contractible to a K_5 -triangulation $\mathcal{M}_1$, and G_D to a 5-wheel W_5 . We say that a rim C of G is *minimal* if G_D has no rim of G other than C .

Lemma 15 *If C is a minimal rim of G , then C has no chord in G_D .*

Proof Since C is contracted to a boundary 5-cycle of W_5 , C is decomposed into five disjoint paths L_0, L_1, L_2, L_3, L_4 which contract the five distinct vertices of the 5-cycle, respectively. Since G_D is contracted to W_5 , there must be a connected subgraph U of $G_D - V(C)$ from which there are five disjoint paths Q_i (except at the origin) to each L_i . Moreover, G_M is decomposed into five connected subgraphs J_0, J_1, J_2, J_3, J_4 with $L_i \subset J_i$ for each i such that the contraction of each J_i into a single vertex transforms G_M into a K_5 -triangulation.

Suppose that (G_D, C) has a chord e . By the existence of U , e does not join L_i and L_{i+2} , for any i . Therefore, e joins two vertices of the same L_k or joins L_k and L_{k+1} for some k . Let C' and C'' be the two cycles of G_D such that $E(C') \cap E(C'') = \{e\}$, $E(C') \cup E(C'') = E(C) \cup \{e\}$, and that $L_{k+2} \subset C'$. Observe that C' is a cycle of G_D bounding a fewer number of faces than C and that the Möbius triangulation with boundary C' contracts to a K_5 -triangulation since this contains all J_i 's and C' passes through each L_i . Moreover, since U is contained in the 2-cell region D' bounded by C' , each Q_i can be taken in D' . Hence C' is a rim of G other than C contained in G_D , contrary to the minimality of C . $\square$

Let $\mathcal{M}_1$ be a K_5 -triangulation with vertices v_0, v_1, v_2, v_3, v_4 , where $\partial\mathcal{M}_1 = v_0v_1v_2v_3v_4$ is called the boundary. Consider a splitting of a vertex v_i of $\mathcal{M}_1$. Then there are two ways of splitting v_i into two vertices v_i and v'_i of degree 3. The splitting of v_i is called a *boundary splitting* when both v_i and v'_i lie on the boundary, where we always suppose that v_i, v'_i, v_{i+1} lie on $\partial\mathcal{M}_1$ in this order. Otherwise, it is called an *inner splitting*, where we always suppose that v_i lies on $\partial\mathcal{M}_1$. A *split- K_5* is a map H obtained from $\mathcal{M}_1$ by applying either a boundary splitting, an inner splitting, or no splitting to each vertex of $\mathcal{M}_1$, and subdividing edges. We call a vertex of H whose degree is greater than 2 is a *node* and a path in H containing only two nodes as its two endpoints a *segment*. Clearly, G_M contains a split- K_5 H as a subgraph so that $\partial H = \partial G_M$. We say that a vertex v of G_M is called a *boundary split node* (resp., an *inner split node*) if v is a node of H arisen by a boundary (resp., inner) splitting. Moreover, $\{v_i, v'_i\}$ is called a *boundary split pair* (resp., an *inner split node*) if they arose by a boundary (resp., inner) splitting. (See Fig. 4.)

Fig. 4 K_5 and split- K_5

Throughout this paper, let $P(\alpha, \beta)$ denote the path of H contained in a segment joining two vertices α and β . The following is the main result in this section.

Lemma 16 *Let G be a projective triangulation contractible to $\mathcal{P}_1$. Then G contains a rim C separating G into a Möbius triangulation G_M and a near triangulation G_D both of whose boundaries are C such that*

- (i) G_M has a split- K_5 H with either none or a single boundary split pair such that $\partial H = C$ (in the former, the nodes of H lying on C are v_0, v_1, v_2, v_3, v_4 , and in the latter, those are $v_0, v'_0, v_1, v_2, v_3, v_4$, where $\{v_0, v'_0\}$ is a boundary split pair), and
- (ii) (G_D, C) has an inner vertex $v \in V(G_D - C)$ which has five disjoint paths Q_i to v_i intersecting C only at v_i , for $i = 0, 1, 2, 3, 4$, such that Q_0 has just one edge

Proof Observe that G has a rim since G is contractible to $\mathcal{P}_1$. Then we can choose a minimal rim, denoted by C . We shall prove that C satisfies both conditions (i) and (ii) of the lemma.

We first consider (i). By the definition of a rim, G_M has a split- K_5 H with boundary C . If H has at least two pairs of boundary split pairs, then we shall modify H , as follows. Suppose that $\{a', a''\}$ is one of the boundary split pairs which is arisen from a node a contained in a triangle abc . (See Fig. 5.) Now we shall find a new split- K_5 H' in which a' is not a boundary split node, b possibly becomes a boundary split node, but no other new boundary pairs are introduced. Iterating this process, we can finally make c be the possible new boundary split node.

Let Q be the region with nodes a', a'', b, c corresponding to a triangle abc . Note that if we take $P(a', a'')$ to be shortest in G , then Q has no edge yz with $y \in V(P(a', c)) - \{a'\}$ and $z \in V(P(a', a'')) - \{a'\}$. (For otherwise, taking z instead of a' , we can take a smaller region corresponding to Q .) Therefore, by Lemma 8, we can take an internal (a', x) -path P for some x on either $P(a'', b) - \{a''\}$ or $P(b, c) - \{b, c\}$. In the former case, putting $H' = H - P(a'', x) \cup P$, we can decrease the number of boundary split pairs. In the latter case, consider $H' = H - P(a'', b) \cup P$, in which b can be regarded as a boundary split node but a' is not.

Fig. 5 Move the position of a boundary split node

Fig. 6 Structures of D

Hence we can determine five nodes v_0, v_1, v_2, v_3, v_4 lying on C in this order if there is no boundary split pair. Otherwise, we can fix six nodes $v_0, v'_0, v_1, v_2, v_3, v_4$ lying on C in this order such that $\{v_0, v'_0\}$ is a unique boundary split pair.

Now consider (ii). For a simple notation, we put $G_D = D$. If (D, C) has only one inner vertex, then (D, C) is a wheel, by Lemmas 7 and 15 (since C is a minimal rim of G). Therefore, (D, C) has at least two inner vertices. Pick up any neighbor v of v_0 with $v \notin V(C)$. This is possible since C has no chord in (D, C) . Since v and v_0 are adjacent in D , we want to take the remaining four disjoint paths Q_i from v to v_i such that Q_i intersects C only at v_i , for $i = 1, 2, 3, 4$. If this is impossible, one of the following holds for any v , by Lemma 11, where $D' = D - V(C - \{v_0, v_1, v_2, v_3, v_4\})$:

- (1) There is $S \subset V(D - C) - \{v\}$ with $|S| \leq 3$ separating $\{v\}$ from $\{v_1, v_2, v_3, v_4\}$ in D' for any v
- (2) There is $S \subset V(D - C) - \{v\}$ with $|S| \leq 2$ separating $\{v, v_1\}$ from $\{v_2, v_3, v_4\}$ (or $\{v, v_4\}$ from $\{v_1, v_2, v_3\}$) in D' for any v , or
- (3) There is $S \subset V(D - C) - \{v\}$ with $|S| \leq 1$ separating $\{v, v_1, v_4\}$ from $\{v_2, v_3\}$, ($\{v, v_1, v_2\}$ from $\{v_3, v_4\}$, or $\{v, v_3, v_4\}$ from $\{v_1, v_2\}$) in D' for any v

We shall prove that in each of the above cases, we can take another rim C' in D bounding a fewer number of faces than C , in order to get a contradiction to the minimality of C . To avoid a repetition of similar arguments, we omit the cases described in the parenthesis of Cases (2) and (3).

(1) Let $S = \{p, q, r\}$. (The case when $|S| \leq 2$ is similar.) Then there is a path $L = apqrb$ of length exactly four, where $a \in V(P_0) - \{v_0, v_1\}$ and $b \in V(P_4) - \{v_0, v_4\}$. Here we choose a and b to be nearest to v_0 on P_0 and P_4 , respectively. See Fig. 6(1).

Take the unique chordless path J joining a and b and consisting of neighbors of p, q, r lying on the side containing v_0 with respect to the cut L of (D, C) , and let $C' = J \cup P(a, v_1) \cup P_1 \cup P_2 \cup P_3 \cup P(v_4, b)$. Now we shall prove that C' is a rim of G . By the obstruction of L , C' has no chord. Thus, contracting all inner vertices of the 2-cell region bounded by C' into a single vertex, we get a wheel with boundary C' , by Lemma 7. On the other hand, if H has no v'_0 , then taking b, v_1, v_2, v_3, v_4 as new boundary nodes, we can find a new split- K_5 H' with boundary C' , which is obtained from $H \cup J$ by removing the edges of $P(a, v_0)$. Then suppose that H has v'_0 . If v'_0 is an inner vertex of $P(v_0, a)$, then we take boundary nodes $\{a, b, v_1, v_2, v_3, v_4\}$ in which $\{a, b\}$ is a boundary split pair. In this case, H' is obtained from $H \cup J$ by removing the edges of $P(v_0, v'_0)$. Otherwise (i.e., $v'_0 \in V(P(a, v_1)) - \{v_1\}$), H' has boundary nodes $\{b, v'_0, v_1, v_2, v_3, v_4\}$ in which $\{b, v'_0\}$ is a boundary split pair. In this case, H' is obtained from $H \cup J$ by removing the edges of $P(v_0, a)$.

(2) Let $S = \{p, q\}$. (The case when $|S| = 1$ is similar.) Similarly to Case (1), we can find a path $L = apqb$ of length exactly three, where $a \in V(P_1) - \{v_2\}$ and $b \in V(P_4) - \{v_0, v_4\}$. See Fig. 6(2). Take the unique chordless path J consisting of neighbors of p and q lying on the side containing v_0 with respect to L , and consider the smaller near triangulation D' whose boundary C' contains J , and a, b, v_2, v_3, v_4 as nodes. By the obstruction of L , C' has no chord, and hence, contracting all inner vertices of D' , we can get a wheel with boundary C' . On the other hand, take $\{a, v_2, v_3, v_4, b\}$ as the boundary nodes of a new split- K_5 H' . In this case, H' is obtained from $H \cup J$ by removing the edges of P_0 (resp., $P(v_0, v'_0)$) if there is v'_0 (resp., no v'_0) in H .

(3) Take $S = \{p\}$ and consider the path $L = apb$, where $a \in V(P_1) - \{v_2\}$ and $b \in V(P_3) - \{v_3\}$. See Fig. 6(3). Take the unique path J consisting of the neighbors of p lying on the side containing v_0 with respect to L , and consider the smaller near triangulation D' with boundary C' containing J . Similarly to Cases (1) and (2), C' has no chord, and hence we can transform D' into a wheel with boundary C' by contracting all inner vertices of D' into a single vertex. Let us take boundary nodes of a new split- K_5 H' . Consider a 2-cell region bounded by $P_0 \cup P(v_1, a) \cup J \cup P(b, v_4) \cup P_4$ and find its internal (v_0, u) -path P_{u, v_0} for some vertex $u \in V(J) - \{a, b\}$. This is possible, by Lemma 8, since C has no chord. Take u, a, v_2, v_3, b as the boundary nodes of H' . Then H' is obtained from $H \cup J \cup P_{u, v_0}$ by removing the edges of P_4 and P_0 (resp., P_4 and $P(v'_0, v_1)$) if there is v'_0 (resp., not v'_0) in H .

Consequently we have found contradictions in all the three cases, and hence these cases cannot happen if the rim C is minimal. Therefore, the lemma follows. $\square$

6 Lemmas for Geometric Realization

Let H be a split- K_5 on the Möbius band and let v be a node of degree 4 on ∂H . Suppose that H has a geometric realization $\hat{H}$. Let A, B and C be three flat faces of $\hat{H}$ incident to v in this order. In $\hat{H}$, v is said to be of *type I* if C bends to the same side of A with respect to B , and of *type II* otherwise. (See Fig. 7, where the left and the right show nodes v of type I and II, respectively.) In $\hat{\mathcal{M}}_1$ shown in Fig. 3, only the vertex a is of type II and others are of type I.

Fig. 7 Vertices of degree 4 of type I and II

Fig. 8 Inner and boundary splittings of a vertex of type I

Suppose that a node v is of type I, where $A \cap B = e$ and $B \cap C = e'$, and $a = A \cap \partial H$ and $c = C \cap \partial H$. For $\hat{H}$, take a half-plane π_A parallel to A with boundary the straight line a but not containing e , and a half-plane π_C parallel to C with boundary the straight line c but not containing e' . It is easy to see that if π_A and π_C share a half-line, $\ell_{A,C}$, with endpoint v , then an inner splitting of v in $\hat{H}$ can be realized without changing the positions of all nodes of $\hat{H}$. Such a condition is called an *inner splitting condition* of v in $\hat{H}$.

Observation 17 A node v of type I satisfies an inner splitting condition if A and C are contained in the same half-space of $\mathbb{R}^3$ with respect to a plane containing a and c .

See Fig. 8, where (a) is an inner splitting and (b) is a boundary splitting of a vertex of type I. (On the other hand, the left of Fig. 7 is a vertex of type I which does not satisfies an inner splitting condition, since A and C lie in distinct sides with respect to a plane containing e and e' .)

Let f be a face of a geometric realization $\hat{H}$ such that the boundary of f contains exactly k nodes. We say that f is *convex* if f is a convex k -gon whose nodes are vertices of the k -gon f . We say that $\hat{H}$ is *convex* if each face of $\hat{H}$ is convex.

Lemma 18 Suppose that H' is obtained from H by a single inner splitting of a vertex v of degree 4, and H has a convex geometric realization $\hat{H}$. If v is of type I satisfying an inner splitting condition in $\hat{H}$, then H' has a convex geometric realization.

Proof Put v' on $\ell_{A,C}$ close to v , as shown in Fig. 8. $\square$

Lemma 19 *Suppose that H' is obtained from H by a single boundary splitting of a vertex v of degree 4, and H has a convex geometric realization $\hat{H}$. Then H' always has a convex geometric realization.*

Proof Put v' on e' , and move the position of v along e slightly, as shown in Fig. 8. $\square$

7 Projective Triangulations Contractible to $\mathcal{P}_1$

In this section, we shall construct a geometric realization of a projective triangulation G contractible to $\mathcal{P}_1$ with one face removed. By Lemma 16, G has a split- K_5 H with boundary C , in which C has exactly five nodes v_0, v_1, v_2, v_3, v_4 or six nodes $v_0, v'_0, v_1, v_2, v_3, v_4$. Let G_M and G_D be the Möbius triangulation and the near triangulation obtained from G by cutting along C . Let P_i be the path on C joining v_i and v_{i+1} , not containing v_{i+2} , for $i = 0, 1, 2, 3, 4$.

Lemma 20 *G_M has a geometric realization $\hat{G}_M$ such that each segment on C is a straight-line segment, and that from some fixed point x in $\mathbb{R}^3$, an edge, say e , on P_4 incident to v_0 is half seen, and all other edges on $\partial\hat{G}_M$ can be completely seen. In particular, if P_0 bends at v'_0 , then the convex hull of P_0 can be completely seen from x .*

Proof For constructing a geometric realization of G_M , we use that of $\mathcal{M}_1$ shown in the top of Fig. 3, where b and e are of type I satisfying the inner splitting condition, by Remark 6 and Observation 17. Note that G_M has a split- K_5 H with at most one boundary split pair on C , by Lemma 16. Moreover, we may suppose that each segment of H is chordless. (If some segment P of H has a chord, then we can find a shorter path P' joining the endpoints of P and passing through the chord. Hence we can take P' instead of P .)

We consider the following three cases, depending on which nodes of H are inner-split.

Case 1 None of $v_0, \dots, v_4$ are inner split nodes.

If C has no boundary split pair either, then H is homeomorphic to $\mathcal{M}_1$, and hence H has a geometric realization $\hat{H}$, as shown in the left of Fig. 9. Note that $\hat{H}$ is convex since each face of H has exactly three nodes. In $\hat{H}$, fix the position of the vertices of C so that only one edge e of C can be half seen and all other edges of C can be completely seen from a point $x = (1, -1, 20)$ in $\mathbb{R}^3$. Since the boundary of each face of H gives an exhibition in $\mathbb{R}^3$, we can get a required geometric realization of G_M , by Lemma 9.

Let H' be a split- K_5 obtained from the above H by a boundary splitting of v_0 . By Lemma 19, a boundary splitting at v_0 can be applied freely so that the resulting geometric realization $\hat{H}'$ is convex. In particular, the convex hull of the bent P_0 in $\hat{H}'$ can be completely seen, since the segment v_0v_2 of $\hat{H}$ can be seen from $(1, -1, 20)$.

Fig. 9 Geometric realizations of H and H'

Therefore, by Lemma 9, G_M has a required geometric realization, as shown in the right of Fig. 9.

Case 2 Both v_0 and v_4 are inner split nodes.

If each of v_1, v_2, v_3 is applied no splitting, then see Fig. 10, where we give coordinates of the vertices, as follows:

$$\begin{array}{llll} v_0 = (-1, 3, 3), & v'_0 = (1, 4, 4), & v_1 = (-4, 0, 0), & v_2 = (0, 0, 6), \\ v_3 = (4, 0, 0), & v_4 = (1, -3, 3), & v'_4 = (-1, -4, 4). \end{array}$$

Observe that two quadrilateral regions $v_1v_3v_4v'_4$ and $v_1v_3v'_0v_0$ of $\hat{H}$ are flat, since they are contained in two planes $y + z = 0$ and $y - z = 0$, respectively, and that they are convex in $\mathbb{R}^3$ since their xz -coordinates are $(-4, 0), (4, 0), (\pm 1, 3), (\mp 1, 4)$. Let R be the pentagonal region $v_0v'_0v_2v'_4v_4$ of H . Then the embedding of ∂R in $\mathbb{R}^3$ is an exhibition, since the projection of R into the plane $x = 0$ is a convex pentagon whose vertices have yz -coordinates: $(3, 3), (4, 4), (0, 6), (-4, 4), (-3, 3)$. Hence, since other two regions of H are triangular in $\mathbb{R}^3$, we get a geometric realization of G_M , by Lemma 9. However, seeing the body from $(1, -1, 20)$, then the convex hull of ∂R hides P_4 completely, and hence the condition (ii) of the theorem might not be satisfied.

Hence we find an auxiliary path of G_M , as follows:

Claim We may suppose that G_M has an internal (v_0, x) -path for R , denoted by P , for some vertex $x \in P(v_4, v'_4) - \{v_4\}$. (See the left of Fig. 10.)

Proof We may suppose that H is chosen in G_M so that the number of inner vertices in R is minimal. By the symmetry, we shall prove that there is an internal (v_0, x) -path for some $x \in V(P(v_4, v'_4)) - \{v_4\}$ or an internal (v_4, x') -path for some $x' \in V(P(v_0, v'_0)) - \{v_0\}$. If neither of them exist, then we have an edge zw such that z is an inner vertex of P_4 and w is an inner vertex of $P(v'_0, v_2) \cup P(v_2, v'_4)$, by Lemma 8. By the symmetry, we may suppose that $w \in V(P(v'_0, v_2)) - \{v'_0\}$. Let $H' = H - E(P(w, v'_0)) \cup zw$, which is a split- K_5 with a boundary split node v_0 . This case will be considered in Case 3. $\square$

Fig. 10 H with inner split nodes v_0 and v_4

Since the internal (v_0, x) -path divides R into a triangle xv_0v_4 and a region, say R' , bounded by $v_0v'_0v_2v'_4x$, where we might have $x = v'_4$. Hence, take x as an internal point of the straight segment $v_4v'_4$ in $\mathbb{R}^3$ and apply Lemma 9 to R' . Then the convex hull of $\partial R'$ no longer hides P_4 completely, if we see the body from $(1, -1, 20)$. (Figure 10 shows the case when $x = v'_4$ and P is an internal (v'_0, v'_4) -path.) Therefore, if we let H' be H with all edges of R' added, then H' has a required convex geometric realization.

Let's check that v_1 and v_3 satisfy the inner splitting condition in $\hat{H}'$, since they are nodes of type I. Since $\hat{H}'$ is symmetric, it suffices to deal with only v_3 . For a plane π in $\mathbb{R}^3$ containing v_2 , v_3 and v_4 (i.e., $\pi = \{(x, y, z) \in \mathbb{R}^3 : 3x - y + 2z = 12\}$), all other nodes lies in the same half-space with respect to π . Therefore, by Observation 17, v_1 and v_3 must satisfy the inner splitting condition, and hence, even if v_1 or v_3 are inner split nodes in H' , then the corresponding H' has a convex geometric realization, by Lemma 18.

Finally, we consider an inner splitting of v_2 in H' . We put $v'_2 = (0, 0, 6 - \frac{1}{10})$, for example, and fix all other nodes as above. Then we can check that the regions bounded by $v_2v'_2v'_4v_1$ (or $v_2v'_2v'_4v'_1$) and $v_2v'_2v'_0v_3$ (or $v_2v'_2v'_0v'_3$) are exhibited, since they are projected to the plane $z = 0$ as a triangle or a convex quadrilateral. Therefore G_M has a required geometric realization, by Lemma 9.

Case 3 Other cases.

If exactly one of v_0 and v_4 is an inner split node, then we may suppose that v_4 is, by symmetry. Since the convex hull of $v_0v_2v'_4v_4$ does not hide P_4 , we can deal with this case similarly to Case 2. (Figure 11 shows H with v_0 and v_4 applied a boundary splitting and an inner splitting respectively, and an example of its geometric realization.) The case when none of v_0 and v_4 are applied inner splittings can be dealt with similarly to Case 2, too. $\square$

The following is our main result in this section.

Proposition 21 *If a projective triangulation G is contractible to a K_6 -triangulation $\mathcal{P}_1$, then $G - f$ has a geometric realization for some face f of G .*

Fig. 11 H with a boundary split node v_0 and an inner split node v_4

Fig. 12 Pasting G_D with one face removed to $\partial \hat{G}_M$

Proof By Lemma 16, G contains a split- K_5 H with boundary C containing five nodes v_0, v_1, v_2, v_3, v_4 (or six nodes $v_0, v'_0, v_1, v_2, v_3, v_4$), in which the near triangulation, denoted by (G_D, C) , admits five chordless disjoint paths Q_i ($i = 0, 1, 2, 3, 4$) from some inner vertex v to v_i such that Q_0 is just an edge. These five paths divide G_D into five plane graphs, denoted by F_0, F_1, F_2, F_3, F_4 , where each F_i is incident to Q_i and Q_{i+1} , for $i = 0, 1, 2, 3, 4$. Here we remove a face f bounded by vv_0p from F_4 incident to the unique edge of Q_0 , and let $F'_4 = F_4 - f$, where $p \notin \{v, v_0\}$ is a vertex of F_4 . (See the right side of Fig. 12.)

Let G_M denote the Möbius triangulation with boundary C . Then, by Lemma 20, G_M has a geometric realization $\hat{G}_M$ such that each P_i is a straight-line segment (only P_0 might bend), and that from some fixed point $x \in \mathbb{R}^3$, an edge, say e , on P_4 incident to v_0 is half seen, and all other edges on $\partial \hat{G}_M$ can be completely seen. (See the left side of Fig. 12.)

Now, using the fixed point $x \in \mathbb{R}^3$, we first put F'_4 in $\hat{G}_M$, as follows. If $p = v_4$, then we have $F_4 = f$, and hence we have nothing to do. If p is an inner vertex of F_4 , then we put F'_4 in $\hat{G}_M$ so that F'_4 does not collide with the body, moving the position of inner vertices on P_4 , by Lemma 10. If p is an inner point of P_4 , then we apply Lemma 9. Since the edge e incident to f is half seen and all other edges on P_4 are

completely seen from x , we can put F'_4 in $\hat{G}_M$ naturally. If p is an inner point of Q_4 , then fixing p on Q_4 in the position seeing v_0 , we put F'_4 in the body, by Lemma 9.

Finally, we shall put F_0, F_1, F_2, F_3 in the body of $G_M \cup F'_4$, making them be flat triangular disks by Lemma 9. (If P_0 bends at v'_0 , then the convex hull of v_0, v'_0, v_1 can be completely seen from v , by Lemma 20. Hence we can apply Lemma 9 to F_0 , since the positions of x, v_0, v'_0, v_1 of ∂F_0 gives an exhibition.) $\square$

8 Proof of the Theorem

Proof of Theorem 1 Let G be a projective triangulation. By Lemma 14, either G contains a K_4 -quadrangulation as a subgraph or it is contractible to K_6 . In the former case, apply Proposition 13, and in the latter case, apply Proposition 21. $\square$

Acknowledgements The authors are grateful to the anonymous two referees for reading the paper carefully and giving several suggestions improving the expressions in the paper.

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