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# An Inscribing Model for Random Polytopes

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**Abstract** For convex bodies $K$ with $\mathcal{C}^2$ boundary in $\mathbb{R}^d$, we explore random polytopes with vertices chosen along the boundary of $K$. In particular, we determine asymptotic properties of the volume of these random polytopes. We provide results concerning the variance and higher moments of this functional, as well as an analogous central limit theorem.

## 1 Introduction

Let $X$ be a set in $\mathbb{R}^d$ and let $t_1, \ldots, t_n$ be independent random points chosen according to some distribution $\mu$ on $X$. The convex hull of the $t_i$'s is called a *random polytope* and its study is an active area of research which links together combinatorics, geometry and probability. This study traces its root to the middle of the nineteenth century with Sylvester's famous question about the probability of four random points in the plane forming a convex quadrangle [17], and has become a mainstream research area since the mid 1960s, following the investigation of Rényi and Sulanke [13] and Efron [8].

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Throughout this paper, if not otherwise mentioned, we fix a convex body $K \in \mathcal{K}^2_+$, where $\mathcal{K}^2_+$ is the set of compact, convex bodies in $\mathbb{R}^d$ which have non-empty interior and whose boundaries are $\mathcal{C}^2$ and have everywhere positive Gauß–Kronecker curvature. The reader who is interested in the case of general $K$, e.g. when $K$ is a polytope, is referred to [7, 18, 19]. Without loss of generality, we also assume $K$ has volume 1. For a set $X \subset \mathbb{R}^d$ we define $[X]$ to be the convex hull of $X$.

A standard definition for the notion of a random polytope is as follows. Let $t_1, \dots, t_n$ be independent random points chosen according to the uniform distribution on $K$. We let $K_n = [\{t_1, \dots, t_n\}]$. Here and later we write $K_n = \{t_1, \dots, t_n\}$ instead to simplify notations without causing much confusion. Another one, which we call the "inscribing polytope" model, also begins with a convex body $K$, but the points are chosen from the surface of $K$ with respect to a properly defined measure. The main goal of the theory of random polytopes is to understand the asymptotic behavior ($n \to \infty$) of certain key functionals on $K_n$, such as the volume or the number of faces.

For most of these functionals, the expectations have been estimated (either approximately or up to a constant factor) for a long time, due to collective results of many researchers (we refer the interested reader to [5, 20] and [15] for surveys). The main open question is thus to understand the distributions of these functionals around their means, as coined by Weil and Wieacker's survey from the Handbook of Convex Geometry (see the concluding paragraph of [20])

> *We finally emphasize that the results described so far give mean values hence first-order information on random sets and point processes. This is due to the geometric nature of the underlying integral geometric results. There are also some less geometric methods to obtain higher-order informations or distributions, but generally the determination of variance, e.g., is a major open problem.*

The last few years have seen several developments in this direction, thanks to new methods and tools from modern probability. Let us first discuss the model $K_n$ where the points are chosen inside $K$. Reitzner [11], using the Efron–Stein inequality, shows that

$$\operatorname{Var} \mathrm{Vol_d}(K_n) = O(n^{-\frac{d+3}{d+1}}),$$
$$\operatorname{Var} f_i(K_n) = O(n^{\frac{d-1}{d+1}}),$$

where $\mathrm{Vol_d}$ is the standard volume measure on $\mathbb{R}^d$, $f_i$ denotes the number of $i$-dimensional faces. For convenience, we let $Z = \mathrm{Vol_d}(K_n)$. Using martingale techniques, Vu [18] proves the following tail estimate

$$\mathbb{P}\left(|Z - \mathbb{E}Z| \geq \sqrt{\lambda n^{-\frac{d+3}{d+1}}}\right) \leq \exp(-c\lambda) + \exp(-c'n)$$

for any $0 < \lambda < n^{\alpha}$, where $c, c'$ and $\alpha$ are positive constants. A similar bound also holds for $f_i$ with the same proof. From this tail estimate, one can deduce the above variance bound and also bounds for any fixed moments. These moment bounds are sharp, up to a constant, as shown by Reitzner in [10]. Thus, the order of magnitudes of all fixed moments are determined.

Another topic where a significant development has been made is central limit theorems. It has been conjectured that the key functionals such as the volume and number of faces satisfy a central limit theorem.

**Conjecture** (CLT conjecture) *Let $K_n$ be the random polytope determined by n random points chosen in $K$. Then there is a function $\epsilon(n)$ tending to zero with n such that for every x*

$$\left|\mathbb{P}\left(\frac{Z-\mathbb{E}Z}{\sqrt{\operatorname{Var} Z}}\le x\right)-\Phi(x)\right|\le \epsilon(n),$$

*where $\Phi$ denotes the distribution function of the standard normal distribution.*

Reitzner [10], using an inequality due to Rinott [14] (which proved a central limit theorem for a sum of weakly dependent random variables), showed that a central limit theorem really holds for the volume and number of faces of the so-called Poisson random polytope. This is a variant of $K_n$, where the number of random points is not $n$, but a Poisson random variable with mean $n$. This model has the advantage that the numbers of points found in disjoint regions of $K$ are independent, a fact which is technically useful. Combining the above tail estimate and Reitzner's result, Vu [19] proved the CLT conjecture.

The above results together provide a fairly comprehensive picture about $K_n$ when the points are chosen inside $K$. We refer the reader to the last section of [19] for a detailed summary. The main goal of this paper is to provide such a picture for the inscribing model, where points are chosen on the surface of $K$.

Before we may speak about selecting points on the boundary $\partial K$, we need to specify the probability measure on $\partial K$. One wants the random polytope to approximate the original convex body $K$ in the sense that the symmetric difference of the volume of $K$ and $K_n$ is as small as possible. Hence, intuitively, a measure that puts more weight on regions of higher curvature is desired. A good discussion on this can be found in [16]. Let $\mu_{d-1}$ be the $(d-1)$-dimensional Hausdorff measure restricted to $\partial K$. We let $\mu$ be a probability measure on $\partial K$ such that

$$d\mu = \rho d\mu_{d-1}, \tag{1}$$

where $\rho : \partial K \to \mathbb{R}_+$ is a positive, continuous function with $\int_{\partial K} \rho d\mu_{d-1} = 1$.

Note that the assumption $\rho > 0$ is essential, as otherwise we might have a measure that causes $K_n$ to always lie in at most half (or any portion) of $K$ with probability 1.

With the boundary measure properly defined, we can choose $n$ random points on the boundary of $K$ independently according to $\mu_\rho$ on $\partial K$. Denote the convex hull of these $n$ points by $K_n$ and we call it *random inscribed polytope*. For this model, the volume is perhaps the most interesting functional (as the number of vertices is always $n$), and it will be the focus of the present work. For notational convenience, we denote $Z$ for $\mathrm{Vol_d}(K_n)$ throughout this paper.

The inscribing model is somewhat more difficult to analyze than the model where points are chosen inside $K$. Indeed, sharp estimates on the volume were obtained only recently, thanks to the tremendous effort of Schütt and Werner, in a long and highly technical paper [16]. We have

$$\mathbb{E}Z = 1 - (c_K + o(1))n^{-\frac{2}{d-1}} \tag{2}$$

where $c_K$ is a constant depending on $K$ (the 1 here represents the volume of $K$).

It is worth recalling that in the model where points are chosen uniformly inside $K$ it is known that $\mathbb{E}\mathrm{Vol_d}(K - K_n) = O(n^{-\frac{2}{d+1}})$. Observe that by inserting $n^{\frac{d+1}{d-1}}$ for $n$ in this result we obtain a function $O(n^{-\frac{2}{d-1}})$, which is the correct growth rate found in (2). We can explain this (at least intuitively) by noting that in the uniform model, the expected number of vertices is $\Theta(n^{\frac{d-1}{d+1}})$. However, in the inscribing model all points are vertices. Thus we may view the uniform model on $n$ points as yielding the same type of behavior as the inscribing model on $n^{\frac{d-1}{d+1}}$ points. Further evidence for this behavior is given by Reitzner in [12] where he obtains estimates (which are sharp up to a constant factor) for all intrinsic volumes.

Reitzner gives an upper bound on the variance [11]:

$$\mathrm{Var}\, Z = O(n^{-\frac{d+3}{d-1}}).$$

The first result we show in this paper is that the variance estimate is sharp, up to a constant factor.

**Theorem 1.1** (Variance) *Given* $K \in \mathcal{K}^2_+$,

$$\mathrm{Var}\, Z = \Omega(n^{-\frac{d+3}{d-1}}),$$

*where the implicit constant depends on dimension $d$ and the convex body $K$ only.*

The next result in this paper shows that the volume has exponential tail.

**Theorem 1.2** (Concentration) *For a given convex body* $K \in \mathcal{K}^2_+$, *there are positive constants* $\alpha$ *and* $c$ *such that the following holds. For any constant* $0 < \eta < \frac{d-1}{3d+1}$ *and* $0 < \lambda \le \frac{\alpha}{4} n^{\frac{d-1}{3d+1} + \frac{2(d+1)\eta}{d-1}} < \frac{\alpha}{4} n$, *we have*

$$\mathbb{P}(|Z - \mathbb{E}Z| \ge \sqrt{\lambda V_0}) \le 2\exp(-\lambda/4) + \exp(-cn^{\frac{d-1}{3d+1} - \eta}), \tag{3}$$

*where* $V_0 = \alpha n^{-\frac{d+3}{d-1}}$.

It is easy to deduce from this theorem the following:

**Corollary 1.3** (Moments) *For any given convex body $K$ and $k \ge 2$, the $k$th moments of $Z$ satisfies*

$$M_k = O((n^{-\frac{d+3}{d-1}})^{k/2}).$$

To emphasize the dependence of $Z = \mathrm{Vol_d} K_n$ on $n$, we write $Z_n$ instead of $Z$ in the following result:

**Corollary 1.4** (Rate of convergence)

$$\lim_{n\to\infty}\left|\left(\frac{Z_n}{\mathbb{E}Z_n}-1\right)f(n)\right|=0$$

*almost surely, for*

$$f(n)=\delta(n)(n^{-\frac{d+3}{d-1}}\ln n)^{-1/2}$$

*where $\delta(n)$ is a function tending to zero arbitrarily slowly as $n\to\infty$.*

Finally, we obtain the central limit theorem for the Poisson model. Let $K\in\mathcal{K}^2_+$, and let $\mathrm{Pois}(n)$ be a Poisson point process with intensity $n$. Then the intersection of $\mathrm{Pois}(n)$ and $\partial K$ consists of random points $\{t_1,\dots,t_N\}$ where the number of points $N$ is Poisson distributed with mean $n\mu(\partial K)=n$. We write $\Pi_n=[x_1,\dots,x_N]$.

**Theorem 1.5** *Given $K\in\mathcal{K}^2_+$, we have*

$$\left|\mathbb{P}\left(\frac{\mathrm{Vol_d}(\Pi_n)-\mathbb{E}\mathrm{Vol_d}(\Pi_n)}{\sqrt{\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n)}}\le x\right)-\Phi(x)\right|=o(1),$$

*where the $o(1)$ term is of order $O(n^{-\frac{1}{4}}\ln^{\frac{d+2}{d-1}}n)$ as $n\to\infty$.*

We hope this result will infer a central limit theorem for $K_n$, which indeed is the case for random polytopes where the points are chosen inside $K$, as mentioned earlier (see [10, 19]). However, for random inscribing polytopes, some difficulties remain. We are, however, able to prove that the two models are very close in the sense that the expectations of volume for the two models are asymptotically equivalent, and the variances are only off by constant multiplicative factor (see Theorem 5.5).

In the rest of the paper, we present the proof of the above theorems in Sects. 3, 4, and 5, respectively; Sect. 2 is devoted to notations; we also present proofs of some crucial technical lemmas in the appendix, along with statements of many other lemmas whose proofs can either be found or deduced relatively easily from the literature (see, e.g., [5, 10–12, 18]).

## 2 Notations

### 2.1 Geometry

The vectors $e_1,\dots,e_d$ always represent a fixed orthonormal basis of $\mathbb{R}^d$. The discussions in this paper, unless otherwise specified, are all based on this basis. For a vector $x$, we denote its coordinate by $x^1,\dots,x^d$, i.e. $x=(x^1,\dots,x^d)$. By $B^i(x,r)$ we indicate the $i$-dimensional Euclidean closed ball of radius $r$ centered at $x$, i.e.

$$B^i(x,r)=\{y\in\mathbb{R}^i\mid\|x-y\|\le r\}.$$

The norm $\|\cdot\|$ is the Euclidean norm. When the dimension is $d$, we sometimes simply write $B(x,r)$.

For points $t_1, \dots, t_n \in \mathbb{R}^d$, the convex hull of them is defined by

$$[t_1, \dots, t_n] = \left\{ \lambda_1 t_1 + \cdots + \lambda_n t_n \,\middle|\, 0 \le \lambda_i \le 1,\ 1 \le i \le n, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

In particular, the closed line segment between two points $x$ and $y$ is

$$[x, y] = \{\lambda x + (1 - \lambda) y \mid 0 \le \lambda \le 1\}.$$

To analyze the geometry, it is necessary to introduce the following. For any $y \in \mathbb{R}^d$ write $y = (y^1, \dots, y^d)$ for the coordinates with respect to some fixed basis $e_1, \dots, e_d$. For unit vector $u \in \mathbb{R}^d$, let $H(u, h) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = h\}$, where here $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^d$. Further, the halfspace associated to this hyperplane we denote by $H^+(u, h) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle \ge h\}$. Since $K$ is smooth, for each point $y \in \partial K$, there is some unique outward normal $u_y$. We thus may define the *cap* $C = C(y, h)$ of $K$ to be $H^+(u_y, h_K(y) - h) \cap K$, where $h_K(y)$ is the support function such that $H^+(u_y, h_K(y))$ intersects $K$ in the point $y$ only. In general, one should think of a cap as $K \cap H^+$ where $H^+$ is some closed half space. Throughout this paper, we also use the notion of $\epsilon$*-cap* to emphasize that $\mathrm{Vol_d}(C) = \mathrm{Vol_d}(K \cap H^+) = \epsilon$. Similarly, we call $C = K \cap H^+$ an $\epsilon$*-boundary* cap to emphasize that $\mu(\partial K \cap H^+) = \epsilon$.

We define the $\epsilon$*-wet part* of $K$ to be the union of all caps that are $\epsilon$-boundary caps of $K$ and we denote it by $F_\epsilon^c$. The complement of the $\epsilon$-wet part in $K$ is said to be the $\epsilon$*-floating body* of $K$, which we denote by $F_\epsilon$. This notion comes from the mental picture that when $K$ is a three dimensional convex body containing $\epsilon$ units of water, the floating body is the part that floats above water (see [6]). Finally, consider the floating body $F_\epsilon$ and a point $x \in F_\epsilon^c$. We say that $x$ *sees* $y$ if the chord $[x, y]$ does not intersect $F_\epsilon$. Set $S_{x,\epsilon}$ to be the set of those $y \in K$ seen by $x$. We then define

$$g(\epsilon) = \sup_{x \in F_\epsilon^c} \mathrm{Vol_d}(S_{x,\epsilon}).$$

In particular, we note that $S_{x,\epsilon}$ is the union of all $\epsilon$-boundary caps containing $x$.

Since $K$ is smooth, it is well known that $g(\epsilon) = \Theta(\mathrm{Vol_d}(\epsilon\text{-boundary cap}))$ (see [6]).

### 2.2 Asymptotic Notation

We shall always assume $n$ is sufficiently large, without comment. We use the notation $\Omega, O, \Theta$ etc. with respect to $n \to \infty$, unless otherwise indicated. All constants are assumed to depend on at most the dimension $d$, the body $K$, and $\rho$.

## 3 Variance

In this section, we provide a proof of Theorem 1.1. It follows an argument first used by Reitzner in [10], which has also been utilized by Bárány and Reitzner [4] to prove a lower bound of the variance in the case where the convex body is a polytope. Essentially, we condition on arrangements of our vertices where they can be perturbed in such a way that the resulting change in volume is independent for each vertex in question.

Choosing the vertices along the boundary according to a given distribution, as opposed to uniformly in the body, adds technical complication and requires greater use of the boundary structure. The key to the study is the boundary approximation mentioned both in this section and in Appendix 1.

## 3.1 Small Local Perturbations

We begin by establishing some notation. Define the standard paraboloid $E$ to be

$$E = \{z \in \mathbb{R}^d \mid z^d \geq (z^1)^2 + \cdots + (z^{d-1})^2\}.$$

Hence we have $2E = \{z \in \mathbb{R}^d \mid z^d \geq \frac{1}{2}((z^1)^2 + \cdots + (z^{d-1})^2)\}$ and observe that we have the inclusion

$$E \subset 2E.$$

We now choose a simplex $S$ in the cap $C(0,1)$ of $E$. Choose the base of the simplex to be a regular simplex with vertices in $\partial E \cap H(e_d, h_d)$ and the origin ($h_d$ to be determined later). We shall denote by $v_0, v_1, \ldots, v_d$ the vertices of this simplex, singling out $v_0$ to be the apex of $S$ (i.e. the origin). The important point here is that for sufficiently small $h_d$, the cone $\{\lambda x \in \mathbb{R}^d \mid \lambda \geq 0, x \in S\}$ contains $2E \cap H(e_d, 1)$. Indeed, as the radius of $E \cap H(e_d, h_d)$ is $\sqrt{h_d}$, the inradius of base of the simplex is $\sqrt{h_d/d^2}$, hence for $h_d < 1/2d^2$ our above inclusion holds.

Now, look at the orthogonal projection of the vertices of the simplex to the plane spanned by $\{e_1, \ldots, e_{d-1}\}$, which we think of as $\mathbb{R}^{d-1}$ and denote the relevant operator as

$$\mathrm{proj} : \mathbb{R}^d \to \mathbb{R}^{d-1}.$$

Around the origin we center a ball $B_0$ of radius $r$, and around each projected point (except the origin) we can center a ball in $\mathbb{R}^{d-1}$ of radius $r'$, both to be chosen later. We label these balls $B_1, \ldots, B_d$, where $B_i$ is the ball about $\mathrm{proj}(v_i)$. We can form the corresponding sets $B_i'$ to be the inverse image of these sets on $\partial E$ under the projection operator. In other words, if $b : \mathbb{R}^{d-1} \to \mathbb{R}$ is the quadratic form whose graph defines $E$, $\tilde{b} : \mathbb{R}^{d-1} \to \partial E$ the map induced by $b$, then

$$B_i' = \tilde{b}(B_i), \quad i = 0, \ldots, d.$$

We note that if we choose $r$ sufficiently small, then for any choice of random points $Y \in B_0'$ and $x_i \in B_i', i = 1, \ldots, d$ the cone on these points is close to the cone on the simplex in the sense that

$$\{\lambda x \mid x \in [Y, x_1, \ldots, x_d],\ \lambda \geq 0\} \supset 2E \cap H(e_d, 1).$$

We may also think of $Y$ being chosen randomly, according to the distribution induced from the $(d-1)$-dimensional Hausdorff measure on $E$, say. Then, passing to a smaller $r$ if necessary, we see that for any choice of $x_i \in B_i', i = 1, \ldots, d,$ we have

$$\mathrm{Var}_Y(\mathrm{Vol_d}([Y, x_1, \ldots, x_d])) \geq c_0 > 0.$$

All the above follows from continuity. We hope results of this type to be true for arbitrary caps of $\partial K$, and indeed our current construction will serve both model and computational tool for similar constructions on arbitrary caps.

We now consider the general paraboloid

$$Q = \left\{ z \in \mathbb{R}^d \,\middle|\, z^d \geq \frac{1}{2}(k_1(z^1)^2 + \cdots + k_{d-1}(z^{d-1})^2) \right\},$$

where here $k_i > 0$ for all $i$ and let the curvature be $\kappa = \prod k_i$. We now transform the cap $C(0,1)$ of $E$ to the cap $C(0,h)$ of $Q$ by the (unique) linear map $A$ which preserves the coordinate axes. Let $D_i$ be the image of $B_i$ under this affinity. We find that the volume of the $D_i$ scales to give

$$\mu(D_i) = c_1 h^{\frac{d-1}{2}}, \quad i = 1, \ldots, d, \tag{4}$$

where here $c_1$ is some positive constant only depending on the curvature $\kappa = \prod k_i$ and our choice of $r$ and $r'$.

Next, for each point $x \in \partial K$ we identify our general paraboloid $Q$ with the approximating paraboloid $Q_x$ of $K$ at $x$ (in particular, we identify $\mathbb{R}^{d-1}$ with the tangent hyperplane at $x$ and the origin with $x$). We thus write $D_i(x)$ to indicate the set $D_i, i = 1, \ldots, d$, corresponding to $Q_x$. Analogously to the construction of the $\{B_i'\}$ we can construct the $\{D_i'(x)\}$ as follows. Let $f^x : \mathbb{R}^{d-1} \to \mathbb{R}$ be the function whose graph locally defines $\partial K$ at $x$ (this exists for $h$ sufficiently small, see Lemma 6.1), $\tilde{f} : \mathbb{R}^{d-1} \to \partial K$ the induced function. Let

$$D_i'(x) = \tilde{f}(D_i(x)).$$

We note here that in general the sets $D_i'(x)$ are *not* the images of $B_i'$ under $A$ as $A(B_i')$ may not lie on the boundary $\partial K$ in general.

Because the curvature is bounded above and below by positive constants, as is $\rho$, we see that the volume of $D_i(x)$ is given by

$$c_3 h^{\frac{d-1}{2}} \leq \mu(D_i(x)) \leq c_4 h^{\frac{d-1}{2}}, \tag{5}$$

where $c_3, c_4$ are constants depending only on $K$.

We now wish to get bounds for $\mathrm{Var}_Y(\mathrm{Vol_d}([Y, x_1, \ldots, x_d]))$ where $x_i \in D_i'(x), i = 1, \ldots, d$, and we choose $Y$ randomly in $D_0'(x)$ according to the distribution on the boundary. To begin with, we'll need the following technical lemma.

**Lemma 3.1** *There exists a $r_0 > 0$ and $r_0'$ such that for all $r_0 > r > 0$ and $r_0' > r' > 0$ we have an $h_r > 0$ such that for any choice of $x_i \in D_i'(x), i = 1, \ldots, d$, and $h_r > h > 0$:*

$$c_5 h^{d+1} \leq \mathrm{Var}_Y([Y, x_1, \ldots, x_d]) \leq c_6 h^{d+1}, \tag{6}$$

*where $c_5, c_6$ are positive constants depending only on $K$ and $r$.*

The proof of this lemma is given in Appendix 1. Assuming this lemma is true, we proceed with our analysis as follows.

Fix some choice for $h_d < 1/2d^2$. Let $v_0, \dots, v_d$ denote the vertices of the simplex $S$. Then by continuity we know that there is some $\eta > 0$ such that choosing $x_i$ in $\eta$-balls $B(v_i, \eta)$ centered at the vertices preserves our desired inclusion, namely

$$\{\lambda x \mid x \in [x_0, x_1, \dots, x_d], \lambda \geq 0\} \supset 2E \cap H(e_d, 1). \tag{7}$$

We now desire to set $r' > 0$ such that $D'_i(x) \subset A(B(v_i, \eta))$ for all $x \in \partial K$. As a consequence, we will obtain the inclusion, for $x_i \in D'_i(x)$,

$$\{\lambda x \mid x \in [x_0, x_1, \dots, x_d], \lambda \geq 0\} \supset 2Q_x \cap H(u_x, h) \supset K \cap H(u_x, h).$$

Choose $\epsilon > 0$ such that

$$\begin{aligned} U_i = \{(x, y) \in \mathbb{R}^d \mid x \in B(\mathrm{proj} v_i, \eta/2) \subset \mathbb{R}^{d-1} \text{ and} \\ (1+\epsilon)^{-1} b_E(x) \leq y \leq (1+\epsilon) b_E(x)\} \subset B(v_i, \eta) \end{aligned} \tag{8}$$

for each $i$, where $b_E$ is the quadratic form defining our standard paraboloid $E$. Appealing to Lemma 6.1 we take $h$ sufficiently small such that for all $x \in \partial K$,

$$(1+\epsilon)^{-1} b_x(y) \leq f_x(y) \leq (1+\epsilon) b_x(y).$$

Choosing $r' < \eta/2$ forces the $B_i$ to be balls of radius $r'$ about $\mathrm{proj} v_i$, which by the above causes $D'_i(x) \subset A(U_i) \subset A(B(v_i, \eta))$.

With these choices for $r, r'$ and some constant $h_0 > 0$ to enforce the condition that $h$ is sufficiently small above, we now proceed to the body of our argument.

### 3.2 Proof of Lower Bound on Variance

Choose $n$ points $t_1, \dots, t_n$ randomly in $\partial K$ according to the probability induced by the distribution. Choose $n$ points $y_1, \dots, y_n \in \partial K$ and corresponding disjoint caps according to Lemma 6.6. In each cap $C(y_j, h_n)$ (of $K$) establish sets $\{D_i(y_j)\}$ and $\{D'_i(y_j)\}$ for $i = 0, \dots, d$ and $j = 1, \dots, n$ as in the above discussion.

We let $A_j$, $j = 1, \dots, n$ be the event that exactly one random point is contained in each of the $D_i(y_j)$, $i = 0, \dots, d$ and every other point is outside $C(y_j, h_n) \cap \partial K$. We calculate the probability as

$$\begin{aligned} P(A_j) &= n(n-1)\cdots(n-d)\mathbb{P}(t_i \in D'_i(y_j), i = 0, \dots, d) \\ &\quad \times \mathbb{P}(t_i \notin C(y_j, h_n) \cap \partial K, i \geq d+1) \\ &= n(n-1)\cdots(n-d) \prod_{i=0}^{d} \mu(D'_i(y_j)) \prod_{k=d+1}^{n} (1 - \mu(C(y_j, h_n) \cap \partial K)). \end{aligned}$$

We can give a lower bound for this quantity with (5) and Lemma 6.6 , and noting specifically that $h_n = \Theta(n^{-2/(d-1)})$:

$$\mathbb{P}(A_j) \geq c_7 n^{d+1} n^{-d-1} (1 - c_8 n^{-1})^{n-d-1} \geq c_9 > 0, \tag{9}$$

where $c_7, c_8, c_9$ are positive constants. In particular, denoting by $\mathbf{1}_A$ the indicator function of event $A$. We obtain that

$$\mathbb{E}\left(\sum_{j=1}^{n} \mathbf{1}_{A_j}\right) = \sum_{j=1}^{n} \mathbb{P}(A_j) \geq c_9 n. \tag{10}$$

Now we denote by $\mathcal{F}$ the position of all points of $\{t_1, \dots, t_n\}$ except those which are contained in $D_0'(y_j)$ with $\mathbf{1}_{A_j} = 1$. We then use the conditional variance formula to obtain a lower bound:

$$\operatorname{Var} Z = \mathbb{E} \operatorname{Var}(Z \mid \mathcal{F}) + \operatorname{Var} \mathbb{E}(Z \mid \mathcal{F}) \geq \mathbb{E} \operatorname{Var}(Z \mid \mathcal{F}).$$

Now we look at the case where $\mathbf{1}_{A_j}$ and $\mathbf{1}_{A_k}$ are both 1 for some $j, k \in \{1, \dots, n\}$. Assume without loss of generality that $t_j$ and $t_k$ are the points in $D_0'(y_j)$ and $D_0'(y_k)$, respectively. We note that by construction there can be no edge between $t_j$ and $t_k$, so the volume change affected by moving $t_j$ within $D_0'(y_j)$ is independent of the volume change of moving $t_k$ within $D_0'(y_k)$. This independence allows us to write the conditional variance as the sum

$$\operatorname{Var}(Z|\mathcal{F}) = \sum_{j=1}^{n} \operatorname{Var}_{t_j}(Z) \mathbf{1}_{A_j},$$

where here each variance is taken over $t_j \in D_0'(y_j)$. We now invoke Lemma 3.1, equation (10), and the bound $h_n \approx n^{-2/(d-1)}$ to compute

$$\begin{aligned}\mathbb{E} \operatorname{Var}(Z \mid \mathcal{F}) &= \mathbb{E}\left(\sum_{j=1}^{n} \operatorname{Var}_{t_j}(Z) \mathbf{1}_{A_j}\right) \geq c_5 h^{d+1} \mathbb{E}\left(\sum_{j=1}^{n} \mathbf{1}_{A_j}\right) \\ &\geq c_{10} (n^{-2/(d-1)})^{d+1} c_6 n = c_{11} n^{-(d+3)/(d-1)}.\end{aligned}$$

Thus, the above provides the promised lower bound on $\operatorname{Var} Z$.

## 4 Concentration

Our concentration result shows that $\operatorname{Vol_d}(K_n)$ is highly concentrated about its mean. Namely, we obtain a bound of the form

$$\mathbb{P}(|Z - \mathbb{E}Z| \geq \sqrt{\lambda \operatorname{Var} Z}) \leq c_1 \exp(-c_2 \lambda) \tag{11}$$

for positive constants $c_1, c_2$. Such an inequality indicates that $Z$ has an exponential tail, which proves sufficient to provide information about the higher moments of $Z$ and the rate of convergence of $Z$ to its mean.

### 4.1 Discrete Geometry

We now set up some basic geometry which will be the subject of our analysis. Let $L$ be a finite collection of points. For a point $x \in K$, define

$$\Delta_{x,L} = \mathrm{Vol_d}([L \cup x]) - \mathrm{Vol_d}([L]).$$

A key property is the following observation.

**Lemma 4.1** *Let $L$ be a set whose convex hull contains the floating body $F_\epsilon$. Then for any $x \in K$,*

$$\Delta_{x,L} \le g(\epsilon).$$

The major geometry result which allows for our analysis is the following lemma quantifying the fact that $K_n$ contains the floating body $F_\epsilon$ with high probability.

**Lemma 4.2** *There are positive constants $c$ and $c'$ such that the following holds for every sufficiently large $n$. For any $\epsilon \ge c' \ln n/n$, the probability that $K_n$ does not contain $F_\epsilon$ is at most* $\exp(-c\epsilon n)$.

The proof of this result can be done using the notion of VC-dimension, similar details of which can be found in [18].

### 4.2 A Slightly Weaker Result

The proof of Theorem 1.2 is rather technical. So we will first attempt a simpler one of a slightly weaker result, which represents one of the main methodology used in this paper.

Put $G_0 = 3g(\epsilon)$ and $V_0 = 36ng(\epsilon)^2$, where $g(\epsilon)$ is as defined in the previous subsection. We show:

**Theorem 4.3** *For a given $K \in \mathcal{K}_+^2$ there are positive constants $\alpha$, $c$, and $\epsilon_0$ such that the following holds: for any $\alpha \ln n/n < \epsilon \le \epsilon_0$ and $0 < \lambda \le V_0/4G_0^2$, we have*

$$\mathbb{P}(|Z - \mathbb{E}Z| \ge \sqrt{\lambda V_0}) \le 2\exp(-\lambda/4) + \exp(-c\epsilon n).$$

We note that the constants used in the definition of $G_0$ and $V_0$ are chosen for convenience and can be optimized, though we make no effort to do so.

To compare Theorem 4.3 with Theorem 1.2, we first compute $V_0$. $V_0 = 36ng(\epsilon)^2 = \Theta(\epsilon^{(d+1)/(d-1)})$, from definition of $g(\epsilon)$ and by Lemma 6.2. So, setting $\epsilon = \alpha \ln n/n$ for some positive constant $c$ satisfying Lemma 6.2 and greater than a given $\alpha$ gives

$$\begin{aligned} V_0 &= 36ng(\epsilon)^2 = 36n\Theta(\epsilon^{(d+1)/(d-1)})^2 = \Theta(nn^{-2(d+1)/(d-1)}(\ln n)^{2(d+1)/(d-1)}) \\ &= \Theta(n^{-(d+3)/(d-1)}(\ln n)^{2(d+1)/(d-1)}). \end{aligned} \tag{12}$$

So, up to a logarithmic factor $V_0$ is comparable to $\mathrm{Var}\, Z$.

To obtain Theorem 1.2 we utilize a martingale inequality (Lemma 4.4). This inequality, which is a generalization of an earlier result of Kim and Vu [9], appears to be a new and powerful tool in the study of random polytopes. It was first used by Vu in [18], and seems to provide a very general framework for the study of key functionals. The reader who is familiar with other martingale inequalities, most notably that of Azuma [2], will be familiar with the general technique (see also [1]).

Recall $K_n = [t_1, \dots, t_n]$, where $t_i, i = 1, \dots, n$, are independent random points in $\partial K$. Let the sample space be $\Omega = \{t \mid t = (t_1, \dots, t_n), t_i \in \partial K\}$ and let $Z = Z(t_1, \dots, t_n) = \mathrm{Vol}_{\mathrm{d}}(K_n)$ a function of these points, we may define the *(absolute) martingale difference sequence*

$$G_i(t) = |\mathbb{E}(Z \mid t_1, \dots, t_{i-1}, t_i) - \mathbb{E}(Z \mid t_1, \dots, t_{i-1})|.$$

Thus, $G_i(t)$ is a function of $t = (t_1, \dots, t_n)$ that only depends on the first $i$ points. We then set

$$V_i(t) = \int G_i^2(t)\partial t_i, \qquad V(t) = \sum_{i=1}^{n} V_i(t),$$

$$G_i'(t) = \sup_{t_i} G_i(t) \quad \text{and} \quad G(t) = \max_i G_i'(t).$$

Note also that $|Z - \mathbb{E}Z| \le \sum_i G_i$. The key to our proof is the following concentration lemma, which was derived using the so-called divide-and-conquer martingale technique (see [18]).

**Lemma 4.4** *For any positive $\lambda$, $G_0$ and $V_0$ satisfying $\lambda \le V_0/4G_0^2$, we have*

$$\mathbb{P}(|Z - \mathbb{E}Z| \ge \sqrt{\lambda V_0}) \le 2\exp(-\lambda/4) + \mathbb{P}(V(t) \ge V_0 \textit{ or } G(t) \ge G_0). \tag{13}$$

The proof of this lemma can be found in [18].

Comparing Lemma 4.4 to Theorem 4.3 we find that the technical difficulty comes in bounding the term $\mathbb{P}(V(t) \ge V_0$ or $G(t) \ge G_0)$, which corresponds to the error term $p_{NT}$.

Set $V' = n^{-1}V_0 = 36g(\epsilon)^2$. We find that we can replace $\exp(-c\epsilon n)$ with $n\exp(-c'\epsilon n)$ by adjusting the relevant constant $c'$ so that $n\exp(-c'\epsilon n) < \exp(-c\epsilon n)$. Thus, we're going to prove that

$$\mathbb{P}(G(t) \ge G_0 \text{ or } V(t) \ge V_0) \le n\exp(-c\epsilon n)$$

for some positive constant $c$.

To do this, we'll prove the following claim.

**Claim 4.5** *There is a positive constant $c$ such that for any $1 \le i \le n$,*

$$\mathbb{P}(G_i'(t) \ge G_0 \textit{ or } V_i(t) \ge V') \le \exp(-c\epsilon n).$$

From this claim the trivial union bound gives

$$\mathbb{P}(G(t) \ge G_0 \text{ or } V(t) \ge V_0) \le n\exp(-c\epsilon n),$$

hence quoting Lemma 4.4 finishes our proof of Theorem 4.3.

### 4.3 Proof of Claim 4.5

Recall that $Z = Z(t_1, \ldots, t_n) = \mathrm{Vol_d}(K_n)$ for points $t_i \in \partial K$.

The triangle inequality gives us

$$G_i(t) = |\mathbb{E}(Z \mid t_1, \ldots, t_{i-1}, t_i) - \mathbb{E}(Z \mid t_1, \ldots, t_{i-1})| \\ \leq \mathbb{E}_x \big|\mathbb{E}(Z \mid t_1, \ldots, t_{i-1}, t_i) - \mathbb{E}(Z \mid t_1, \ldots, t_{i-1}, x)\big|,$$

where $\mathbb{E}_x$ denotes the expectation over a random point $x$. The analysis for the two terms in the last inequality is similar, so we will estimate the first one. Let us fix (arbitrarily) $t_1, \ldots, t_{i-1}$. Let $L$ be the union of $\{t_1, \ldots, t_{i-1}\}$ and the random set of points $\{t_{i+1}, \ldots, t_n\}$. Since

$$\mathrm{Vol_d}([L \cup t_i]) = \mathrm{Vol_d}([L]) + \Delta_{t_i, L},$$

we have

$$\mathbb{E}(Z \mid t_1, \ldots, t_{i-1}, t_i) = \mathbb{E}(\mathrm{Vol_d}([L]) \mid t_1, \ldots, t_{i-1}) + \mathbb{E}(\Delta_{t_i, L} \mid t_1, \ldots, t_{i-1}).$$

The key inequality of the analysis is the following:

$$\mathbb{E}(\Delta_{t_i, L} \mid t_1, \ldots, t_{i-1}) \leq \mathbb{P}(F_\epsilon \not\subseteq [L] \mid t_1, \ldots, t_{i-1}) + g(\epsilon). \qquad (14)$$

The inequality (14) follows from two observations:

- If $F_\epsilon \not\subseteq [L]$, $\Delta_{t_i, L}$ is at most 1.
- If $[L]$ contains $F_\epsilon$, $\Delta_{t_i, L} \leq g(\epsilon)$ by Lemma 4.1.

We denote by $\Omega_{(j)}$ and $\Omega^{<j>}$ the spaces spanned by $\{t_1, \ldots, t_j\}$ and $\{t_j, \ldots, t_n\}$, respectively.

Set $\delta = n^{-4}$. We say that the set $\{t_1, \ldots, t_{i-1}\}$ is *typical* if

$$\mathbb{P}_{\Omega^{\langle i+1\rangle}}(F_\epsilon \subseteq [L] \mid t_1, \ldots, t_{i-1}) \geq 1 - \delta.$$

The rest of the proof has two steps. In the first step, we show that if $\{t_1, \ldots, t_{i-1}\}$ is typical then $G_i'(t) \leq G_0$ and $V_i(t) \leq V'$. In the second step, we bound the probability that $\{t_1, \ldots, t_{i-1}\}$ is not typical.

*First step.* Assume that $\{t_1, \ldots, t_{i-1}\}$ is typical, so $\mathbb{P}_{\Omega^{<i+1>}}(F_\epsilon \not\subseteq [L] \mid t_1, \ldots, t_{i-1}) \leq \delta = n^{-4}$. We first bound $G_i'(t)$. Observe that

$$\begin{aligned} G_i(t) &\leq \mathbb{E}_x|\mathbb{E}(Z \mid t_1, \ldots, t_{i-1}, t_i) - \mathbb{E}(Z \mid t_1, \ldots, t_{i-1}, x)| \\ &\leq \mathbb{E}_x|\mathbb{E}(\Delta_{t_i, L} \mid t_1, \ldots, t_{i-1}) - \mathbb{E}(\Delta_{x, L} \mid t_1, \ldots, t_{i-1})| \\ &\leq \mathbb{E}(\Delta_{t_i, L} \mid t_1, \ldots, t_{i-1}) + \mathbb{E}_x\mathbb{E}(\Delta_{x, L} \mid t_1, \ldots, t_{i-1}) \\ &\leq 2g(\epsilon) + 2n^{-4} \leq 3g(\epsilon) = G_0 \quad \text{(by (14))}. \end{aligned}$$

In the last inequality we use the fact that $\epsilon = \Omega(\ln n/n)$, $g(\epsilon) = \Omega(\epsilon^{(d+1)/(d-1)}) \gg n^{-4}$. Thus it follows that

$$G_i'(t) = \max_{t_i} G_i(t) \leq G_0.$$

Calculating $V_i(t)$ using the above bound on $G_i(t)$ it follows that

$$V_i(t) = \int G_i(t)^2 d\mu(t_i) \leq \int 9g(\epsilon)^2 d\mu(t_i) = 9g(\epsilon)^2 < V'.$$

*Second step.* In this step, we bound the probability that $\{t_1, \ldots, t_{i-1}\}$ is not typical. First of all, we will need a technical lemma as follows. Let $\Omega'$ and $\Omega''$ be probability spaces and set $\Omega'''$ to be their product. Let $A$ be an event in $\Omega'''$ which occurs with probability at least $1 - \delta'$, for some $0 < \delta' < 1$.

**Lemma 4.6** *For any* $1 > \delta > \delta'$

$$\mathbb{P}_{\Omega'}(\mathbb{P}_{\Omega''}(A \mid x) \leq 1 - \delta) \leq \delta'/\delta,$$

*where $x$ is a random point in $\Omega'$ and $\mathbb{P}_{\Omega'}$ and $\mathbb{P}_{\Omega''}$ are the probabilities over $\Omega'$ and $\Omega''$, respectively.*

*Proof* Recall that $\mathbb{P}_{\Omega'''}(A) \geq 1 - \delta'$. However,

$$\mathbb{P}_{\Omega'''}(A) = \int_{\Omega'} \mathbb{P}_{\Omega''}(A \mid x) \partial x \leq 1 - \delta \mathbb{P}_{\Omega'}(\mathbb{P}_{\Omega''}(A \mid x) \leq 1 - \delta).$$

The claim follows. □

Recall that $L = \{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}$. Lemma 4.2 yields

$$\mathbb{P}(F_\epsilon \nsubseteq [L]) \leq \exp(-c_0 \epsilon n),$$

for some positive constant $c_0$ depending only on $K$. Applying Lemma 4.6 with $\Omega' = \Omega_{(i-1)}$, $\Omega'' = \Omega^{\langle i+1 \rangle}$, $\delta' = \exp(-c\epsilon n)$ and $\delta = n^{-4}$, we have

$$\begin{aligned} &\mathbb{P}_{\Omega_{(i-1)}}(\{t_1, \ldots, t_{i-1}\} \text{ is not typical}) \\ &\quad = \mathbb{P}_{\Omega_{(i-1)}}(\mathbb{P}_{\Omega^{\langle i+1 \rangle}}(F_\epsilon \nsubseteq [L] \mid t_1, \ldots, t_{i-1}) \leq 1 - \delta) \\ &\quad \leq \delta'/\delta = n^4 \exp(-c_0 \epsilon n) \leq \exp(-c\epsilon n) \end{aligned}$$

for $c = c_0/2$, given $c_0 \epsilon n \geq 8 \ln n$. This final condition can be satisfied by setting the $\alpha$ involved in the lower bound of $\epsilon$ to be sufficiently large. Thus, our proof is complete.

### 4.4 A Better Bound on Deviation

By using more of the smooth boundary structure, we can obtain a better result. As we shall see at the end of the proof, this result implies Theorem 1.2.

**Theorem 4.7** *For any smooth convex body $K$ with distribution $\mu$ along the boundary, there are constants $c, c', \alpha, \epsilon_0$ such that the following holds. For any $V_0 \geq$*

$\alpha n^{-(d+3)/(d-1)}$, $\epsilon_0 \geq \epsilon > \alpha \ln n/n$, $G_0 \geq 3\epsilon^{(d+1)/(d-1)}$, *and* $0 < \lambda \leq V_0/4G_0^2$, *we have*

$$\mathbb{P}(|Z - \mathbb{E}Z| \geq \sqrt{\lambda V_0}) \leq 2\exp(-\lambda/4) + p_{NT},$$

*where*

$$p_{NT} = \exp(-c\epsilon n) + \exp(-c' n^{\frac{d-1}{3d+1} - \eta}),$$

*and* $\eta$ *is any small positive constant less than* $\frac{d-1}{3d+1}$.

The proof of Theorem 4.7 follows from more careful estimates concerning $\Delta_{x,L}$. An analogous result for random polytopes can be found in Sect. 2.5 of [18].

The key difference between this result and Theorem 4.3 is that here $V_0$ is independent of $\epsilon$, so we can set $V_0 = \alpha n^{-(d+3)/(d-1)}$ without affecting the tail estimate. If we also set $\epsilon = n^{-\frac{2d+2}{3d+1} - \eta}$, then the two error terms in $p_{NT}$ are the same (up to a constant factor). Since $G_0 = 3g(\epsilon) = 3\Theta(\epsilon^{(d+1)/(d-1)})$, we have $\lambda < V_0/4G_0^2 \leq c'' n^{\frac{d-1}{3d+1} + \frac{2(d+1)\eta}{d-1}}$ for some constant $c''$. Hence Theorem 1.2.

## 5 Central Limit Theorem

### 5.1 Poisson Central Limit Theorem

Before we prove the theorem, we should give a brief review of the Poisson point process. Let $K \in \mathcal{K}_+^2$, and let $\text{Pois}(n)$ be a Poisson point process with intensity $n$ concentrated on $K$. Then applying $\text{Pois}(n)$ on $K$ gives us random points $\{x_1, \dots, x_N\}$ where the number of points $N$ is Poisson distributed with intensity $n\mu(\partial K) = n$. We write $\Pi_n = [x_1, \dots, x_N]$. Conditioning on $N$, the points $x_1, \dots, x_N$ are independently uniformly distributed in $\partial K$. For two disjoint subsets $A$ and $B$ of $\partial K$, their intersections with $\text{Pois}(n)$, i.e. the point sets $A \cap \text{Pois}(n) = \{x_1, \dots, x_N\}$ and $B \cap \text{Pois}(n) = \{y_1, \dots, y_M\}$, are independent. This means $N$ and $M$ are independently Poisson distributed with intensity $n\mu(A)$ and $n\mu(B)$ respectively, and $x_i$ and $y_j$ are chosen independently.

The following standard estimates of the tail of Poisson distribution will be used repeatedly throughout this section. Let $X$ be a Poisson random variable with mean $\lambda$. Then

$$\begin{aligned}\mathbb{P}\left(X \leq \frac{\lambda}{2}\right) &= \sum_{k=0}^{\lambda/2} e^{-\lambda}\frac{\lambda^k}{k!} \leq e^{-\lambda} + \sum_{k=1}^{\lambda/2} e^{-\lambda}\left(\frac{e\lambda}{k}\right)^k \\ &\leq \frac{\lambda+1}{2}e^{-\lambda}(2e)^{\lambda/2} \leq \frac{\lambda+1}{2}\left(\frac{e}{2}\right)^{-\lambda/2} = \Theta\left(\left(\frac{e}{2}\right)^{-\lambda/2}\right), \end{aligned} \tag{15}$$

where the last equality holds when $\lambda$ is large. Similarly,

$$\mathbb{P}(X \geq 3\lambda) \leq \sum_{k=3\lambda}^{\infty} e^{-\lambda}\left(\frac{e\lambda}{k}\right)^k \leq \sum_{k=0}^{\infty} e^{-\lambda}\left(\frac{e}{3}\right)^k = ce^{-\lambda}, \tag{16}$$

where $c$ is a small constant.

The key ingredient of the proof is the following theorem:

**Theorem 5.1** (Baldi and Rinott [3]) *Let $G$ be the dependency graph of random variables $Y_i$'s, $i = 1, \dots, m$, and let $Y = \sum_i Y_i$. Suppose the maximal degree of $G$ is $D$ and $|Y_i| \le B$ a.s., then*

$$\left|\mathbb{P}\left(\frac{Y - \mathbb{E}Y}{\sqrt{\operatorname{Var} Y}} \le x\right) - \Phi(x)\right| = O(\sqrt{S}),$$

*where $\Phi(x)$ is the standard normal distribution and $S = \frac{mD^2B^3}{(\sqrt{\operatorname{Var} Y})^3}$.*

Here the dependency graph of random variables $Y_i$'s is a graph on $m$ vertices such that there is no edge between any two disjoint subsets, $A_1$ and $A_2$, of $\{Y_i\}_{i=1}^m$ if these two sets of random variables are independent.

Because we can divide the convex body $K$ into Voronoi cells according to the cap covering Lemma 6.6, we will study $\operatorname{Vol_d}(\Pi_n)$ as a sum of random variables which are volumes of the intersection of $\Pi_n$ with each of the Voronoi cell. And the theorem above allows us to prove central limit theorem for sums of random variables that may have small dependency on each other.

First we let

$$m = \left\lfloor \frac{n}{4d \ln n} \right\rfloor.$$

By Lemma 6.6, given $K \in \mathcal{K}_+^2$, we can choose $m$ points, namely $y_1, \dots, y_m$, on $\partial K$. And the Voronoi cells $\operatorname{Vor}(y_i)$ of these points dissect $K$ into $m$ parts. Let

$$Y_i = \operatorname{Vol_d}(\operatorname{Vor}(y_i) \cap K) - \operatorname{Vol_d}(\operatorname{Vor}(y_i) \cap \Pi_n),$$

$i = 1, \dots, m$. So

$$Y = \sum_i Y_i = \operatorname{Vol_d}(K) - \operatorname{Vol_d}(\Pi_n). \tag{17}$$

Moreover, these Voronoi cells also dissect the boundary of $K$ into $m$ parts, and each contains a cap $C_i$ with $d$-dimensional volume

$$\operatorname{Vol_d}(C_i) = \Theta(m^{-\frac{d+1}{d-1}}),$$

by Lemma 6.6. Now by Lemma 6.2 it is a boundary cap with $(d-1)$-dimensional volume

$$\mu(C_i \cap \partial K) = \Theta(m^{-1}) = \Theta\left(\frac{4d \ln n}{n}\right).$$

Denote by $A_i$ $(i = 1, \dots, m)$ the number of points generated by the Poisson point process of intensity $n$ contained in $C_i \cap \partial K$, hence $A_i$ is Poisson distributed with mean $\lambda = n\mu(C_i \cap \partial K) = \Theta(4d \ln n)$. Then

$$\mathbb{P}(A_i = 0) = e^{-\lambda} = O(n^{-4d}).$$

And by (15),

$$\mathbb{P}(A_i \geq 3\lambda) = \mathbb{P}(A_i \geq 12d \ln n) = O(n^{-4d}).$$

Now let $A^m$ be the event that there is at least one point and at most $12d \ln n$ points in every $A_i$ for $i = 1, \ldots, m$. Then

$$1 \geq \mathbb{P}(A^m) = \mathbb{P}(\cap_i \{1 \leq A_i \leq 12d \ln n\}) \geq 1 - \Omega(n^{-4d+1}). \tag{18}$$

The rest of the proof is organized as follows. We first prove the central limit theorem for $\mathrm{Vol_d}(\Pi_n)$ when we condition on $A^m$, then we show removing the condition doesn't affect the estimate much, as $A^m$ holds almost surely. Let $\widetilde{\mathbb{P}}$ denote the conditional probability measure induced by the Poisson point process $X(n)$ on $\partial K$ given $A^m$, i.e.

$$\widetilde{\mathbb{P}}(\mathrm{Vol_d}(\Pi_n) \leq x) = \mathbb{P}(\mathrm{Vol_d}(\Pi_n) \leq x | A^m).$$

Similarly, we define the corresponding conditional expectation and variance to be $\widetilde{\mathbb{E}}$ and $\widetilde{\mathrm{Var}}$, then

**Lemma 5.2**

$$\left| \widetilde{\mathbb{P}}\left( \frac{\mathrm{Vol_d}(\Pi_n) - \widetilde{\mathbb{E}}\mathrm{Vol_d}(\Pi_n)}{\sqrt{\widetilde{\mathrm{Var}}\mathrm{Vol_d}(\Pi_n)}} \leq x \right) - \Phi(x) \right| = O(n^{-\frac{d+1}{4(d-1)}} \ln^{\frac{d+2}{d-1}} n). \tag{19}$$

*Proof* Note that by (17), $\mathrm{Vol_d}(\Pi_n) - \widetilde{\mathbb{E}}\mathrm{Vol_d}(\Pi_n) = \widetilde{\mathbb{E}}Y - Y$, and $\widetilde{\mathrm{Var}}Y = \widetilde{\mathrm{Var}}\mathrm{Vol_d}(\Pi_n) = \Theta(n^{-\frac{d+3}{d-1}})$, by Theorem 5.5. Hence it suffices to show $Y$ satisfies the Central Limit Theorem under $\widetilde{\mathbb{P}}$.

Given $A^m$, we define the dependency graph on random variables $Y_i, i = 1, \ldots, m$ as follows: we connect $Y_i$ and $Y_j$ if $\mathrm{Vor}(y_i) \cap C(y_j, c, m^{-\frac{2}{d-1}}) \neq \emptyset$ for some constant $c$ which satisfies Lemma 6.8. To check dependency, we see that if $Y_i \nsim Y_j$, then $\mathrm{Vor}(y_i) \cap C(y_j, c, m^{-\frac{2}{d-1}}) = \emptyset$. Thus, for any point $P_1 \in \mathrm{Vor}(y_i) \cap \partial K$, $P_2 \in \mathrm{Vor}(y_j) \cap \partial K$, the line segment $[P_1, P_2]$ cannot be contained in the boundary of $\Pi_n$. Otherwise, it would be a contradiction to Lemma 6.8. Therefore, there is no edge of $\Pi_n$ between vertices in $\mathrm{Vor}(y_i)$ and $\mathrm{Vor}(y_j)$, hence $Y_i$ and $Y_j$ are independent given $A^m$.

To apply Theorem 5.1 to $Y$, we are left to estimate parameters $D$ and $B$.

By Lemma 6.7, $C(y_i, c, m^{-\frac{2}{d-1}})$ $(i = 1, \ldots, m)$ can intersect at most $O(1)$ many $\mathrm{Vor}(y_i)$'s. Hence $D = O(1)$.

By Lemma 6.8, for any point $x_i$ in $C_i$, $i = 1, \ldots, m$,

$$\delta^H(K, \Pi_n) \leq \delta^H(K, [x_1, \ldots, x_m]) = O(m^{-\frac{2}{d-1}}).$$

So

$$\mathrm{Vor}(y_i) \backslash \Pi_n \subseteq C(y_i, h'), \tag{20}$$

where $h' = O(m^{-\frac{2}{d-1}})$. By Lemma 6.5 and (20),

$$Y_i \leq \mathrm{Vol_d}(C(y_i, h')) = O(m^{-\frac{d+1}{d-1}}) = O\left(\left(\frac{4d\ln n}{n}\right)^{\frac{d+1}{d-1}}\right) := B.$$

Hence by the Baldi–Rinott Theorem, the rate of convergence in (19) is $n^{-\frac{d+1}{4(d-1)}} \times (\ln n)^{\frac{d+2}{d-1}}$, and we finish the proof. $\square$

Now, we will remove the condition $A^m$. First observe an easy fact.

**Proposition 5.3** *For any events $A$ and $B$,*

$$|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq \mathbb{P}(A^c).$$

Hence we can deduce:

**Lemma 5.4**

$$|\widetilde{\mathbb{P}}(\mathrm{Vol_d}(\Pi_n) \leq x) - \mathbb{P}(\mathrm{Vol_d}(\Pi_n) \leq x)| = O(n^{-4d+1}), \tag{21}$$

$$|\widetilde{\mathbb{E}}\mathrm{Vol}_\mathrm{d}^k(\Pi_n) - \mathbb{E}\mathrm{Vol}_\mathrm{d}^k(\Pi_n)| = O(n^{-4d+1}), \tag{22}$$

$$|\widetilde{\mathrm{Var}}\mathrm{Vol_d}(\Pi_n)) - \mathrm{Var}\,\mathrm{Vol_d}(\Pi_n)| = O(n^{-4d+1}). \tag{23}$$

The proofs of these three equations follow more or less from Proposition 5.3 with $\mathbb{P}((A^m)^c) = O(n^{-4d+1})$, and can be found in [10]. As a result of Lemma 5.4, we can remove the condition $A^m$ and obtain Theorem 1.5 as follows. For notational convenience, we denote $\mathrm{Vol_d}(\Pi_n)$ by $X$ temporarily. For each $x$, let $\tilde{x}$ be such that

$$\mathbb{E}X + x\sqrt{\mathrm{Var}\,X} = \widetilde{\mathbb{E}}X + \tilde{x}\sqrt{\widetilde{\mathrm{Var}}X},$$

then

$$|x - \tilde{x}| = O(n^{-4d+1+\frac{d+3}{2(d-1)}}) + |x|O(n^{-4d+1+\frac{d+3}{d-1}}), \tag{24}$$

by (21) and Lemma 5.2. We have

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leq \mathbb{E}X + x\sqrt{\mathrm{Var}\,X}) = \widetilde{\mathbb{P}}(X \leq \widetilde{\mathbb{E}}X + \tilde{x}\sqrt{\widetilde{\mathrm{Var}}}) + O(n^{-4d+1}) \\ &= \Phi(\tilde{x}) + O(n^{-\frac{d+1}{4(d-1)}} \ln^{\frac{d+2}{d-1}} n) + O(n^{-4d+1}). \end{aligned}$$

But $|\Phi(x) - \Phi(\tilde{x})| = O(n^{-1})$, since $|\Phi(x) - \Phi(\tilde{x})| \leq |x - \tilde{x}| \leq O(n^{-1})$ when $|x| \leq n$ and by (24) $|\tilde{x}| \geq cn$ when $|x| \geq n$ which implies $|\Phi(x) - \Phi(\tilde{x})| \leq \Phi(n) + \Phi(cn)$. So $|F_X(x) - \Phi(x)| = |\mathbb{P}(X \leq \mathbb{E}X + x\sqrt{\mathrm{Var}\,X}) - \Phi(x)| = O(n^{-\frac{d+1}{4(d-1)}} \ln^{\frac{d+2}{d-1}} n)$. Hence finishes the proof of Theorem 1.5.

### 5.2 Approximating $K_n$ by $\Pi_n$

As is pointed out in the introduction, $\Pi_n$ approximates $K_n$ quite well, as one might expect.

**Theorem 5.5** *Let $\Pi_n$ be the convex hull of points chosen on $\partial K$ according to the Poisson point process* Pois$(n)$. *Then,*

$$\mathbb{E}\mathrm{Vol_d}(\Pi_n) \approx \mathbb{E}\mathrm{Vol_d}(K_n) \approx 1 - c(K,d)n^{-\frac{2}{d-1}},$$

*as $n \to \infty$, and*

$$\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n) = \Theta(\mathrm{Var}\,\mathrm{Vol_d}(K_n)) = \Theta(n^{-\frac{d+3}{d-1}}).$$

*Proof* Due to the conditioning property of Poisson point process, we have

$$\mathbb{E}\mathrm{Vol_d}(\Pi_n) = \sum_{|k-n|\le n^{7/8}} \mathbb{E}\mathrm{Vol_d}(K_k)e^{-n}\frac{n^k}{k!} + \sum_{|k-n|\ge n^{7/8}} \mathbb{E}\mathrm{Vol_d}(K_k)e^{-n}\frac{n^k}{k!}.$$

For Poisson distribution, the Chebyschev's inequality gives $\mathbb{P}(|k-n| \ge n^{7/8}) \le n^{-3/4}$. Hence the second summand is bounded above by $n^{-3/4}$ since $\mathbb{E}\mathrm{Vol_d}(K_k)$ is at most 1. By 2, $\mathbb{E}\mathrm{Vol_d}(K_k) = 1 - k^{-\frac{2}{d-1}} = 1-(1+o(1))n^{-\frac{2}{d-1}}$, when $|k-n| \le n^{7/8}$.

For the variance, we can rewrite $\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n)$ as follows:

$$\mathrm{Var}\,\mathrm{Vol_d}(\Pi_n) = \mathbb{E}_N\,\mathrm{Var}(\mathrm{Vol_d}(\Pi_n) \mid N) + \mathrm{Var}_N\,\mathbb{E}(\mathrm{Vol_d}(\Pi_n) \mid N).$$

By (15), the second term in the above equation becomes:

$$\begin{aligned}
&\mathrm{Var}\,\mathbb{E}(\mathrm{Vol_d}(\Pi_n) \mid N)\\
&\quad = \mathbb{E}_N\mathbb{E}^2\mathrm{Vol_d}(K_N) - (\mathbb{E}_N\mathbb{E}\mathrm{Vol_d}(K_N))^2\\
&\quad = \sum_{j=\frac{n}{2}}^{\infty}\sum_{k=\frac{n}{2}}^{\infty}(\mathbb{E}^2\mathrm{Vol_d}(K_k) - \mathbb{E}\mathrm{Vol_d}(K_k)\mathbb{E}\mathrm{Vol_d}(K_j))e^{-2n}\frac{n^{k+j}}{k!j!} + O\left(\left(\frac{e}{2}\right)^{-n/2}\right)\\
&\quad = \sum_{j=\frac{n}{2}}^{\infty}\sum_{k=j}^{\infty}(\mathbb{E}\mathrm{Vol_d}(K_k) - \mathbb{E}\mathrm{Vol_d}(K_j))^2 e^{-2n}\frac{n^{k+j}}{k!j!} + O\left(\left(\frac{e}{2}\right)^{-n/2}\right),
\end{aligned}$$

where the third equality is due to (15). By Lemma 6.9, $\mathbb{E}\mathrm{Vol_d}(K_{j+1}) - \mathbb{E}\mathrm{Vol_d}(K_j) = c(K,d)j^{-\frac{d+1}{d-1}}$ when $j \to \infty$, hence

$$\mathbb{E}\mathrm{Vol_d}(K_k) - \mathbb{E}\mathrm{Vol_d}(K_j) = \sum_{i=j}^{k-1}\mathbb{E}\mathrm{Vol_d}(K_{i+1}) - \mathbb{E}\mathrm{Vol_d}(K_i) \le c(K,d)(k-j)j^{-\frac{d+1}{d-1}},$$

and

$$\operatorname{Var}\mathbb{E}(\mathrm{Vol_d}(\Pi_n)\mid N) \le c(K,d)\sum_{j=\frac{n}{2}}^{\infty}\sum_{k=j}^{\infty}(k-j)^2 j^{-\frac{2d+2}{d-1}} e^{-2n}\frac{n^{k+j}}{k!j!} + O\left(\left(\frac{e}{2}\right)^{-n/2}\right)$$

$$\le cn^{-\frac{2d+2}{d-1}}\operatorname{Var}N + O\left(\left(\frac{e}{2}\right)^{-n/2}\right) = O(n^{-\frac{d+3}{d-1}}).$$

Now, $\operatorname{Var}\mathrm{Vol_d}(K_n) = \Theta(n^{-\frac{d+3}{d-1}})$, so by (15) and (16), we have

$$\begin{aligned}\mathbb{E}\operatorname{Var}\mathrm{Vol_d}(\Pi_n|N) &= \mathbb{E}(\Theta(N^{-\frac{d+3}{d-1}}))\\ &= O\left(\mathbb{P}\left(N\le\frac{n}{2}\right)\right) + \mathbb{E}(N^{-\frac{d+3}{d-1}}\chi_{\{\frac{n}{2}<N\le 3n\}}) + O(\mathbb{P}(3n<N))\\ &= \Theta(n^{-\frac{d+3}{d-1}}).\end{aligned}$$

□

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## Appendix 1 Geometric Toolkit

### 6.1 Boundary Approximation

We begin with some basic notions and notation. For $K\in\mathcal{K}_+^2$,at each point $x\in\partial K$ there is a unique paraboloid $Q_x$, given by a quadratic form $b_x$, osculating $\partial K$ at $x$. We may describe $Q_x$ and $b_x$ by identifying the tangent hyperplane of $\partial K$ at $x$ with $\mathbb{R}^{d-1}$ and $x$ with the origin. This is a well known fact, see e.g. [10]. In a neighborhood of $x$, we can represent $\partial K$ as the graph of a $\mathcal{C}^2$, convex function $f:\mathbb{R}^{d-1}\to\mathbb{R}$, i.e. each point in $\partial K$ near $x$ can be written in the form $(y, f_x(y))$, where $y\in R^{d-1}$ the form $(y^1,\dots,y^{d-1})$. Thus, we may write

$$b_x(y) = \frac{1}{2}\sum_{1\le i,j\le d-1}\frac{\partial f_x}{\partial y^i\partial y^j}(0)y^i y^j \quad \text{and}$$

$$Q_x = \{(y,z)\mid z\ge b_x(y),\ \ y\in\mathbb{R}^{d-1}, z\in\mathbb{R}\},$$

here $\frac{\partial f_x}{\partial y^i\partial y^j}(0)$ denote the second partial derivative of $f_x$ at the origin with respect to $y^i$ and $y^j$. The main thrust of the above is that these paraboloids approximate the boundary structure. The formulation given here is due to Reitzner, who provides a proof in [12].

**Lemma 6.1** *Let $K\in\mathcal{K}_+^2$ and choose $\delta>0$ sufficiently small. Then there exists a $\lambda>0$, depending only on $\delta$ and $K$, such that for each point $x\in\partial K$ the following holds: If we identify the tangent hyperplane to $\partial K$ at $x$ with $\mathbb{R}^{d-1}$ and $x$ with the origin, then we may define the $\lambda$–neighborhood $U^\lambda$ of $x\in\partial K$ by* $\operatorname{proj}U^\lambda = B^{d-1}(0,\lambda)$.

*$U^\lambda$ can be represented by a convex function $f_x(y) \in \mathcal{C}^2$, for $y \in B^{d-1}(0,\lambda)$. Furthermore,*

$$(1+\delta)^{-1} b_x(y) \le f_x(y) \le (1+\delta) b_x(y) \quad \text{and} \tag{25}$$

$$\sqrt{1+|\nabla f_x(y)|^2} \le (1+\delta), \tag{26}$$

*for $y \in B^{d-1}(0,\lambda)$, where $b_x$ is defined as above and $\nabla f_x(y)$ stands for the gradient of $f_x(y)$.*

This lemma proves that at each point $x \in \partial K$, the deviation of the boundary of the approximating paraboloid $\partial Q_x$ from $\partial K$ is uniformly bounded in a small neighborhood of $x$.

We use this lemma to show how one can relate $\epsilon$-caps to $\epsilon$-boundary caps. This relationship is used repeatedly throughout the paper as it allows us to work with volumes of different dimensions.

**Lemma 6.2** *For a given $K \in \mathcal{K}^2_+$, there exists constants $\epsilon_0, c, c' > 0$ such that for all $0 < \epsilon < \epsilon_0$ we have that for any $\epsilon$-cap $C$ of $K$,*

$$c^{-1}\epsilon^{(d-1)/(d+1)} \le \mu(C \cap \partial K) \le c\epsilon^{(d-1)/(d+1)}$$

*and for any $\epsilon$-boundary cap $C'$ of $K$,*

$$c'^{-1}\epsilon^{(d+1)/(d-1)} \le \mathrm{Vol_d}(C') \le c'\epsilon^{(d+1)/(d-1)}.$$

*Proof* We shall prove the first statement. Fix some $\delta > 0$ for Lemma 6.1.

Consider in $\mathbb{R}^d$ the paraboloid given by the equation

$$z^d \ge (z^1)^2 + (z^2)^2 + \cdots + (z^{d-1})^2.$$

Intersecting this paraboloid with the halfspace defined by the equation $z^d \le 1$ gives an object which we shall call the standard cap, $E$. We form $(1+\delta)^{-1}E$ and $(1+\delta)E$ similarly by the equations $z^d \ge (1+\delta)^{-1}((z^1)^2 + (z^2)^2 + \cdots, (z^{d-1})^2)$ and $z^d \ge (1+\delta)((z^1)^2 + (z^2)^2 + \cdots, (z^{d-1})^2)$, using the same halfspace as before. We note the inclusions

$$(1+\delta)^{-1}E \supset E \supset (1+\delta)E.$$

Let $c_1 = \mathrm{Vol_d}((1+\delta)^{-1}E)$ and $c_2 = \mathrm{Vol_d}((1+\delta)E)$, and further set $c_3 = \mu(\mathrm{proj}((1+\delta)^{-1}E))$ and $c_4 = \mu(\mathrm{proj}((1+\delta)E))$ where here proj is the orthogonal projection onto the hyperplane spanned by the first $(d-1)$ coordinates.

Now, let $C$ be our $\epsilon$-cap. Let $x$ be the unique point in $\partial K$ whose tangent hyperplane is parallel to the hyperplane defining $C$. Assuming that Lemma 6.1 applies, we may equate the tangent hyperplane of $\partial K$ at $x$ with $\mathbb{R}^{d-1}$, and view $C \cap \partial K$ as being given by some convex function $f : \mathbb{R}^{d-1} \to \mathbb{R}$. Further, let $Q_x$ be the unique paraboloid osculating $\partial K$ at $x$. Let $A$ be a linear transform that takes $E$ to $Q_x$. We observe that $Q_x$ is the paraboloid defined by the set $z^d \ge b_x(z^1, \ldots, z^{d-1})$ intersected with the halfspace $z^d \le h$, for some $h > 0$. We can define $(1+\delta)^{-1}Q_x$ (resp.

$(1+\delta)Q_x$) to be the set defined by the intersection of this same half space and the points given by $z^d \geq (1+\delta)^{-1}b_x(z^1,\dots,z^{d-1})$ (resp. $z^d \geq (1+\delta)b_x(z^1,\dots,z^{d-1})$). Observe that $A((1+\delta)^{-1}E) = (1+\delta)^{-1}Q_x$ and $A((1+\delta)E) = (1+\delta)Q_x$.

Appealing to Lemma 6.1, we see that

$$(1+\delta)^{-1}Q_x \supset C \supset (1+\delta)Q_x.$$

This gives

$$c_1|\det A| \geq \epsilon \geq c_2|\det A|. \tag{27}$$

Let $\tilde{f}:\mathbb{R}^{d-1} \to \partial K$ be the function induced by $f$, i.e. $\tilde{f}(y) = (y, f_x(y))$. Using the inclusion

$$\tilde{f}(\mathrm{proj}((1+\delta)^{-1}Q_x)) \supset C \cap \partial K \supset \tilde{f}(\mathrm{proj}((1+\delta)Q_x))$$

and the bound

$$(1+\delta) \geq \sqrt{1+|\nabla f|^2} \geq 1$$

furnished by Lemma 6.1, if $A'$ represents the restriction of $A$ to the first $(d-1)$ coordinates, we obtain

$$c_3|\det A'|(1+\delta) \geq \mu(C \cap \partial K) \geq c_4|\det A'|. \tag{28}$$

A simple computation shows $|\det A| = 2^{(d-1)/2}\kappa^{-1/2}h^{(d+1)/2}$ and $|\det A'| = 2^{(d-1)/2}\kappa^{-1/2}h^{(d-1)/2}$, where $\kappa$ is the Gauß–Kronecker curvature of $\partial K$ at $x$. Using this and (27) gives upper and lower bounds on $h$, and this bound with (28) gives

$$c_5\epsilon^{(d-1)/(d+1)} \geq \mu(C \cap \partial K) \geq c_6\epsilon^{(d-1)/(d+1)},$$

where here $c_5, c_6$ are constants depending only on $\kappa$. As $K$ is compact and $\kappa$ is always positive we can assume we can change $c_5$ and $c_6$ to be independent of $\kappa$, and hence $x$.

Finally, we return to the issue of values of $\epsilon$ (hence $h$) for which Lemma 6.1 applies. We note that in general every quadratic form $b_x$ can be given by

$$b_x(y) = \frac{1}{2}\sum_i k_i(y^i)^2,$$

where $k_i$ are the principal curvatures. We observe that as the Gauß–Kronecker curvature is positive then there are positive constants $k'$ and $k''$ depending only on $K$ such that $0 < k' < k_i < k''$. This bounds the possible geometry of $Q_x$, and implies the existence of an $\epsilon_0$ such that for $0 < \epsilon < \epsilon_0$, such that $\mathrm{proj}((1+\delta)^{-1}Q_x) \subset B(0,\lambda)$ ($\lambda$ as given in Lemma 6.1), allowing us to apply Lemma 6.1. This completes the proof of the first statement. The second statement is similar. Relaxing constants allows the statement as given. □

*Remark 6.3* It is important to note that the above is *not* true for general convex bodies. In particular, any polytope $P$ provides an example of a convex body with caps $C$ such that the quantities $\mathrm{Vol_d}(C)$ and $\mu(C \cap \partial P)$ are unrelated.

### 6.2 Caps and Cap Covers

Lemma 6.4 through 6.8 and their proofs below can be found in [10].

**Lemma 6.4** *Given $K \in \mathcal{K}_+^2$, there exist constants $d_1, d_2$ such that for each cap $C(x,h)$ with $h \leq h_0$, we have*

$$\partial K \cap B(x, d_1 h^{\frac{1}{2}}) \subset C(x,h) \subset B(x, d_2 h^{\frac{1}{2}}).$$

**Lemma 6.5** *Given $K \in \mathcal{K}_+^2$, there exists a constant $d_3$ such that for each cap $C(x,h)$ with $h \leq h_0$, we have*

$$\mathrm{Vol_d}(C(x,h)) \leq d_3 h^{\frac{d+1}{2}}.$$

**Lemma 6.6** (Cap covering) *Given $m \geq m_0$ and $K \in \mathcal{K}_+^2$, there are points $y_1, \ldots, y_m \in \partial K$, and caps $C_i = C(y_i, h_m)$ and $\overline{C}_i = C(y_i, (2d_2/d_1)^2 h_m)$ with*

$$C_i \subset B(y_i, d_2 h_m^{1/2}) \subset \mathrm{Vor}(y_i),$$
$$\mathrm{Vor}(y_i) \cap \partial K \subset B(y_i, 2d_2 h_m^{1/2}) \cap \partial K \subset \overline{C}_i \quad \text{and}$$
$$h_m = \Theta(m^{-\frac{2}{d-1}}).$$

*Here* Vor$(y_i)$ *is the Voronoi cell of $y_i$ in $K$ defined by*:

$$\mathrm{Vor}(y_i) = \{x \in K : \| x - y_i \| \leq \| x - y_k \| \text{ for all } k \neq i\},$$

*and we have*

$$\mathrm{Vol_d}(C_i) = \Theta(m^{-\frac{d+1}{d-1}}),$$

*for all $i = 1, \ldots, m$.*

*Proof* The proof follows from the fact that given $m$, for a suitable $r_m$, we can find balls $B(y_i, r_m)$, $i = 1, \ldots, m$ such that they form a maximal packing of $\partial K$, hence $B(y_i, 2r_m)$ form a covering of $\partial K$. Use Lemma 6.4, one can convert between the height of cap $h_m$ and radius of the ball $r_m$. □

**Lemma 6.7** *Let $K, m$ be given, and $y_i, i = 1, \ldots, m$ be chosen as in Lemma 6.6. The number of Voronoi cells* Vor$(y_j)$ *intersecting a cap $C(y_i, h)$ is $O((h^{\frac{1}{2}} m^{\frac{1}{d-1}} + 1)^{d+1})$, $i = 1, \ldots, m$.*

**Lemma 6.8** *Let $m$, $K$ and $y_i, C_i, i = 1, \ldots, m$ be chosen as in Lemma 6.6. Choose on the boundary within each cap $C_i$ an arbitrary point $x_i$ (i.e. $x_i \in C_i \cap \partial K$), then*

$$\delta^H(K, [x_1, \ldots, x_m]) = O(m^{-\frac{2}{d-1}}),$$

*and there is a constant $c$ such that for any $y \in \partial K$ with $y \notin C(y_i, cm^{-\frac{2}{d-1}})$, the line segment $[y, x_i]$ intersects the interior of the convex hull $[x_1, \ldots, x_m]$.*

**Lemma 6.9** *For large n*,

$$\mathbb{E}\mathrm{Vol_d}(K_{n+1}) - \mathbb{E}\mathrm{Vol_d}(K_n) = O(n^{-(d+1)/(d-1)}).$$

This lemma can be proved using techniques from integral geometry similar to that found in [11]. Alternatively, one case use the notion of $\epsilon$-floating bodies to give an appropriate bound. We give a proof sketch below of a slightly weaker version below, and note that through techniques similar to that used to prove Theorem 1.2 and in [18], we can remove the logarithmic factor.

*Sketch of the Proof* Following the notation found in the concentration proof, let $\Omega' = \{t = (t_1, \ldots, t_n) \mid t_i \in \partial K\}$, and put $L = \{t_1, \ldots, t_n\}$.

Observe that we can write

$$\begin{aligned}
&\mathbb{E}\mathrm{Vol_d}(K_{n+1}) - \mathbb{E}\mathrm{Vol_d}(K_n)\\
&\quad = \int_{\Omega'}\int_{\partial K} \mathrm{Vol_d}([t_1, \ldots, t_n, t_{n+1}]) - \mathrm{Vol_d}([t_1, \ldots, t_n]) dt_{n+1} dt\\
&\quad = \int_{\Omega'}\int_{\partial K} \Delta_{t_{n+1},L} dt_{n+1} dt.
\end{aligned}$$

Let $A$ be the event that $F_\epsilon \subseteq [L]$. The integrand can be estimated by

$$\Delta_{t_{n+1},L} \le g(\epsilon)\chi_A + \chi_{\bar{A}}.$$

Here, we use $g(\epsilon)$ as an upperbound for $\Delta_{t_{n+1},L}$ when $F_\epsilon \subseteq [L]$ and 1 otherwise. This bound is independent of $t_{n+1}$, so our integral is upper bounded by

$$\int_{\Omega'} g(\epsilon)\chi_A + \chi_{\bar{A}} dt \le g(\epsilon) + P(F_\epsilon \not\subseteq [L]).$$

Setting $\epsilon = c \ln n/n$ so that it satisfies Lemma 4.2 we find that $g(\epsilon) = \Theta(\epsilon^{(d+1)/(d-1)}) = \Theta(n^{-(d+1)/(d-1)} poly(\ln n))$ and $P(F_\epsilon \not\subseteq [L]) = \exp(-c'\epsilon n) = n^{-cc'}$. Choosing $c$ to be sufficiently large we find that

$$\mathbb{E}\mathrm{Vol_d}(K_{n+1}) - \mathbb{E}\mathrm{Vol_d}(K_n) = O(n^{-(d+1)/(d-1)} poly(\ln n)). \qquad \square$$

## Appendix 2 Proof of Corollaries 1.3 and 1.4

*Proof of Corollary 1.3* Let $\lambda_0 = \frac{\alpha}{4} n^{\frac{d-1}{3d+1} + \frac{2(d+1)\eta}{d-1}}$ be the upper bound for $\lambda$ given in Theorem 1.2. So for $\lambda > \lambda_0$, by (1.2)

$$\begin{aligned}
\mathbb{P}(|Z - \mathbb{E}Z| \ge \sqrt{\lambda V_0}) &\le \mathbb{P}(|Z - \mathbb{E}Z| \ge \sqrt{\lambda_0 V_0})\\
&\le 2\exp(-\lambda_0/4) + \exp(-cn^{\frac{d-1}{3d+1} - \eta}).
\end{aligned}$$

Combining (1.2) and the above, we get for any $\lambda > 0$,

$$\mathbb{P}(|Z - \mathbb{E}Z| \ge \sqrt{\lambda V_0}) \le 2\exp(-\lambda/4) + 2\exp(-\lambda_0/4) + \exp(-cn^{\frac{d-1}{3d+1} - \eta}). \quad (29)$$

We then compute the $k$th moment $M_k$ of $Z$, beginning with the definition:

$$M_k = \int_0^\infty t^k d\mathbb{P}(|Z - \mathbb{E}Z| < t).$$

If we set $\gamma(t) = \mathbb{P}(|Z - \mathbb{E}Z| \geq t)$ then we can write

$$\begin{aligned} M_k &= \int_0^\infty t^k d\mathbb{P}(|Z - \mathbb{E}Z| < t) = -\int_0^\infty t^k d\gamma(t) \\ &= (-t^k\gamma(t))|_0^\infty + \int_0^\infty kt^{k-1}\gamma(t)dt = \int_0^1 kt^{k-1}\gamma(t)dt. \end{aligned}$$

Note that the limits of integration can be limited to [0, 1] because we've assumed the volume of $K$ is normalized to 1.

Setting $t = \sqrt{\lambda V_0}$ we get

$$\begin{aligned} &\int_0^1 kt^{k-1}\gamma(t)dt \\ &= \int_0^{1/V_0} k(\sqrt{\lambda V_0})^{k-1}\mathbb{P}(|Z - \mathbb{E}Z| \geq \sqrt{\lambda V_0})\frac{\sqrt{V_0}}{2\sqrt{\lambda}}d\lambda \\ &\leq \frac{k}{2}V_0^{k/2}\int_0^{1/V_0} \lambda^{\frac{k}{2}-1} 2\exp(-\lambda/4) + 2\exp(-\lambda_0/4) + \exp(-cn^{\frac{d-1}{3d+1}-\eta})d\lambda \\ &\hfill \text{by (29).} \end{aligned}$$

We may now evaluate each term separately.

For the first term we observe that

$$\int_0^{1/V_0} 2\lambda^{\frac{k}{2}-1}\exp(-\lambda/4)d\lambda \leq \int_0^\infty 2\lambda^{\frac{k}{2}-1}\exp(-\lambda/4)d\lambda = c_k,$$

where $c_k$ is a constant depending only on $k$.

Since

$$V_0 = \alpha n^{-\frac{d+3}{d-1}} \gg n^{-5},$$

we can compute the second term:

$$\begin{aligned} \int_0^{1/V_0} \lambda^{\frac{k}{2}-1} 2\exp(-\lambda_0/4)d\lambda &\leq \frac{2}{k}2\exp(-\lambda_0/4)V_0^{-\frac{k}{2}} \\ &\leq \frac{2}{k}2\exp\left(-\frac{\alpha}{16}n^{\frac{d-1}{3d+1}+\frac{2(d+1)\eta}{d-1}}\right)n^{\frac{5k}{2}} = o(1). \end{aligned}$$

The last term can be computed similarly and gives $o(1)$ again. Hence,

$$M_k \leq (c_k + o(1))kV_0^{k/2} = O(V_0^{k/2}). \qquad \square$$

*Proof of Corollary 1.4*

$$\mathbb{P}\left(\left|\frac{Z_n}{\mathbb{E}Z_n} - 1\right| f(n) \geq \delta(n)\right) \leq \mathbb{P}\left(|Z_n - \mathbb{E}Z_n| \geq \mathbb{E}Z_n\sqrt{32n^{-\frac{d+3}{d-1}}\ln n}\right)$$

$$\le \mathbb{P}(|Z_n - \mathbb{E}Z_n| \ge \sqrt{8 \ln n V_0})$$
$$\le 2\exp(-8\ln n/4) + \exp(-cn^{\frac{d-1}{3d+1}-\eta})$$
$$\le 3\exp(-2\ln n) \le 3n^{-2},$$

by Theorem 1.2. The second inequality above is due to the fact that $\mathbb{E}Z_n = 1 - c_K n^{-\frac{2}{d-1}} > 1/2$ when $n$ is large. Since $\sum n^{-2}$ is convergent, by Borel–Cantelli, $|\frac{Z_n}{\mathbb{E}Z_n} - 1| f(n)$ converges to 0 almost surely, hence the corollary. □

## Appendix 3 Proof of Lemma 3.1

We first prove the following claim. The notation follows that found in Section 3.1

**Claim 8.1** *Let $x \in \partial K$. There is some $h(K) > 0$ such that for $h(K) > h > 0$ there exists a constant $c(r) > 0$ depending only on $r$ and $K$ such that*

$$\frac{1}{2}|\det A|^2 c(r) \le \mathrm{Var}_Y(\mathrm{Vol_d}([Y, Av_1, \dots, Av_d])) \le 2|\det A|^2 c(r),$$

*and $Y$ is a random point chosen in $D_0'(x)$ according to the distribution on $\partial K$.*

*Proof* To prove this claim, we compute. Recall that $A$ is the linear map which takes $E$ to the paraboloid $Q_x$. We shall denote by $A'$ the map $A$ restricted to $\mathbb{R}^{d-1}$. We shall denote by $f : T_x(\partial K) \approx \mathbb{R}^{d-1} \to \mathbb{R}$ the function whose graph defines $\partial K$ locally, and $\tilde{f} : \mathbb{R}^{d-1} \to \partial K$ the function induced by $f$. Thus, we have:

$$\begin{aligned}
&\mathbb{E}_Y(\mathrm{Vol_d}([Y, Av_1, \dots, Av_d])) \\
&\quad = \frac{\int_{D_0} \mathrm{Vol_d}([\tilde{f}(Y), Av_1, \dots, Av_d])\rho(\tilde{f}(Y))\sqrt{1 + f_{Y^1}^2 + \dots + f_{Y^{d-1}}^2} dY}{\int_{A'(C_0)} \rho(f'(Y))\sqrt{1 + f_{Y^1}^2 + \dots + f_{Y^{d-1}}^2}\, dY} \\
&\quad = \Bigg(|\det A'| \int_{C_0} \mathrm{Vol_d}([\tilde{f}(AX), Av_1, \dots, Av_d])\rho(\tilde{f}(AX)) \\
&\qquad \times \sqrt{1 + f_{Y^1}^2 + \dots + f_{Y^{d-1}}^2}(AX) dX\Bigg) \Bigg/ \Bigg(|\det A'| \int_{C_0} \rho(f'(AX)) \\
&\qquad \times \sqrt{1 + f_{Y^1}^2 + \dots + f_{Y^{d-1}}^2}(AX) dY\Bigg).
\end{aligned} \tag{30}$$

Observe that if we set $A^{-1} \circ \tilde{f}(AX) = f^*$ to be the pullback of $\tilde{f}$ under $A$ then $\mathrm{Vol_d}([\tilde{f}(AX), Av_1, \dots, Av_d]) = |\det A| \cdot \mathrm{Vol_d}([f^*(X), v_1, \dots, v_d])$. Letting $b : \mathbb{R}^{d-1} \to \mathbb{R}$ denote the quadratic form defining $E$, $\tilde{b} : \mathbb{R}^{d-1} \to \partial E$ the induced function, we then use Lemma 6.1 to get the bound

$$2^{-1} b \le f \circ A' \le 2b, \tag{31}$$

when $h$ is sufficiently small. Thus, we get the bound

$$\begin{aligned}\mathrm{Vol_d}([2^{-1}b(X), v_1, \dots, v_d]) &\geq \mathrm{Vol_d}([f^*(X), v_1, \dots, v_d]) \\ &\geq \mathrm{Vol_d}([2b(X), v_1, \dots, v_d]),\end{aligned}$$

which follows from the geometry. Now, since $v_1, \dots, v_d$ form a $(d-1)$ simplex parallel to the plane $\mathbb{R}^{d-1}$ we can write $\mathrm{Vol_d}([b(X), v_1, \dots, v_d]) = c_d(1 - b(X))$, where $c_d$ is some positive constant depending only on dimension. We may write $b(X) = |X|^2$, and this allows us to see that

$$\begin{aligned}&\mathrm{Vol_d}([2^{-1}\tilde{b}(X), v_1, \dots, v_d]) \\ &\quad = \mathrm{Vol_d}([\tilde{b}(X), v_1, \dots, v_d])(1 - 2^{-1}|X|^2)/(1 - |X|^2) \\ &\quad = \mathrm{Vol_d}([\tilde{b}(X), v_1, \dots, v_d])(1 - 2^{-1}|X|^2)(1 + |X|^2 + |X|^4 + \cdots) \\ &\quad = \mathrm{Vol_d}([\tilde{b}(X), v_1, \dots, v_d])(1 + o_r(1)).\end{aligned}$$

Here, $o_r(1)$ indicates a function which goes to 0 as $r$ goes to 0. Similarly, we have

$$\mathrm{Vol_d}([2\tilde{b}(X), v_1, \dots, v_d]) = \mathrm{Vol_d}([\tilde{b}(X), v_1, \dots, v_d])(1 + o_r(1)).$$

Thus, we may write

$$\begin{aligned}&\frac{\int_{C_0} \mathrm{Vol_d}([f^*(X), v_1, \dots, v_d])\rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \cdots + f_{Y^{d-1}}^2}(AX)dX}{\int_{C_0} \rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \cdots + f_{Y^{d-1}}^2}(AX)dY} \\ &\quad \geq (1 + o_r(1)) \\ &\qquad \times \frac{\int_{C_0} \mathrm{Vol_d}([\tilde{b}(X), v_1, \dots, v_d])\rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \cdots + f_{Y^{d-1}}^2}(AX)dX}{\int_{C_0} \rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \cdots + f_{Y^{d-1}}^2}(AX)dY}.\end{aligned} \tag{32}$$

Setting $F(X) = \rho(\tilde{f}(AX))\sqrt{1 + f_{Y^1}^2 + \cdots + f_{Y^{d-1}}^2}(AX)$ the above is thus

$$\geq (1 + o_r(1)) \cdot \frac{\min_{C_0} F(X)}{\max_{C_0} F(X)} \cdot \frac{\int_{C_0} \mathrm{Vol_d}([b(X), v_1, \dots, v_d])dX}{\int_{C_0} dX}.$$

Now, if we can show that the term $\frac{\min_{C_0} F(X)}{\max_{C_0} F(X)} \geq (1 + o_{r,h}(1))$, only depending on $r$ and $h$, then from our earlier observation we can conclude that (30) is bounded below by

$$|\det A| \cdot (1 + o_{r,h}(1)) \cdot \frac{\int_{C_0} \mathrm{Vol_d}([b(X), v_1, \dots, v_d])dX}{\int_{C_0} dX}.$$

Note $o_{r,h}(1)$ denotes a function which goes to 0 as both $r$ and $h$ go to 0.

Invoking Lemma 6.1, we observe that we may make the term

$$\sqrt{1+f_{Y^1}^2+\cdots+f_{Y^{d-1}}^2}(AX)$$

sufficiently less than $(1+\delta)$, for any $\delta>0$, by choosing $r,h$ both sufficiently small (independent of $f$). Thus, we may write $\sqrt{1+f_{Y^1}^2+\cdots+f_{Y^{d-1}}^2}(AX)=(1+o_{r,h}(1))$.

Next, we note that $\rho$ is a uniformly continuous function on $K$. It is not too hard to see that the function $\min_{C_0}\rho(f'(AX))/\max_{C_0}\rho(f'(AX))=(1+o_{r,h}(1))$, where again the $o(1)$ function is independent of the basepoint. Using the fact that

$$\begin{aligned}&\min\rho(f'(AX))\sqrt{1+f_{Y^1}^2+\cdots+f_{Y^{d-1}}^2}(AX)\\&\quad\ge(\min\rho(f'(AX)))(\min\sqrt{1+f_{Y^1}^2+\cdots+f_{Y^{d-1}}^2}(AX))\end{aligned}$$

(similarly for max) we thus find that

$$(1+o_{r,h}(1))\ge\frac{\min_{C_0}F(X)}{\max_{C_0}F(X)}\ge(1+o_{r,h}(1)),$$

where the functions in question are independent of basepoint.

If we let

$$\phi_1(r)=\frac{\int_{C_0}\mathrm{Vol}_{\mathrm{d}}([\tilde{b}(X),v_1,\ldots,v_d])dX}{\int_{C_0}dX}$$

then we can summarize our findings as, independent of basepoint,

$$\lim_{h\to0}\frac{\mathbb{E}_Y(\mathrm{Vol}_{\mathrm{d}}([Y,Av_1,\ldots,Av_d]))}{|\det A|\phi_1(r)}=(1+o_r(1)).\tag{33}$$

By an identical argument, if we set $\phi_2(r)=\frac{\int_{C_0}\mathrm{Vol}_{\mathrm{d}}^2([\tilde{b}(X),v_1,\ldots,v_d])dX}{\int_{C_0}dX}$ then we have

$$\lim_{h\to0}\frac{\mathbb{E}_Y(\mathrm{Vol}_{\mathrm{d}}^2([Y,Av_1,\ldots,Av_d]))}{|\det A|^2\phi_2(r)}=(1+o_r(1)).\tag{34}$$

Using (33) and (34) we can compute:

$$\begin{aligned}&\lim_{h\to0}\mathrm{Var}_Y([Y,Av_1,\ldots,Av_d])/|\det A|^2\\&\quad=\lim_{h\to0}\mathbb{E}_Y(\mathrm{Vol}_{\mathrm{d}}^2([Y,Av_1,\ldots,Av_d]))/|\det A|^2\\&\qquad-\lim_{h\to0}\mathbb{E}_Y^2(\mathrm{Vol}_{\mathrm{d}}([Y,Av_1,\ldots,Av_d]))/|\det A|^2\\&\quad=\phi_2(r)(1+o_r(1))-\phi_1^2(r)(1+o_r(1))^2=(\phi_2(r)-\phi_1^2(r))(1+o_r(1)).\end{aligned}\tag{35}$$

Thus, by letting $r$ become sufficiently small so that the final $(1+o_r(1))>0$ we note that (35) is positive, since this quantity $\phi_2(r)-\phi_1^2(r)$ is just the variance of

$\mathrm{Vol_d}([b(X), v_1, \dots, v_d])$ where $X$ is taken over $C_0$, thus always positive. This proves there exists $c_1 > 0$ such that for $h$ sufficiently small,

$$\mathrm{Var}_Y([Y, Av_1, \dots, Av_d]) \geq c_1 |\det A|^2.$$

By the same arguments we also get

$$\mathrm{Var}_Y([Y, Av_1, \dots, Av_d]) \leq c_2 |\det A|^2.$$

So the claim is proved. □

With the preceding claim, we now prove Lemma 3.1. Instead of the convex hull of $[Y, Av_1, \dots, Av_d]$ we shall study the convex hull $[Y, x_1, \dots, x_d]$, where $x_i \in D_i'$, using the fact that the $x_i$ are close to the $Av_i$ when $h$ is small. To do this, we'll need a second claim.

**Claim 8.2** *There exists a* $\delta > 0$ *such that If for each* $i$, $x_i \in B(v_i, d)$, *then*

$$\mathrm{Vol_d}([2^{-1} b(X), x_1, \dots, x_d]) = \mathrm{Vol_d}([b(X), x_1, \dots, x_d])(1 + o_r(1))$$

*and*

$$\mathrm{Vol_d}([2b(X), x_1, \dots, x_d]) = \mathrm{Vol_d}([b(X), x_1, \dots, x_d])(1 + o_r(1)),$$

*where the hidden functions depend only on* $r$ (*i.e. they are not functions of the* $x_i$).

*Proof* We simply note that there exists a $\delta > 0$ such that for any fixed choice of $x_i$,

$$\frac{\mathrm{Vol_d}([2^{-1} b(X), x_1, \dots, x_d])}{\mathrm{Vol_d}([b(X), x_1, \dots, x_d])} \to 1 \quad \text{as } X \to 0.$$

We also note that $X, x_1, \dots, x_d$ lie in $C_0 \times B(v_1, \delta) \times \dots \times B(v_d, \delta)$, a compact set. These two conditions guarantee that the maximum of the ratio, taken over all $x_1, \dots, x_d$, converges to 1 as $X \to 0$. Thus, the ratio converges to 1 independently of the choice of $x_1, \dots, x_d$, and hence the claimed result.

The statement for $\mathrm{Vol_d}([2b(X), x_1, \dots, x_d])$ is analogous. □

With this claim, we can adapt Claim 8.1 to work for any $x_i \in B(v_i, \delta)$, by using the above claim in place of (32). With this we can show that for $h$ sufficiently small we can choose $r$ sufficiently small such that

$$\begin{aligned} &\frac{1}{2} |\det A|^2 \mathrm{Var}_X(\mathrm{Vol_d}([b(X), x_1, \dots, x_d])) \\ &\quad \leq \mathrm{Var}_Y(\mathrm{Vol_d}([Y, Ax_1, \dots, Ax_d])) \\ &\quad \leq 2 |\det A|^2 \mathrm{Var}_X(\mathrm{Vol_d}([b(X), x_1, \dots, x_d])), \end{aligned} \tag{36}$$

where here the quantity $\mathrm{Var}_X(\mathrm{Vol_d}([b(X), x_1, \dots, x_d]))$ is the variance taken over $C_0$. But as $\mathrm{Var}_X(\mathrm{Vol_d}([b(X), v_1, \dots, v_d]))$ is positive, continuity guarantees that

$$c' > \mathrm{Var}_X(\mathrm{Vol_d}([b(X), x_1, \dots, x_d])) > c > 0$$

if the $x_i$ are sufficiently close to the $v_i$, say $x_i \in B(v_i, \eta)$ for all $i$, for some $\eta > 0$. Then,

$$\frac{1}{2}|\det A|^2 c' \leq \mathrm{Var}_Y(\mathrm{Vol_d}([Y, Ax_1, \ldots, Ax_d])) \leq 2|\det A|^2 c, \tag{37}$$

if $x_i \in B(v_i, \eta)$ for all $i$.

Now, we need to verify that we can choose $C_i$ sufficiently small such that points in $D_i'$ always map into $B(v_i, \eta)$, which will complete the lemma. To do this, note that if we set $r' < \eta/2$, then we can choose $\epsilon > 0$ such that

$$\begin{aligned} U_i = \{(x, y) \in \mathbb{R}^d \mid x \in B(\mathrm{proj} v_i, \eta/2) \subset \mathbb{R}^{d-1} \text{ and } \\ (1+\epsilon)^{-1} b(x) \leq y \leq (1+\epsilon) b(x)\} \subset B(v_i, \eta) \end{aligned} \tag{38}$$

for each $i$. By Lemma 6.1 we can take $h$ to be sufficiently small such that for all $x \in \partial K$

$$(1+\epsilon)^{-1} b_x(y) \leq f_x(y) \leq (1+\epsilon) b_x(y)$$

in all caps of height $h$. So if we thus choose $C_i$ to be the $\eta/2$ ball about $\mathrm{proj} v_i$, then we note that $D_i' \subset A(U_i)$. Thus, any $y_i \in D_i'$ can be written as $Ax_i$ for some $x_i \in U_i \subset B(v_i, \eta)$, and thus (37) holds. Hence, the lemma.

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