A Polyhedral Model in Euclidean 3-Space of the Six-Pentagon Map of the Projective Plane

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Received: 14 February 2007 / Revised: 2 August 2007 /

Published online: 19 October 2007

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Abstract In a private communication, Branko Grünbaum asked: “*I wonder whether you know anything about the possibility of realizing as a polyhedron in Euclidean 3-space the family of six pentagons, that is a model of the projective plane arising by identifying antipodal points of the regular dodecahedron. Naturally, any realization must have some self-intersections—but is there any realization that is not completely contained in a plane?*”

We show that it is possible to realize this polyhedron; in our realization five of the six faces are simple polygons. In this model there are sets of three faces, which form a realization of the Möbius strip without self-intersections. There are four variants of the model. We conjecture that in any model of this polyhedron there must be at least one self-intersecting face.

Keywords Model of the real projective plane · Polyhedron · Möbius strip

Theorem *The family of six pentagons, that is a model of the projective plane arising by identifying antipodal points of the regular dodecahedron, can be realized as a polyhedron in Euclidean 3-space such that five of the six faces are simple planar pentagons and the sixth has a single self-intersection.*

Remark It is an open question whether there is a polyhedral realization in Euclidean 3-space such that all of the six faces are simple planar pentagons. We conjecture that there must be at least one self-intersecting face.

In what follows we give a constructive proof of our Theorem.

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Fig. 1 Two drawings of the complete graph with six vertices embedded in the real projective plane

Let us embed the complete graph K_6 in the real projective plane, so that we obtain a map A having $V = 6$ vertices, $E = \frac{6 \cdot 5}{2} = 15$ edges and $F = \frac{6 \cdot 5}{3} = 10$ faces (triangles).

The vertices are $1, \dots, 6$. The list FA contains the 10 regions of the map. Let us denote by B the dual of the map A ; then FB is the list of the regions of B . The vertices of B are denoted by the labels of the triangles in the list FA .

List FA	List FB	List VA	List VB
1[1, 2, 3]	1[1, 2, 3, 4, 5]	1(8, 8, 16)	1(6, -16, 6)
2[1, 3, 4]	2[1, 5, 6, 7, 8]	2(4, 2, 4)	2(-16, -16, 17)
3[1, 4, 5]	3[1, 8, 9, 10, 2]	3(0, -1, 0)	3(-16, 14, 2)
4[1, 5, 6]	4[3, 2, 10, 6, 7]	4(-1, 0, 0)	4(20, 14, -16)
5[1, 6, 2]	5[3, 7, 8, 9, 4]	5(1, 2, 2)	5(6, 28, -16)
6[2, 4, 6]	6[4, 5, 6, 10, 9]	6(0, 0, -1)	6(-16, 72, -16)
7[2, 4, 5]			7(-16, -8, 24)
8[2, 5, 3]			8(-24, -16, 36)
9[3, 5, 6]			9(80, -16, -16)
10[3, 6, 4]			10(-16, -16, -16)

We note that the map A and B is known as the regular map $\{3, 5\}/2$ and $\{5, 3\}/2$, respectively [1]; the latter arises as an embedding of the Petersen graph in the projective plane.

Our aim is to construct a polyhedron in Euclidean 3-space having the same combinatorial structure as the map B . This polyhedron is also denoted by B .

At first we construct a polyhedron that realizes the map A ; we give “suitable” coordinates to the vertices of A . We show these coordinates in list VA .

Now we construct the dual of the polyhedron A by applying a polarity with respect to a sphere. The center is $O(0, 0, 0)$, the radius is $r = 4$. The dual of A will be the desired polyhedron B .

Fig. 2 Two drawings of the decomposition of the real projective plane into six pentagonal regions (any two regions are adjacent)

Fig. 3 The face $1[1, 2, 3, 4, 5]$ is self-intersecting, the faces $3[1, 8, 9, 10, 2]$, $4[3, 2, 10, 6, 7]$ and $6[4, 5, 6, 10, 9]$ are simple pentagons

By “suitable” above we mean that we try to find coordinates for the vertices of A such that most of the faces of B will be simple.

In this investigation the software *Euler3D* provided great help (<http://www.euler3d.hu/>).

We obtain the vertices of polyhedron B using the data in lists FA , VA , FB . The coordinates of these vertices are shown in list VB .

The polyhedron B consists of five simple pentagons and a self-intersecting one. Three pentagons can be chosen which are adjacent to each other so that they form a non-intersecting Möbius strip. There are four Möbius strips in polyhedron B .

The structure of polyhedron B is shown in Figs. 3, 4 and 5, and the Möbius strips are shown in Figs. 6, 7, 8, 9 and 10. In addition, in Fig. 11 a net of our model is given.

Fig. 4 The faces 1, 3, 4, 6 and $2[1, 5, 6, 7, 8]$; 2 is a simple polygon, too

Fig. 5 The six pentagons. The common edge of faces 2 and 5 $[3, 7, 8, 9, 4]$ is the edge $[7, 8]$, which continues along the self-intersecting line $(7, x)$ of the surface

Fig. 6 The faces 3, 4 and 5, which together form a Möbius strip. Its boundary is the closed line $[1, 2, 3, 4, 9, 10, 6, 7, 8]$

Fig. 7 The faces 2, 3 and 4, which together form a Möbius strip. Its boundary is the closed line [1, 2, 3, 7, 8, 9, 10, 6, 5]

Fig. 8 The faces 2, 3 and 6, which together form a Möbius strip. Its boundary is the closed line [1, 2, 10, 6, 7, 8, 9, 4, 5]

Fig. 9 The faces 4, 5 and 6, which together form a Möbius strip. Its boundary is the closed line [2, 3, 4, 5, 6, 7, 8, 9, 10]

Fig. 10 A Möbius strip can be constructed from four convex quadrilaterals, as well as from three quadrilaterals, two of which, however, are concave

Fig. 11 The net of the model

Reference

1. Coxeter, H.S.M., Moser, W.O.J.: Generators and Relations for Discrete Groups. Springer, Berlin (1972)