Rigidity and the Lower Bound Theorem for Doubly Cohen–Macaulay Complexes

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Abstract We prove that for $d \geq 3$, the 1-skeleton of any $(d - 1)$ -dimensional doubly Cohen–Macaulay (abbreviated 2-CM) complex is generically d -rigid. This implies that Barnette’s lower bound inequalities for boundary complexes of simplicial polytopes (Barnette, D. *Isr. J. Math.* 10:121–125, 1971; Barnette, D. *Pac. J. Math.* 46:349–354, 1973) hold for every 2-CM complex of dimension ≥ 2 (see Kalai, G. *Invent. Math.* 88:125–151, 1987). Moreover, the initial part (g_0, g_1, g_2) of the g -vector of a 2-CM complex (of dimension ≥ 3) is an M -sequence. It was conjectured by Björner and Swartz (*J. Comb. Theory Ser. A* 113:1305–1320, 2006) that the entire g -vector of a 2-CM complex is an M -sequence.

1 Introduction

The g -theorem gives a complete characterization of the f -vectors of boundary complexes of simplicial polytopes. It was conjectured by McMullen in 1970 and proved by Billera and Lee [5] (sufficiency) and by Stanley [13] (necessity) in 1980. A major open problem in f -vector theory is the g -conjecture, which asserts that this characterization holds for all homology spheres. The open part of this conjecture is to show that the g -vector of every homology sphere is an M -sequence, i.e. it is the f -vector of some order ideal of monomials. Based on the fact that homology spheres are doubly Cohen–Macaulay (abbreviated 2-CM) and that the g -vector of some other classes of 2-CM complexes is known to be an M -sequence (e.g. [14]), Björner and Swartz [14] recently suspected that

Conjecture 1.1 ([14], a weakening of Problem 4.2.) *The g -vector of any 2-CM complex is an M -sequence.*

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We prove a first step in this direction, namely:

Theorem 1.2 *Let K be a $(d - 1)$ -dimensional 2-CM simplicial complex (over some field) where $d \geq 4$. Then $(g_0(K), g_1(K), g_2(K))$ is an M -sequence.*

This theorem follows from the following theorem, combined with an interpretation of rigidity in terms of the face ring (Stanley–Reisner ring), due (implicitly) to Lee [10].

Theorem 1.3 *Let K be a $(d - 1)$ -dimensional 2-CM simplicial complex (over some field) where $d \geq 3$. Then K has a generically d -rigid 1-skeleton.*

Kalai [8] showed that if a simplicial complex K of dimension ≥ 2 satisfies the following conditions then it satisfies Barnette’s lower bound inequalities:

- (a) K has a generically $(\dim(K) + 1)$ -rigid 1-skeleton.
- (b) For each face F of K of codimension > 2 , its link $lk_K(F)$ has a generically $(\dim(lk_K(F)) + 1)$ -rigid 1-skeleton.
- (c) For each face F of K of codimension 2, its link $lk_K(F)$ (which is a graph) has at least as many edges as vertices.

Kalai used this observation to prove that Barnette’s inequalities hold for a large class of simplicial complexes.

Observe that the link of a vertex in a 2-CM simplicial complex is 2-CM, and that a 2-CM graph is 2-connected. Combining it with Theorem 1.3 and the above result of Kalai we conclude:

Corollary 1.4 *Let K be a $(d - 1)$ -dimensional 2-CM simplicial complex where $d \geq 3$. For all $0 \leq i \leq d - 1$ $f_i(K) \geq f_i(n, d)$ where $f_i(n, d)$ is the number of i -faces in a (equivalently every) stacked d -polytope on n vertices. (Explicitly, $f_{d-1}(n, d) = (d - 1)n - (d + 1)(d - 2)$ and $f_i(n, d) = \binom{d}{i}n - \binom{d+1}{i+1}i$ for $1 \leq i \leq d - 2$.)*

Theorem 1.3 is proved by decomposing K into a union of minimal $(d - 1)$ -cycle complexes (Fogelsanger’s notion [6]). Each of these pieces has a generically d -rigid 1-skeleton ([6]), and the decomposition is such that gluing the pieces together results in a complex with a generically d -rigid 1-skeleton. The decomposition is detailed in Theorem 3.4.

This paper is organized as follows: In Sect. 2 we give the necessary background from rigidity theory, explain the connection between rigidity and the face ring, and reduce the results mentioned in the Introduction to Theorem 3.4. In Sect. 3 we give the necessary background on 2-CM complexes, prove Theorem 3.4 and discuss related problems and results.

2 Rigidity

The presentation of rigidity here is based mainly on the one in Kalai [8].

Let $G = (V, E)$ be a graph. A map $f : V \rightarrow \mathbb{R}^d$ is called a d -embedding. It is *rigid* if any small enough perturbation of it which preserves the lengths of the edges is induced by an isometry of $\mathbb{R}^d$. Formally, f is called *rigid* if there exists an $\varepsilon > 0$ such that if $g : V \rightarrow \mathbb{R}^d$ satisfies $d(f(v), g(v)) < \varepsilon$ for every $v \in V$ and $d(g(u), g(w)) = d(f(u), f(w))$ for every $\{u, w\} \in E$, then $d(g(u), g(w)) = d(f(u), f(w))$ for every $u, w \in V$ (where $d(a, b)$ denotes the Euclidean distance between the points a and b).

G is called *generically d -rigid* if the set of its rigid d -embeddings is open and dense in the topological vector space of all of its d -embeddings.

Let $V = [n]$, and let $\text{Rig}(G, f)$ be the $dn \times |E|$ matrix which is defined as follows: for its column corresponding to $\{v < u\} \in E$ put the vector $f(v) - f(u)$ (resp. $f(u) - f(v)$) at the entries of the d rows corresponding to v (resp. u) and zero otherwise. G is generically d -rigid iff $\text{Im}(\text{Rig}(G, f)) = \text{Im}(\text{Rig}(K_V, f))$ for a generic f , where K_V is the complete graph on V . $\text{Rig}(G, f)$ is called the *rigidity matrix* of G (its rank is independent of the generic f that we choose).

Let G be the 1-skeleton of a $(d-1)$ -dimensional simplicial complex K . We define d generic degree-one elements in the polynomial ring $A = \mathbb{R}[x_1, \dots, x_n]$ as follows: $\Theta_i = \sum_{v \in [n]} f(v)_i x_v$ where $f(v)_i$ is the projection of $f(v)$ on the i -th coordinate, $1 \leq i \leq d$. Then the sequence $\Theta = (\Theta_1, \dots, \Theta_d)$ is a linear system of parameters for the face ring $\mathbb{R}[K] = A/I_K$ (I_K is the ideal in A generated by the monomials whose support is not an element of K). Let $H(K) = \mathbb{R}[K]/(\Theta) = H(K)_0 \oplus H(K)_1 \oplus \dots$ where (Θ) is the ideal in A generated by the elements of Θ and the grading is induced by the degree grading in A . Consider the multiplication map $\omega : H(K)_1 \rightarrow H(K)_2$, $m \rightarrow \omega m$ where $\omega = \sum_{v \in [n]} x_v$. Lee [10] proved that

$$\dim_{\mathbb{R}} \text{Ker}(\text{Rig}(G, f)) = \dim_{\mathbb{R}} H(K)_2 - \dim_{\mathbb{R}} \omega(H(K)_1). \quad (1)$$

Assume that G is generically d -rigid. Then $\dim_{\mathbb{R}} \text{Ker}(\text{Rig}(G, f)) = f_1(K) - \text{rank}(\text{Rig}(K_V, f)) = g_2(K) = \dim_{\mathbb{R}} H(K)_2 - \dim_{\mathbb{R}} H(K)_1$. Combining with (1), the map ω is injective, and hence $\dim_{\mathbb{R}}(H(K)/(\omega))_i = g_i(K)$ for $i = 2$; clearly this holds for $i = 0, 1$ as well. Hence $(g_0(K), g_1(K), g_2(K))$ is an M -sequence. We conclude that Theorem 1.3 implies Theorem 1.2, via the following algebraic result:

Theorem 2.1 *Let K be a $(d-1)$ -dimensional 2-CM simplicial complex (over some field) where $d \geq 3$. Then the multiplication map $\omega : H(K)_1 \rightarrow H(K)_2$ is injective.*

In order to prove Theorem 1.3, we need the concept of minimal cycle complexes, introduced by Fogelsanger [6]. We summarize his theory below.

Fix a field k (or more generally, any Abelian group) and consider the formal chain complex on a ground set $[n]$, $C = (\bigoplus \{kT : T \subseteq [n]\}, \partial)$, where $\partial(1T) = \sum_{t \in T} \text{sign}(t, T) T \setminus \{t\}$ and $\text{sign}(t, T) = (-1)^{|\{s \in T : s < t\}|}$. Define *subchain*, *minimal d -cycle* and *minimal d -cycle complex* as follows: $c' = \sum \{b_T T : T \subseteq [n], |T| = d+1\}$ is a *subchain* of a d -chain $c = \sum \{a_T T : T \subseteq [n], |T| = d+1\}$ iff for every such T , $b_T = a_T$ or $b_T = 0$. A d -chain c is a *d -cycle* if $\partial(c) = 0$, and is a *minimal d -cycle* if its only subchains which are cycles are c and 0. A simplicial complex K which is spanned by the support of a *minimal d -cycle* is called a *minimal d -cycle complex* (over k), i.e. $K = \{S : \exists T \ S \subseteq T, a_T \neq 0\}$ for some minimal d -cycle c as above. For example, triangulations of connected manifolds without boundary are minimal cycle complexes—fix $k = \mathbb{Z}_2$ and let the cycle be the sum of all facets.

The following is the main result in Fogelsanger's thesis.

Theorem 2.2 (Fogelsanger [6]) *For $d \geq 3$, every minimal $(d - 1)$ -cycle complex has a generically d -rigid 1-skeleton.*

We will need the following gluing lemma, due of Asimov and Roth, who introduced the concept of generic rigidity of graphs [1].

Theorem 2.3 (Asimov and Roth [2]) *Let G_1 and G_2 be generically d -rigid graphs. If $G_1 \cap G_2$ contains at least d vertices, then $G_1 \cup G_2$ is generically d -rigid.*

Now we are ready to conclude Theorem 1.3 from the decomposition theorem, Theorem 3.4.

Proof of Theorem 1.3 Consider a decomposition sequence of K as guaranteed by Theorem 3.4, $K = \bigcup_{i=1}^m S_i$. By Theorem 2.2 each S_i has a generically d -rigid 1-skeleton. By Theorem 2.3 for all $2 \leq i \leq m$ $\bigcup_{j=1}^i S_j$ has a generically d -rigid 1-skeleton, in particular K has a generically d -rigid 1-skeleton ($i = m$). $\square$

Remark One can verify that Theorems 2.2 and 2.3, and hence also Theorem 1.3, continue to hold when replacing “generically d -rigid” by the notion “ d -hypperconnected”, introduced by Kalai [7]. Both of these assertions have an interpretation in terms of algebraic shifting, introduced by Kalai (see e.g. his survey [9]), namely: for both the exterior and symmetric shifting operators over the field $\mathbb{R}$, denoted by Δ , $\{d, n\} \in \Delta(K)$. The existence of this edge in the shifted complex implies the non-negativity of $g_2(K)$.

3 Decomposing a 2-CM Complex

Definition 3.1 A simplicial complex K is 2-CM (over a fixed field k) if it is Cohen–Macaulay and for every vertex $v \in K$, $K - v$ is Cohen–Macaulay of the same dimension as K .

Here $K - v$ is the simplicial complex $\{T \in K : v \notin T\}$. By a theorem of Reisner [11], a simplicial complex L is Cohen–Macaulay iff it is pure and for every face $T \in L$ (including the empty set) and every $i < \dim(lk_L(T))$, $\tilde{H}_i(lk_L(T); k) = 0$ where $lk_L(T) = \{S \in L : T \cap S = \emptyset, T \cup S \in L\}$ and $\tilde{H}_i(M; k)$ is the reduced i -th homology of M over k . The proof of Theorem 3.4 is by induction on $\dim(K)$. Let us first consider the case where K is 1-dimensional.

A (simple finite) graph is 2-connected if after a deletion of any vertex from it, the remaining graph is connected and nontrivial (i.e. is not a single vertex nor empty). Note that a graph is 2-CM iff it is 2-connected.

Lemma 3.2 *A graph G is 2-connected iff there exists a decomposition $G = \bigcup_{i=1}^m C_i$ such that each C_i is a simple cycle and for every $1 < i \leq m$, $C_i \cap (\bigcup_{j < i} C_j)$ contains an edge.*

Moreover, for each $i_0 \in [m]$ the C_i 's can be reordered by a permutation $\sigma : [m] \rightarrow [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $C_{\sigma^{-1}(i)} \cap (\bigcup_{j < i} C_{\sigma^{-1}(j)})$ contains an edge.

Proof Whitney [15] showed that a graph G is 2-connected iff it has an open ear decomposition, i.e. there exists a decomposition $G = \bigcup_{i=0}^m P_i$ such that each P_i is a simple open path, P_0 is an edge, $P_0 \cup P_1$ is a simple cycle and for every $1 < i \leq m$ $P_i \cap (\bigcup_{j < i} P_j)$ equals the 2 end vertices of P_i .

Assume that G is 2-connected and consider an open ear decomposition as above. Let $C_1 = P_0 \cup P_1$. For $i > 1$ choose a simple path $\tilde{P}_i$ in $\bigcup_{j < i} P_j$ that connects the 2 end vertices of P_i , and let $C_i = P_i \cup \tilde{P}_i$. $(C_1, \dots, C_m)$ is the desired decomposition sequence of G .

Let C be the graph whose vertices are the C_i 's and two of them are neighbors iff they have an edge in common. Thus, C is connected, and hence the 'Moreover' part of the Lemma is proved.

The other implication, that such a decomposition implies 2-connectivity, will not be used in the sequel, and its proof is omitted. $\square$

For the induction step we need the following cone lemma. For v a vertex not in the support of a $(d-1)$ -chain c , let $v * c$ denote the following d -chain: if $c = \sum \{a_T T : v \notin T \subseteq [n], |T| = d\}$ where $a_T \in k$ for all T , then $v * c = \sum \{\text{sign}(v, T) a_T T \cup \{v\} : v \notin T \subseteq [n], |T| = d\}$ where $\text{sign}(v, T) = (-1)^{|\{t \in T : t < v\}|}$.

Lemma 3.3 *Let s be a minimal $(d-1)$ -cycle and let c be a minimal d -chain such that $\partial(c) = s$, i.e. c has no proper subchain c' such that $\partial(c') = s$. For v a vertex not in any face in $\text{supp}(c)$, the support of c , define $\tilde{s} = c - v * s$. Then $\tilde{s}$ is a minimal d -cycle.*

Proof $\partial(\tilde{s}) = \partial(c) - \partial(v * s) = s - (s - v * \partial(s)) = 0$ hence $\tilde{s}$ is a d -cycle. To show that it is minimal, let $\hat{s}$ be a subchain of $\tilde{s}$ such that $\partial(\hat{s}) = 0$. Note that $\text{supp}(c) \cap \text{supp}(v * s) = \emptyset$.

Case 1: v is contained in a face in $\text{supp}(\hat{s})$. By the minimality of s , $\text{supp}(v * s) \subseteq \text{supp}(\hat{s})$. Thus, by the minimality of c also $\text{supp}(c) \subseteq \text{supp}(\hat{s})$ and hence $\hat{s} = \tilde{s}$.

Case 2: v is not contained in any face in $\text{supp}(\hat{s})$. Thus, $\text{supp}(\hat{s}) \subseteq \text{supp}(c)$. As $\partial(\hat{s}) = 0$ then $\partial(c - \hat{s}) = s$. The minimality of c implies $\hat{s} = 0$. $\square$

Theorem 3.4 *Let K be a d -dimensional 2-CM simplicial complex over a field k ($d \geq 1$). Then there exists a decomposition $K = \bigcup_{i=1}^m S_i$ such that each S_i is a minimal d -cycle complex over k and for every $i > 1$, $S_i \cap (\bigcup_{j < i} S_j)$ contains a d -face.*

Moreover, for each $i_0 \in [m]$ the S_i 's can be reordered by a permutation $\sigma : [m] \rightarrow [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $S_{\sigma^{-1}(i)} \cap (\bigcup_{j < i} S_{\sigma^{-1}(j)})$ contains a d -face.

Proof The proof is by induction on d . For $d = 1$, by Lemma 3.2 $K = \bigcup_{i=1}^{m(K)} C_i$ such that each C_i is a simple cycle and for every $i > 1$ $C_i \cap (\bigcup_{j < i} C_j)$ contains an edge. Define $s_i = \sum \{\text{sign}_e(i) e : e \in (C_i)_1\}$, then s_i is a minimal 1-cycle (orient the edges

properly: $\text{sign}_e(i)$ equals 1 or -1 accordingly) whose support spans the simplicial complex C_i . Moreover, by Lemma 3.2 each C_{i_0} , $i_0 \in [m(K)]$, can be chosen to be the first in such a decomposition sequence.

For $d > 1$, note that the link of every vertex in a 2-CM simplicial complex is 2-CM. For a vertex $v \in K$, as $lk_K(v)$ is 2-CM then by the induction hypothesis $lk_K(v) = \bigcup_{i=1}^{m(v)} C_i$ such that each C_i is a minimal $(d-1)$ -cycle complex and for every $i > 1$ $C_i \cap (\bigcup_{j<i} C_j)$ contains a $(d-1)$ -face. Let s_i be a minimal $(d-1)$ -cycle whose support spans C_i . As $K-v$ is CM of dimension d , $\tilde{H}_{d-1}(K-v; k) = 0$. Hence there exists a d -chain c such that $\partial(c) = s_i$ and $\text{supp}(c) \subseteq K-v$.

Take c_i to be such a chain with a support of minimal cardinality. By Lemma 3.3, $\tilde{s}_i = c_i - v * s_i$ is a minimal d -cycle. Let $S_i(v)$ be the simplicial complex spanned by $\text{supp}(\tilde{s}_i)$; it is a minimal d -cycle complex. By the induction hypothesis, for every $i > 1$ $S_i(v) \cap (\bigcup_{j<i} S_j(v))$ contains a d -face (containing v). Thus, $K(v) := \bigcup_{j=1}^{m(v)} S_j(v)$ has the desired decomposition for every $v \in K$. $K = \bigcup_{v \in \text{Ver}(K)} K(v)$ as $st_K(v) \subseteq K(v)$ for every v , where $st_K(v) = \{T \in K : T \cup \{v\} \in K\}$.

Let v be any vertex of K . Since the 1-skeleton of K is connected, we can order the vertices of K such that $v_1 = v$ and for every $i > 1$ v_i is a neighbor of some v_j where $1 \leq j < i$. Let $v_{l(i)}$ be such a neighbor of v_i . By the induction hypothesis we can order the $S_j(v_i)$'s such that $S_1(v_i)$ will contain $v_{l(i)}$, and hence, as K is pure, will contain a d -face which appears in $K(v_{l(i)})$ (this face contains the edge $\{v_i, v_{l(i)}\}$). The resulting decomposition sequence $(S_1(v_1), \dots, S_{m(v_1)}(v_1), S_1(v_2), \dots, S_{m(v_n)}(v_n))$ is as desired.

Moreover, every $S_j(v_{i_0})$ where $i_0 \in [n]$ and $j \in [m(v_{i_0})]$ can be chosen to be the first in such a decomposition sequence. Indeed, by the induction hypothesis $S_j(v_{i_0})$ can be the first in the decomposition sequence of $K(v_{i_0})$, and as mentioned before, the connectivity of the 1-skeleton of K guarantees that each such prefix $(S_1(v_{i_0}), \dots, S_{m(v_{i_0})}(v_{i_0}))$ can be completed to a decomposition sequence of K on the same $S_j(v_i)$'s. $\square$

Theorem 1.3 follows also from the following corollary combined with Theorem 2.2.

Corollary 3.5 *Let K be a d -dimensional 2-CM simplicial complex over a field k ($d \geq 1$). Then K is a minimal cycle complex over the Abelian group $\tilde{k} = k(x_1, x_2, \dots)$ whose elements are finite linear combinations of the (variables) x_i 's with coefficients in k .*

Proof Consider a decomposition $K = \bigcup_{i=1}^m S_i$ as guaranteed by Theorem 3.4, where $S_i = \overline{\text{supp}(c_i)}$ (the closure w.r.t. inclusion of $\text{supp}(c_i)$) for some minimal d -cycle c_i over k . Define $\tilde{c}_i = x_i c_i$, thus $\tilde{c}_i$ is a minimal cycle over $\tilde{k}$. Define $\tilde{c} = \sum_{i=1}^m \tilde{c}_i$. Clearly $\tilde{c}$ is a cycle over $\tilde{k}$ whose support spans K . It remains to show that $\tilde{c}$ is minimal. Let $\tilde{c}'$ be a subchain of $\tilde{c}$ which is a cycle, $\tilde{c}' \neq \tilde{c}$. We need to show that $\tilde{c}' = 0$. Denote by $\tilde{\alpha}_T$ ($\tilde{\alpha}'_T$) the coefficient of the set T in $\tilde{c}$ ($\tilde{c}'$) and by $\tilde{\alpha}_T(i)$ the coefficient of the set T in $\tilde{c}_i$. If $\tilde{\alpha}'_T = 0$ then for every i such that $\tilde{\alpha}_T(i) \neq 0$, the minimality of $\tilde{c}_i$ implies that $\tilde{\alpha}'_T = 0$ whenever $\tilde{\alpha}_T(i) \neq 0$. By assumption, there exists a set T_0 such that $\tilde{\alpha}'_{T_0} = 0 \neq \tilde{\alpha}_{T_0}$. In particular, there exists an index i_0 such

that $\tilde{\alpha}_{T_0}(i_0) \neq 0$, hence $\tilde{\alpha}'_F = 0$ whenever $\tilde{\alpha}_F(i_0) \neq 0$. As $S_{i_0} \cap (\bigcup_{j < i_0} S_j)$ contains a d -face in case $i_0 > 1$, repeated application of the above argument implies $\tilde{\alpha}'_F = 0$ whenever $\tilde{\alpha}_F(1) \neq 0$. Repeated application of the fact that $S_i \cap (\bigcup_{j < i} S_j)$ contains a d -face for $i = 2, 3, \dots$ and of the above argument shows that $\tilde{\alpha}'_F = 0$ whenever $\tilde{\alpha}_F(i) \neq 0$ for some $1 \leq i \leq m$, i.e. $\tilde{c}' = 0$. $\square$

A pure simplicial complex has a *nowhere zero flow* if there is an assignment of integer non-zero weights to all of its facets which forms a $\mathbb{Z}$ -cycle. This generalizes the definition of a nowhere zero flow for graphs (e.g. [12] for a survey).

Corollary 3.6 *Let K be a d -dimensional 2-CM simplicial complex over $\mathbb{Q}$ ($d \geq 1$). Then K has a nowhere zero flow.*

Proof Consider a decomposition $K = \bigcup_{i=1}^m S_i$ as guaranteed by Theorem 3.4. Multiplying by a common denominator, we may assume that each $S_i = \overline{\text{supp}(c_i)}$ for some minimal d -cycle c_i over $\mathbb{Z}$ (instead of just over $\mathbb{Q}$). Let N be the maximal $|\alpha|$ over all nonzero coefficients α of the c_i 's, $1 \leq i \leq m$. Let $\tilde{c} = \sum_{i=1}^m (N^m)^i c_i$. $\tilde{c}$ is a nowhere zero flow for K ; we omit the details. $\square$

Problem 3.7 *Can the S_i 's in Theorem 3.4 be taken to be homology spheres?*

Yhوناتan Iron and I proved (unpublished) the following lemma:

Lemma 3.8 *Let K , L and $K \cap L$ be simplicial complexes of the same dimension $d - 1$. Assume that K and L are weak-Lefschetz, i.e. that multiplication by a generic degree-one element g in $H = H(K), H(L)$, $g : H_{i-1} \rightarrow H_i$, is injective for all $i \leq \lfloor d/2 \rfloor$. If $K \cap L$ is CM then $K \cup L$ is weak-Lefschetz.*

In view of this lemma, if the intersections $S_i \cap (\bigcup_{j < i} S_j)$ in Theorem 3.4 can be taken to be CM, and the S_i 's can be taken to be homology spheres, then Conjecture 1.1 would be reduced to the long standing g -conjecture for homology spheres. Can the intersections be guaranteed to be CM?

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