Covering Spheres with Spheres

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Received: 1 June 2006 / Revised: 14 July 2007 / Published online: 2
September 2007
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Abstract Given a sphere of any radius r in an n -dimensional Euclidean space, we study the coverings of this sphere with solid spheres of radius one. Our goal is to design a covering of the lowest *covering density*, which defines the average number of solid spheres covering a point in a bigger sphere. For growing dimension n , we design a covering that gives the covering density of order $(n \ln n)/2$ for a sphere of any radius $r > 1$ and a complete Euclidean space. This new upper bound reduces two times the order $n \ln n$ established in the classic Rogers bound.

1 Introduction

Spherical coverings. Let $B_r^n(\mathbf{x})$ be a *ball* (solid sphere) of radius r centered at some point $\mathbf{x} = (x_1, \dots, x_n)$ of an n -dimensional Euclidean space $\mathbb{R}^n$:

$$B_r^n(\mathbf{x}) \stackrel{\text{def}}{=} \left\{ \mathbf{z} \in \mathbb{R}^n \mid \sum_{i=1}^n (z_i - x_i)^2 \leq r^2 \right\}.$$

We also use a simpler notation B_r^n if a ball is centered at the origin $\mathbf{x} = 0$. For any subset $A \subseteq \mathbb{R}^n$, we say that a subset $\text{Cov}(A, \varepsilon) \subseteq \mathbb{R}^n$ forms an ε -*covering* (an ε -*net*) of A if A is contained in the union of the balls of radius ε centered at points $\mathbf{x} \in \text{Cov}(A, \varepsilon)$. In this case, we use notation

$$\text{Cov}(A, \varepsilon) : A \subseteq \bigcup_{\mathbf{x} \in \text{Cov}(A, \varepsilon)} B_\varepsilon^n(\mathbf{x}).$$

By changing the scale in $\mathbb{R}^n$, we can always consider the rescaled set A/ε and the new covering $\text{Cov}(A/\varepsilon, 1)$ with unit balls $B_1^n(\mathbf{x})$. Without loss of generality, below

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we consider these (unit) coverings. One of the classical problems is to obtain tight bounds on the covering size $|\text{Cov}(B_r^n, 1)|$ for any ball B_r^n of radius r and dimension n .

Another related covering problem arises for a *sphere*

$$S_r^n \stackrel{\text{def}}{=} \left\{ \mathbf{z} \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = r^2 \right\}.$$

Then a unit ball $B_1^{n+1}(\mathbf{x})$ intersects this sphere with a spherical cap

$$C_r^n(\rho, \mathbf{y}) = S_r^n \cap B_1^{n+1}(\mathbf{x}),$$

which has some center $\mathbf{y} \in S_r^n$, half-chord $\rho \leq 1$, and the corresponding half-angle $\alpha = \arcsin \rho/r$. The biggest possible cap $C_r^n(1, \mathbf{y})$ is obtained if the center $\mathbf{x}$ of the corresponding ball $B_1^{n+1}(\mathbf{x})$ is centered at the distance

$$\|\mathbf{x}\| = \sqrt{r^2 - 1} \quad (1)$$

from the origin. To obtain a minimal covering, we shall consider the biggest caps $C_r^n(1, \mathbf{y})$ assuming that all the centers $\mathbf{x}$ satisfy (1).

Covering density. Given a set $A \subseteq \mathbb{R}^n$, let $|A|$ denote n -dimensional volume (Lebesgue measure) of A . We then consider any unit covering $\text{Cov}(A, 1)$ and define minimum covering density

$$\vartheta(A) = \min_{\text{Cov}(A, 1)} \sum_{\mathbf{x} \in \text{Cov}(A, 1)} \frac{|B_1^n(\mathbf{x}) \cap A|}{|A|}.$$

Minimal coverings have been long studied for the spheres S_r^n and the balls B_r^n . The celebrated Coxeter–Few–Rogers lower bound [1] shows that for a sufficiently large ball B_r^n ,

$$\vartheta(S_r^n) \geq c_0 n. \quad (2)$$

Here and below c_i denote some universal constants. A similar result also holds for any sphere S_r^n of radius $r \geq n$ (see Example 6.3 in [4]).

Various upper bounds on the minimum covering density are obtained for B_r^n and S_r^n by Rogers in the classic papers [2] and [3]. In particular, it follows from these papers that for a sufficiently large radius r , a ball B_r^n and a sphere S_r^n can be covered with density

$$\vartheta \leq \left(1 + \frac{\ln \ln n}{\ln n} + \frac{5}{\ln n} \right) n \ln n. \quad (3)$$

Despite recent improvements obtained in [4] and [5], respectively, for spheres S_r^n and balls B_r^n of a relatively small radius r , the Rogers bound (3) is still the best asymptotic bound known for sufficiently large spheres, balls, and complete spaces $\mathbb{R}^n$ of growing dimension n .

For a sphere S_r^n of any dimension $n \geq 3$ and an arbitrary radius $r > 1$, the best universal upper bound known to date is obtained in [4] (see Corollary 1.2 and Remark 5.1):

$$\vartheta(S_r^n) \leq \left(1 + \frac{2}{\ln n}\right) \left(1 + \frac{\ln \ln n}{\ln n} + \frac{\sqrt{e}}{n \ln n}\right) n \ln n. \quad (4)$$

Our main result is presented in Theorem 1, which reduces about two times the present upper bounds (3) and (4) for $n \rightarrow \infty$.

Theorem 1 *Unit balls can cover a sphere S_r^n of any radius $r > 1$ and any dimension $n \geq 3$ with density*

$$\vartheta(S_r^n) \leq \left(\frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{5}{\ln n}\right) n \ln n. \quad (5)$$

For $n \rightarrow \infty$, there exists $o(1) \rightarrow 0$ such that

$$\vartheta(S_r^n) \leq \left(\frac{1}{2} + \left(\frac{3}{2} + o(1)\right) \frac{\ln \ln n}{\ln n}\right) n \ln n. \quad (6)$$

The following corollary to Theorem 1 (see also [8]) shows that the Rogers bound can also be reduced about two times for the coverings of complete Euclidean spaces $\mathbb{R}^n$.

Corollary 2 *For $n \rightarrow \infty$, unit balls can cover the entire Euclidean space $\mathbb{R}^n$ with density*

$$\vartheta(\mathbb{R}^n) \leq \left(\frac{1}{2} + o(1)\right) n \ln n. \quad (7)$$

2 Preliminaries: Embedded Coverings

In this section, we obtain the estimates on $\vartheta(S_r^n)$ that are similar to (3) and (4). However, we will introduce here a slightly different technique of embedded coverings that will be substantially extended in Section 3 to improve the former bounds (3) and (4). We will also employ most of our calculations performed in this section.

Consider a sphere S_r^n of some dimension $n \geq 3$ and radius $r > 1$. We use notation $C(\rho, \mathbf{y})$ for a cap $C_r^n(\rho, \mathbf{y})$ whenever parameters n and r are fixed; we also use a shorter notation $C(\rho)$ when a specific center $\mathbf{y}$ is of no importance. In this case, $\text{Cov}(\rho)$ will denote any covering of S_r^n with spherical caps $C(\rho)$. By definition, a covering $\text{Cov}(\rho)$ has covering density

$$\vartheta_\rho = \Omega_\rho |\text{Cov}(\rho)|$$

where Ω_ρ is the fraction of the surface of the sphere S_r^n covered by a cap $C(\rho)$,

$$\Omega_\rho = \frac{|C(\rho)|}{|S_r^n|}.$$

For any $\tau < \rho \leq 1$, we extensively use inequality (see Corollary 3.2 (ii) in [4]):

$$\Omega_\tau \geq \Omega_\rho \left(\frac{\tau}{\rho} \right)^n \quad (8)$$

and its particular version $\Omega_\tau \geq \Omega_1 \tau^n$ obtained for $\rho = 1$. We begin with two preliminary lemmas, which will simplify our calculations. Let $f_1(x)$ and $f_2(x)$ be two positive differentiable functions. We say that $f_1(x)$ moderates $f_2(x)$ for $x \geq a$ if for all $x \geq a$,

$$\frac{f'_1(x)}{f_1(x)} \geq \frac{f'_2(x)}{f_2(x)}.$$

Lemma 3 Consider m functions $f_i(x)$ such that $f_1(x)$ moderates each function $f_i(x)$, $i \geq 2$, for $x \geq a$. Then inequality

$$f_1(x) \geq \sum_{i=2}^m f_i(x)$$

holds for any $x \geq a$ if it is valid for $x = a$.

Proof Note that $f_i(x) = f_i(a) \exp\{s_i(x)\}$, where

$$s_i(x) \stackrel{\text{def}}{=} \int_a^x \frac{f'_i(t)}{f_i(t)} dt.$$

Also, $s_i(x) \leq s_1(x)$ for all $i \geq 2$. Therefore,

$$f_1(x) \geq \sum_{i=2}^m f_i(a) \exp\{s_1(x)\} \geq \sum_{i=2}^m f_i(a) \exp\{s_i(x)\} = \sum_{i=2}^m f_i(x),$$

which completes the proof. $\square$

Lemma 4 For any $n \geq 4$,

$$\left(1 - \frac{1}{n \ln n} \right)^{-n} < 1 + 1/\ln n + 1/\ln^2 n. \quad (9)$$

Proof For $n = 4, \dots, 7$, the above inequality is verified numerically. Let $t = \frac{1}{\ln n}$. We take the logarithm of the left part of (9) and use power series. This gives the following inequalities

$$-n \ln \left(1 - \frac{t}{n} \right) = \sum_{i=1}^{\infty} \frac{t^i}{n^{i-1} i} < t + t^2 \sum_{i=1}^{\infty} \left(\frac{1}{2n} \right)^i \leq t + \frac{t^2}{15}. \quad (10)$$

Here we observe that $t < \frac{1}{2}$ for $n \geq 8$, and replace the left-hand side of the first inequality with geometric series in its right-hand side, which in turn is bounded by

taking $n = 8$. We also note that for any $n \geq 8$,

$$t + \frac{t^2}{15} < \frac{1}{2}. \quad (11)$$

For any $z < \frac{1}{2}$ we can also use inequality

$$e^z = \sum_{i=1}^{\infty} \frac{z^i}{i!} \leq 1 + z + \frac{2z^2}{3}. \quad (12)$$

By combining (10) and (12), we obtain the required estimate

$$\left(1 - \frac{1}{n \ln n}\right)^{-n} < 1 + \left(t + \frac{t^2}{15}\right) + \frac{2}{3} \left(t + \frac{t^2}{15}\right)^2 < 1 + t + t^2.$$

The latter inequality can be verified for any $t < 1/2$ directly, or by applying Lemma 3 for $t = \frac{1}{z}$ and $z \geq 2$. $\square$

An embedded algorithm. We employ the following parameters

$$\varepsilon = \frac{1}{n \ln n}, \quad \rho = 1 - \varepsilon, \quad (13)$$

$$\lambda = 1 + \frac{\ln \ln n}{\ln n} + \frac{2}{n}. \quad (14)$$

To design a covering $\text{Cov}(1)$, we also use another covering

$$\text{Cov}(\varepsilon) : S_r^n \subseteq \bigcup_{\mathbf{u} \in \text{Cov}(\varepsilon)} C(\varepsilon, \mathbf{u})$$

with smaller caps $C(\varepsilon, \mathbf{u})$. In our first step, we randomly choose N points $\mathbf{y} \in S_r^n$, where

$$\frac{\lambda n \ln n}{\Omega_\rho} - 1 < N \leq \frac{\lambda n \ln n}{\Omega_\rho}. \quad (15)$$

Consider the set $\{C(\rho, \mathbf{y})\}$ of N caps. In the second step, we take all centers $\mathbf{u}' \in \text{Cov}(\varepsilon)$ that are left uncovered by the set $\{C(\rho, \mathbf{y})\}$ and form the extended set

$$\{\mathbf{x}\} = \{\mathbf{y}\} \cup \{\mathbf{u}'\}.$$

This set $\{\mathbf{x}\}$ covers the entire set $\text{Cov}(\varepsilon)$ and therefore forms a unit covering, if the caps $C(\rho, \mathbf{x})$ are expanded to the caps $C(1, \mathbf{x})$,

$$\text{Cov}(1) : S_r^n \subseteq \bigcup_{\mathbf{x} \in \{\mathbf{x}\}} C(1, \mathbf{x}).$$

Lemma 5 *For any $n \geq 8$, covering $\{\mathbf{x}\}$ has density*

$$\vartheta_* \leq \left(1 + \frac{\ln \ln n}{\ln n} + \frac{3}{\ln n}\right) n \ln n. \quad (16)$$

Proof Any point $\mathbf{u}$ is covered by some cap $C(\rho, \mathbf{y})$ with probability Ω_ρ . Our goal is to estimate the expected number N' of centers $\mathbf{u}'$ left uncovered after N trials. Here

$$N' = (1 - \Omega_\rho)^N \cdot |\text{Cov}(\varepsilon)|.$$

Note that $1 - \Omega_\rho \leq e^{-\Omega_\rho(1+\Omega_\rho/2)}$. Also, according to estimate (15), $N\Omega_\rho > \lambda n \ln n - \Omega_\rho$. Thus,

$$(1 - \Omega_\rho)^N \leq e^{-(\lambda n \ln n - \Omega_\rho)(1+\Omega_\rho/2)} \leq e^{-\lambda n \ln n}. \quad (17)$$

Here the second inequality holds since $\Omega_\rho < \Omega_1 < 1/2$, whereas $\lambda n \ln n > 16$.

Next, we estimate the covering size $|\text{Cov}(\varepsilon)|$. Let this covering have some density ϑ_ε . Then inequality (8) gives the estimates

$$|\text{Cov}(\varepsilon)| = \vartheta_\varepsilon / \Omega_\varepsilon \leq (n \ln n)^n \vartheta_\varepsilon / \Omega_1,$$

$$N' \leq e^{-\lambda n \ln n} (n \ln n)^n \vartheta_\varepsilon / \Omega_1 \leq \vartheta_\varepsilon / (n^2 \Omega_1).$$

According to our design, there exists a covering $\{\mathbf{x}\}$ with caps $C(1, \mathbf{x})$ that has size at most $N + N'$. Next, we combine last inequality with (15), and estimate the covering density of $\{\mathbf{x}\}$:

$$\vartheta_1 = \Omega_1(N + N') \leq (\lambda n \ln n) \Omega_1 / \Omega_\rho + \vartheta_\varepsilon / n^2. \quad (18)$$

Finally, we estimate Ω_1 / Ω_ρ using inequalities (8) and (9):

$$\Omega_1 / \Omega_\rho \leq (1 - \varepsilon)^{-n} \leq 1 + 1/\ln n + 1/\ln^2 n. \quad (19)$$

Thus, we can rewrite our estimate ϑ_1 as

$$\vartheta_1 \leq \vartheta_*(1 - 1/n^2) + \vartheta_\varepsilon / n^2 \quad (20)$$

using notation

$$\vartheta_* = \lambda n \ln n \frac{1 + 1/\ln n + 1/\ln^2 n}{1 - 1/n^2}. \quad (21)$$

For any given n , bound (20) only depends on the density ϑ_ε . Now let us assume that ϑ_1 meets some known upper bound $\tilde{\vartheta}_1$ valid for all (or sufficiently large) radii r . For example, we can use (4) or any weaker estimate. Next, we can change the scale in $\mathbb{R}^{n+1}$ and replace a covering $\text{Cov}(1)$ of a sphere $S_{r/\varepsilon}^n$ with the covering $\text{Cov}(\varepsilon)$ of the sphere S_r^n . This rescaling shows that we can replace ϑ_ε in (20) with any known upper bound $\tilde{\vartheta}_1$. In turn, inequality (20) shows that the bound $\tilde{\vartheta}_1$ will be further reduced by this replacement as long as $\tilde{\vartheta}_1 > \vartheta_*$. Thus, this iteration process yields the upper bound $\vartheta_1 \leq \vartheta_*$.

Our final step is to verify that the upper bound ϑ_* of (21) also satisfies inequality (16). Here we use Lemma 3. Then straightforward calculations show that the right-hand side of (16) exceeds the right-hand side of (21) for $n = 8$ and moderates it for $n \geq 8$, due to the bigger remaining term $3/\ln n$ in (16). $\square$

Remark More detailed arguments show that (16) holds for $n \geq 3$, whereas for $n \geq 8$,

$$\vartheta_* \leq \left(1 + \frac{\ln \ln n}{\ln n} + \frac{2}{\ln n}\right) n \ln n.$$

3 New Covering Algorithm for a Sphere S_r^n

Covering design. In this section, we obtain a covering of the sphere S_r^n with asymptotic density $(n \ln n)/2$. The new design will use both the former covering $\text{Cov}(\varepsilon)$ (with slightly different parameters) and another covering $\text{Cov}(\mu)$ with a larger radius μ that has asymptotic order of $n^{-1/2}$. Namely, we use parameters

$$\begin{aligned}\varepsilon &= \frac{1}{2n \ln n}, \\ \rho &= 1 - \varepsilon, \\ \beta &= \frac{1}{2} + 2 \frac{\ln \ln n}{\ln n}, \\ \lambda &= \beta + \frac{5}{2} \frac{1}{\ln n}, \\ \mu &= \frac{n^{-\beta}}{2\sqrt{3}}, \\ d &= 1 - 2\varepsilon - \mu^2\end{aligned}\tag{22}$$

and proceed as follows.

A. Let a sphere S_r^n be covered with two different coverings $\text{Cov}(\mu)$ and $\text{Cov}(\varepsilon)$:

$$\begin{aligned}\text{Cov}(\mu) : S_r^n &\subseteq \bigcup_{\mathbf{z} \in \text{Cov}(\mu)} C(\mu, \mathbf{z}), \\ \text{Cov}(\varepsilon) : S_r^n &\subseteq \bigcup_{\mathbf{u} \in \text{Cov}(\varepsilon)} C(\varepsilon, \mathbf{u}).\end{aligned}$$

We assume that both coverings have the former density ϑ_* of (16) or less.

B. Randomly choose N points $\mathbf{y} \in S_r^n$ and consider the corresponding spherical caps $C(\rho, \mathbf{y})$, where

$$\frac{\lambda n \ln n}{\Omega_d} - 1 < N \leq \frac{\lambda n \ln n}{\Omega_d}.\tag{23}$$

C. Let $C(\mu, \bar{\mathbf{z}})$ be any cap in $\text{Cov}(\mu)$ that contains at least one center $\mathbf{u} \in \text{Cov}(\varepsilon)$ not covered by the ρ -caps. We consider all such centers $\bar{\mathbf{z}}$ and form the joint set $\{\mathbf{x}\} = \{\mathbf{y}\} \cup \{\bar{\mathbf{z}}\}$. This set covers $\text{Cov}(\varepsilon)$ with ρ -caps and therefore forms the required covering

$$\text{Cov}(1) : S_r^n \subseteq \bigcup_{\mathbf{x} \in \{\mathbf{x}\}} C(1, \mathbf{x}).$$

We now proceed with preliminary discussion, which outlines the main steps of the proof.

Outline of the proof. Let us first assume that we keep the design of Section 2 but apply it to the new covering $\text{Cov}(\mu)$ instead of $\text{Cov}(\varepsilon)$. This will require taking $\rho = 1 - \mu$ to cover the centers of the caps $C(\mu, \mathbf{z})$ and then expanding ρ to 1 to cover the whole μ -caps. Contrary to our former choice of $\rho = 1 - \varepsilon$ in (19), this expansion will exponentially increase the covering density. Namely, straightforward calculations show that

$$\Omega_1/\Omega_{1-\varepsilon} \rightarrow 1, \quad \Omega_1/\Omega_{1-\mu} = \exp\{n^{1/2}\}, \quad n \rightarrow \infty.$$

To circumvent this problem, we keep $\rho = 1 - \varepsilon$ in (22) but change our design as follows.

1. Given any cap $C(\mu, \mathbf{z})$, we say that a cap $C(\rho, \mathbf{y})$ is d -close if $\mathbf{y}$ falls within distance $d < \rho$ to $\mathbf{z}$. In our proof, we refine the selection of the caps $C(\rho, \mathbf{y})$ and count only d -close caps, instead of the ρ -close caps considered in Section 2. It is easy to verify that distance d of (22) is so close to ρ that

$$\Omega_\rho/\Omega_d \rightarrow 1, \quad n \rightarrow \infty. \quad (24)$$

For this reason, counting only d -close caps instead of the former ρ -close caps will carry no overhead to the covering size (23).

2. On the other hand, we will show in Lemma 6 that the μ -cap becomes almost completely covered by a cap $C(\rho, \mathbf{y})$ when the latter becomes d -close instead of ρ -close. Namely, only a vanishing fraction $\omega \approx \exp(-\frac{3}{2} \ln^2 n)$ of a μ -cap is left uncovered in this case.
3. We shall also use the fact that a typical μ -cap is covered by multiple d -close caps. According to (23), the average number $\Omega_d N$ of these caps has the exact order of $\lambda n \ln n$:

$$\lambda n \ln n - \Omega_d < \Omega_d N \leq \lambda n \ln n. \quad (25)$$

In our proof, we first define insufficiently covered μ -caps. Namely, we call a cap $C(\mu, \mathbf{z}')$ non-saturated if it has only s or fewer d -close caps, where

$$s = \lfloor n/q \rfloor, \quad q = 3 \ln \ln n. \quad (26)$$

This choice of s will achieve two goals.

4. We prove in Lemma 7 that non-saturated caps typically form a very small fraction of order $\exp[-\lambda n \ln n]$ among all μ -caps. On the other hand, it is easy to see that the quantity

$$|\text{Cov}(\mu)| \leq \vartheta_*/\Omega_\mu$$

exceeds N by the exponential factor $\Omega_d/\Omega_\mu \sim \exp[\beta n \ln n]$ or less. Then our choice of λ and β in (22) gives the expected number $N' = o(N)$ of non-saturated caps. Thus, non-saturated caps typically form a vanishing fraction of not only μ -caps but also ρ -caps.

5. Next, we proceed with saturated μ -caps and count all those centers $\mathbf{u}'' \in \text{Cov}(\varepsilon)$ that are left uncovered by random ρ -caps. All caps $C(\mu, \mathbf{z}'')$ that contain uncovered centers $\mathbf{u}''$ are called porous. For a given s , we show in Lemma 8 that

Fig. 1 Two intersecting caps $C(\mu, \mathbf{Z})$ and $C(\rho, \mathbf{Y})$ with bases **PQRSA** and **PMRTB**

the set $\{\mathbf{u}''\}$ forms a very small portion of $\text{Cov}(\varepsilon)$, with expected fraction of $\omega^{s+1} \sim \exp[-\frac{3}{2}n \ln n]$. Note that the quantity

$$|\text{Cov}(\varepsilon)| \leq \vartheta_*/\Omega_\varepsilon$$

exceeds N by the exponential factor $\Omega_d/\Omega_\varepsilon \sim \exp[n \ln n]$ or less. Therefore, the expected size of $\{\mathbf{u}''\}$ is $N'' = o(N)$.

- Finally, the centers of all non-saturated and porous caps are combined into the set $\bar{\mathbf{z}} = \{\mathbf{z}', \mathbf{z}''\}$. Then the set $\{\mathbf{x}\} = \{\mathbf{y}, \bar{\mathbf{z}}\}$ completely covers the set $\text{Cov}(\varepsilon)$ with the caps $C(\rho, \mathbf{x})$. Therefore, $\{\mathbf{x}\}$ also covers S_r^n with unit caps.

Main proofs. To prove Theorem 1, we first observe (by numerical comparison) that the existing bound (16) is tighter for $n \leq 100$ than bound (5) of Theorem 1 or even its refined version (39) considered below. Thus, Theorem 1 holds for $n \leq 100$. For this reason, we shall only consider dimensions $n \geq 100$. In the end of the proof, we also address the asymptotic case $n \rightarrow \infty$, wherein the calculations are more straightforward. The proof is based on three lemmas.

Consider two caps $C(\mu, \mathbf{Z})$ and $C(\rho, \mathbf{Y})$ with centers $\mathbf{Y}$ and $\mathbf{Z}$, which are d -close. These caps are represented in Fig. 1. Here the origin $\mathbf{O}$ is the center of S_r^n .

Lemma 6 For any cap $C(\mu, \mathbf{Z})$, a randomly chosen d -close cap $C(\rho, \mathbf{Y})$ fails to cover any given point $\mathbf{x}$ of $C(\mu, \mathbf{Z})$ with probability $p(\mathbf{x}) \leq \omega$, where

$$\omega = \frac{1}{4 \ln n} \left(1 - \frac{3}{n} \ln^2 n \right)^{\frac{n-1}{2}} \leq \frac{1}{4 \ln n} \exp \left(-\frac{3}{2} \ln^2 n \right). \quad (27)$$

Proof The caps $C(\mu, \mathbf{Z})$ and $C(\rho, \mathbf{Y})$ have bases **PQRSA** and **PMRTB**, which form the balls $B_\mu^n(\mathbf{A})$ and $B_\rho^n(\mathbf{B})$. Below we consider the boundary **PQRS** of $C(\mu, \mathbf{Z})$, which forms the sphere $S_\mu^{n-1}(\mathbf{A})$. The bigger cap $C(\rho, \mathbf{Y})$ covers this boundary, with the exception of the cap **PQR** centered at $\mathbf{Q}$. We first consider the case, when $\mathbf{x}$ is a boundary point and belongs to **PQRS**. Then the probability $p(\mathbf{x})$ is the fraction Ω of the entire boundary contained in uncovered cap **PQR**. We first estimate the half-angle $\alpha = \angle \mathbf{PAQ}$ formed by the cap **PQR**.

Let $d(\mathbf{H}, \mathbf{G})$ denote the distance between any two points $\mathbf{H}$ and $\mathbf{G}$. Also, let $\sigma(\mathbf{H})$ be the distance from a point $\mathbf{H}$ to the line $\mathbf{OBY}$ that connects the origin $\mathbf{O}$ with the center $\mathbf{B}$ of the bigger base $B_\rho^n(\mathbf{B})$ and with the center $\mathbf{Y}$ of the cap $C(\rho, \mathbf{Y})$. We use inequalities

$$\sigma(\mathbf{A}) \leq \sigma(\mathbf{Z}) \leq d(\mathbf{Z}, \mathbf{Y}) \leq d.$$

On the other hand, consider the base $\mathbf{PNR}$ of the uncovered cap $\mathbf{PQR}$. Here $\mathbf{N}$ denotes the center of this base. Then both lines $\mathbf{AN}$ and $\mathbf{BN}$ are orthogonal to this base. Also, $d(\mathbf{B}, \mathbf{P}) = \rho$, and $d(\mathbf{N}, \mathbf{P}) \leq d(\mathbf{A}, \mathbf{P}) = \mu$. Thus,

$$d(\mathbf{B}, \mathbf{N}) = \sqrt{d^2(\mathbf{B}, \mathbf{P}) - d^2(\mathbf{N}, \mathbf{P})} \geq \sqrt{\rho^2 - \mu^2} \geq \rho - \mu^2. \quad (28)$$

The latter inequality follows from the trivial inequality $2\rho - 1 \geq \mu^2$. Finally, the center line $\mathbf{OBY}$ is orthogonal to the entire base $\mathbf{PMRTB}$ and its line $\mathbf{BN}$. Then $d(\mathbf{B}, \mathbf{N}) = \sigma(\mathbf{N})$ and

$$d(\mathbf{A}, \mathbf{N}) \geq \sigma(\mathbf{N}) - \sigma(\mathbf{A}) \geq \rho - \mu^2 - d \geq \varepsilon. \quad (29)$$

Now we consider the right triangle $\mathbf{ANP}$ and deduce that

$$\cos \alpha = d(\mathbf{A}, \mathbf{N})/d(\mathbf{A}, \mathbf{P}) \geq \varepsilon/\mu = \sqrt{\frac{3}{n}} \ln n. \quad (30)$$

(Note that the latter expression exceeds 1 for $n < 42$, which simply implies that a d -close cap entirely covers μ -cap for small n .) Given α , we can estimate the fraction Ω of the boundary sphere $S_\mu^{n-1}(\mathbf{A})$ contained in the uncovered cap $\mathbf{PQR}$. This fraction can be defined as (see [4])

$$\Omega < \{2\pi(n-1)\}^{-1/2} \frac{\sin^{n-1} \alpha}{\cos \alpha} < \left\{6\pi \frac{n-1}{n}\right\}^{-1/2} \frac{\sin^{n-1} \alpha}{\ln n} < \frac{\sin^{n-1} \alpha}{4 \ln n}.$$

The last inequality follows from the fact that $6\pi(1 - \frac{1}{n}) > 16$. Now we use (30) and approximation $1 - x \leq e^{-x-x^2/2}$. This gives inequalities

$$\begin{aligned} \sin^{n-1} \alpha &\leq \left(1 - \frac{3}{n} \ln^2 n\right)^{\frac{n-1}{2}} \leq \exp\left\{\left(-\frac{3}{2} \ln^2 n - \frac{9}{4n} \ln^4 n\right)\left(1 - \frac{1}{n}\right)\right\} \\ &\leq \exp\left(-\frac{3}{2} \ln^2 n\right), \\ \Omega &\leq \frac{1}{4 \ln n} \exp\left(-\frac{3}{2} \ln^2 n\right). \end{aligned}$$

Finally, consider the second case, when a point $\mathbf{x}$ does not belong to the boundary $\mathbf{PQRS}$. Therefore, $\mathbf{x}$ is taken from a smaller cap $C(\mu', \mathbf{Z}) \subset C(\mu, \mathbf{Z})$ with the same center $\mathbf{Z}$ and a half-chord $\mu' < \mu$. Similarly to the first case, we define the boundary $S_{\mu'}^{n-1}(\mathbf{A}')$ of $C(\mu', \mathbf{Z})$, where $\mathbf{A}'$ is some center on the line $\mathbf{AZ}$. Again, $p(\mathbf{x})$ is the fraction Ω' of this boundary left uncovered by the bigger cap $C(\rho, \mathbf{Y})$. To obtain the

upper bound on Ω' , we only need to replace μ with a smaller μ' in (30). This gives the smaller angle $\alpha' \leq \arccos(\varepsilon/\mu')$, which is reduced to 0 if $\varepsilon \geq \mu'$. In particular, the center $\mathbf{Z}$ of the cap is always covered by any d -close cap. Thus, we see that any internal layer $S_{\mu'}^{n-1}(\mathbf{A}')$ of the cap $C(\mu, \mathbf{Z})$ has a smaller uncovered fraction $\Omega' \leq \Omega$. This gives the required condition

$$p(\mathbf{x}) \leq \Omega' \leq \Omega < \omega$$

for any point $\mathbf{x}$ and proves our lemma. $\square$

Remark Our choice of d in (22) is central to the above proof, and even a marginal increase in d will completely change our setting. Namely, it can be proven that about half the base of the μ -cap is uncovered if a ρ -cap is $(d + \varepsilon)$ -close.

Our next goal is to estimate the expected number N' of non-saturated caps $C(\mu, \mathbf{z}')$ left after N trials.

Lemma 7 *For $n \geq 100$, the expected number N' of non-saturated caps $C(\mu, \mathbf{z}')$ is*

$$N' < 2^{-n/4} N. \quad (31)$$

Proof Given any center $\mathbf{z}$, a randomly chosen center $\mathbf{y}$ is d -close to $\mathbf{z}$ with the probability Ω_d . Then the probability to obtain s or fewer such caps is

$$P = \sum_{i=0}^s \binom{N}{i} \Omega_d^i (1 - \Omega_d)^{N-i}.$$

Note that for any $i \leq s$,

$$(1 - \Omega_d)^{N-i} \leq \exp\{-(\Omega_d + \Omega_d^2/2)(N - i)\}.$$

Now we use inequality $\Omega_d \leq 1/2$ and the lower bound $N\Omega_d > \lambda n \ln n - \Omega_d$ of (25). Then

$$\begin{aligned} & (\Omega_d + \Omega_d^2/2)(N - i) \\ &= N\Omega_d \left(1 + \frac{\Omega_d}{2}\right) - i\Omega_d \left(1 + \frac{\Omega_d}{2}\right) \\ &\geq (\lambda n \ln n - \Omega_d) + \frac{\Omega_d}{2} [\lambda n \ln n - \Omega_d - s(2 + \Omega_d)] \geq \lambda n \ln n. \end{aligned}$$

Here last inequality follows from the fact that $s(2 + \Omega_d) \leq 5n/(6 \ln \ln n)$ and therefore

$$\lambda n \ln n - \Omega_d - s(2 + \Omega_d) \geq 2.$$

Then

$$P \leq e^{-\lambda n \ln n} \sum_{i=0}^s \frac{(\Omega_d N)^i}{i!} \leq e^{-\lambda n \ln n} \sum_{i=0}^s \frac{(\lambda n \ln n)^i}{i!}. \quad (32)$$

Note that consecutive summands in (32) differ at least $(\lambda n \ln n)/s \geq \lambda q \ln n$ times. Therefore

$$\begin{aligned} \sum_{i=0}^s \frac{(\lambda n \ln n)^i}{i!} &\leq \frac{(\lambda n \ln n)^s}{s!} \sum_{i=0}^{\infty} (\lambda q \ln n)^{-i} \\ &\leq 2 \frac{(\lambda n \ln n)^s}{s!} \leq \frac{(\lambda n \ln n)^s}{(s/e)^s} \leq (e\lambda q \ln n)^{n/q}. \end{aligned}$$

Here the sum of the geometric series $(\lambda q \ln n)^{-i}$ was first bounded from above by $c_n < c_{100} < 2$. Then we used the Sterling formula in the form $s! > (2\pi s)^{1/2} (s/e)^s$ and removed (for simplicity) the vanishing term $2(2\pi s)^{-1/2}$. Finally, the last inequality follows from the fact that its left-hand side increases in s for any $s < \lambda n \ln n$. Summarizing, these substitutions give

$$P \leq \exp\{nh_n - \lambda n \ln n\},$$

where

$$h_n = \frac{\ln(e\lambda q \ln n)}{q} = \frac{1}{3} + \frac{\ln(e\lambda q)}{q}.$$

Next, we recalculate ϑ_* of (16) using parameter λ of (22) and the lower bound of (25). It is easy to verify that for any n ,

$$\vartheta_* \leq 2(\lambda n \ln n - 1/2) \leq 2\Omega_d N.$$

Now we estimate the size of $\text{Cov}(\mu)$ using (8) and (22) as follows

$$|\text{Cov}(\mu)| \leq \vartheta_*/\Omega_\mu \leq 2N\Omega_d/\Omega_\mu \leq 2N\mu^n \leq 2N \exp\{n[\beta \ln n + \ln(12)/2]\}. \quad (33)$$

Thus, the expected number of non-saturated caps is

$$\begin{aligned} N' &\leq |\text{Cov}(\mu)|P \leq 2N \exp\{n[h_n - (\lambda - \beta) \ln n + \ln(12)/2]\} \\ &\leq 2N \exp\{n[h_n - (5 - \ln 12)/2]\}. \end{aligned} \quad (34)$$

Now we see that the quantity Ψ_n in the brackets of (34) consists of the declining positive function h_n and the negative constant. Thus, Ψ_n is a declining function of n . Direct calculation shows that $\Psi_{100} < -0.257$. Therefore estimate (31) holds. $\square$

Consider now the saturated caps $C(\mu, \mathbf{z})$ and the centers $\mathbf{u} \in \text{Cov}(\varepsilon)$ inside them.

Lemma 8 *For any $n \geq 100$, the number of centers $\mathbf{u}'' \in \text{Cov}(\varepsilon)$ left uncovered in all saturated caps $C(\mu, \mathbf{z})$ has expectation*

$$N'' < 2^{-n/2} N.$$

Proof We first estimate the total number $|\text{Cov}(\varepsilon)|$ of centers $\mathbf{u}$. Similarly to (33),

$$|\text{Cov}(\varepsilon)| \leq \vartheta_*/\Omega_\varepsilon \leq 2N\Omega_d/\Omega_\varepsilon \leq 2N(2n \ln n)^n.$$

Any cap $C(\mu, \mathbf{z})$ intersects with at least $s + 1$ randomly chosen caps $C(\rho, \mathbf{y})$. According to Lemma 6, any single ρ -cap fails to cover any given point $x \in C(\mu, \mathbf{z})$ with probability ω or less. Therefore any point $\mathbf{u}'' \in C(\mu, \mathbf{z})$ is not covered with probability ω^{s+1} or less. Note that $\omega^{s+1} < \omega^{n/q}$ where $q = 3 \ln \ln n$. Then we use the upper bound (27) for ω and deduce that

$$N'' \leq |\text{Cov}(\varepsilon)| \cdot \omega^{n/q} \leq 2N \exp\{-n\Phi_n\}, \quad (35)$$

where

$$\Phi_n = \left(\frac{\ln^2 n}{2 \ln \ln n} + \frac{\ln 4}{3 \ln \ln n} \right) - \ln n - \ln \ln n - \ln 2 + \frac{1}{3}. \quad (36)$$

Direct calculation shows that $\Phi_{100} > 0.71$. It is also easy to see that the first term of Φ_n (in parentheses) moderates both terms $\ln n$ and $\ln \ln n$. Thus, $\Phi_n > 0.71$ for all $n \geq 100$, and the lemma is proved. $\square$

Proof of Theorem 1 Consider any cap $C(\mu, \bar{\mathbf{z}})$ that contains at least one uncovered center $\mathbf{u} \in \text{Cov}(\varepsilon)$. Such a cap is either non-saturated or porous and therefore $\{\bar{\mathbf{z}}\} = \{\mathbf{z}'\} \cup \{\mathbf{z}''\}$. (Equivalently, we can directly consider the set $\{\mathbf{z}'\} \cup \{\mathbf{u}''\}$.) Then, according to Lemmas 7 and 8, $\{\bar{\mathbf{z}}\}$ has expected size

$$\bar{N} \leq N' + N'' < 2^{1-n/4} N.$$

Thus, there exist N randomly chosen centers $\mathbf{y}$ that leave at most $2^{1-n/4} N$ centers $\bar{\mathbf{z}}$. The extended set $\{\mathbf{x}\} = \{\mathbf{y}\} \cup \{\bar{\mathbf{z}}\}$ forms a unit covering of S_r^n . This covering has density

$$\vartheta \leq \Omega_1(N + \bar{N}) \leq \Omega_1 N(1 + 2^{1-n/4}) \leq \lambda n \ln n (1 + 2^{1-n/4}) \Omega_1 / \Omega_d.$$

Similarly to inequality (9), we now verify that for $n \geq 100$,

$$\Omega_1 / \Omega_d \leq \left(1 - \frac{1}{n \ln n} - n^{-2\beta} \right)^{-n} < 1 + \frac{1}{\ln n} + \frac{1}{\ln^2 n}. \quad (37)$$

Indeed, replace parameter $t = \frac{1}{\ln n}$ used in the proof of Lemma 4 with the new value

$$t = \frac{1}{\ln n} + n^{1-2\beta} = \frac{1}{\ln n} + \frac{1}{\ln^4 n}.$$

This new t also satisfies inequality (11) for $n \geq 100$, which in turn proves inequality (37). As a secondary remark, note that (37) also proves asymptotic equality (24).

Finally, we take λ of (9) and combine the last inequalities for ϑ and Ω_1 / Ω_d as follows

$$\begin{aligned} \frac{\vartheta}{n \ln n} &\leq \left(\frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{5}{2 \ln n} \right) \left(1 + \frac{1}{\ln n} + \frac{1}{\ln^2 n} \right) (1 + 2^{1-n/4}) \\ &< \frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{5}{\ln n}. \end{aligned}$$

Here we again used Lemma 3. Namely, we verify numerically that the last expression exceeds the previous one at $n = 100$ and moderates it for larger n , due to its bigger remaining term $5/\ln n$. This completes the non-asymptotic case $n \geq 3$.

To complete the proof of Theorem 1, we now present similar estimates for $n \rightarrow \infty$. We take any constant $b > 3/2$ and redefine the parameters in (22) and (26) as follows:

$$\begin{aligned}\beta &= \frac{1}{2} + b \frac{\ln \ln n}{\ln n}, \\ \lambda &= \beta + \frac{3}{4} \frac{1}{\ln n}, \\ \mu &= \frac{1}{2\sqrt{n}} \ln^{-b} n, \\ q &= \ln^2 \ln n.\end{aligned}$$

First, bounds (30) and (27) can be replaced with

$$\begin{aligned}\cos \alpha &\geq \varepsilon/\mu \geq \frac{1}{\sqrt{n}} \ln^{b-1} n, \\ \omega &\leq \left(1 - \frac{1}{n} \ln^{2b-2} n\right)^{\frac{n-1}{2}} \leq \exp\left(-\frac{1}{2} \ln^{2b-2} n\right).\end{aligned}\tag{38}$$

Second, bound (34) can be rewritten as

$$N' \leq 2N \exp\left\{n \left(\frac{\ln(e\lambda q \ln n)}{q} + \ln 2 - \frac{3}{4}\right)\right\}.$$

Note that the first term $\ln(e\lambda q \ln n)/q$ vanishes for $n \rightarrow \infty$, and N' declines exponentially in n . Thus, Lemma 7 holds. Finally, bound (35) is replaced with

$$N'' \leq 2N \exp\left\{n \left[\ln(2n \ln n) - \frac{1}{2} \frac{\ln^{2b-2} n}{\ln^2 \ln n}\right]\right\},$$

which vanishes (faster than exponent in n) for any given $b > 3/2$. Thus, Lemma 8 also holds. Then we proceed similarly to (37). In this case, for sufficiently large n , we obtain the density

$$\frac{\vartheta}{n \ln n} \leq \left(\frac{1}{2} + b \frac{\ln \ln n}{\ln n} + \frac{3}{4} \frac{1}{\ln n}\right) \left(1 + \frac{1}{\ln n} + \frac{1}{\ln^2 n}\right) \leq \frac{1}{2} + b \frac{\ln \ln n}{\ln n} + \frac{3}{2} \frac{1}{\ln n},$$

which proves (6). This completes the proof of Theorem 1. $\square$

For finite n , we can slightly refine bounds (28–29), and also use exact expression for ω in (27). However, these refinements only marginally improve the main bound (5), which is replaced with

$$\frac{\vartheta}{n \ln n} \leq \frac{1}{2} + \frac{2 \ln \ln n}{\ln n} + \frac{4 \ln \ln n}{\ln n}.\tag{39}$$

Finally, note that Theorem 1 directly leads to Corollary 2. Indeed, here we can use the well known fact

$$\vartheta(\mathbb{R}^{n-1}) = \lim_{r \rightarrow \infty} \vartheta(S_r^n)$$

(see a sketch of the proof in [6] or Theorem $\Pi.1$ in [7], where a similar proof is detailed for packings of $\mathbb{R}^n$).

Concluding remarks In summary, we prove in this paper that the classic Rogers bound (3) on covering density of a sphere S_r^n or the Euclidean space $\mathbb{R}^n$ can be reduced about two times for large dimensions n . The main open problem is to further reduce the gap between this bound and its lower counterpart (2), which is linear in n . In this regard, note that our design holds if we employ any constant parameter $\beta > 1/2$ introduced in (22). However, it can be verified that choosing a smaller constant $\beta < 1/2$ will first increase parameter $\mu = n^{-\beta}$ and then reduce parameter d of (22) to the order of $1 - \mu^2/2$. The latter reduction will exponentially increase the covering size, by a factor of

$$\Omega_\rho/\Omega_d \rightarrow \exp\{1 - 2\beta\}.$$

Therefore, our conjecture is that a completely new design is needed for further asymptotic reductions.

Another important problem is to extend the above results to the balls B_r^n of an arbitrary radius r . Our conjecture is that $\vartheta(B_r^n) \leq (\frac{1}{2} + o(1))n \ln n$ for any r and $n \rightarrow \infty$.

Acknowledgements The author is grateful to an anonymous referee for many helpful comments and remarks. This research was supported in part by NSF grants CCF-0622242 and CCF-0635339.

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