On a Generalization of Tarski's Plank Problem*

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Abstract. Answering a question of the author and K. Bezdek we prove that for any decomposition of the disk B^2 of unit radius into finite many convex cells the sum of the diameters of the incircles of the cells is at least 2. This strengthens Tarski's plank problem, where it was shown that the sum of the minimal widths of the cells is at least 2.

1. Introduction and Preliminaries

A *strip* or a *plank* in E^2 is a closed set bounded by two parallel lines. The distance between these lines is called the *width* of the strip. The *minimal width* (in short *width*) of a convex set C is the minimum of the widths of the strips containing C .

The following theorem was conjectured by Tarski [4] and was proved by Bang [1]: If a closed convex set C in E^2 is covered by a finite number of strips, then the sum of their widths is greater than or equal to the width of C . If C is the unit disk, then there is an extremely simple proof. We repeat this proof here, mainly because our main result (Theorem 1) is a generalization of this argument:

Assume that the unit disk in E^2 is covered by strips of widths $w_1, \dots, w_n$. Now consider the positive hemisphere in E^3 raised over the given unit disk and project perpendicularly each strip to this hemisphere. The surface area of the projection of the i th strip is πw_i (this is a well-known fact from calculus, already discovered by Archimedes). Since the projections cover the hemisphere, we have $\sum_1^n w_i \pi \geq 2\pi$.

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Fig. 1. Theorem 1 and Corollary 1.

With K. Bezdek [2]—while working on a related plank problem called Conway’s fried potato problem—we took a closer look at the above calculus fact and stated Theorem 1 as a conjectured generalization. In 1996, independently, we handled some special cases and realized that the complete solution immediately would imply Corollary 1. Let B^2 denote the closed circular disk of radius 1. Let S be the closed hemisphere raised over B^2 .

Theorem 1. *If T is a triangle or a strip in the plane, and r is the inradius of the common part of T and B^2 , then the vertical projection of $T \cap B^2$ to the hemisphere S has an area less than or equal to $2\pi r$ (Fig. 1), where equality holds iff T is a strip of width $2r$ crossing the disk B^2 .*

Corollary 1 (Generalization of Tarski’s Theorem). *Consider a partition of disk B^2 into n convex cells (Fig. 1). If $r_1, \dots, r_n$ are the inradii of the convex cells, then $\sum_1^n r_i \geq 1$.*

Proof. Label the cells by $C_1, \dots, C_n$. Since $C_1, \dots, C_n$ tile the unit disk B^2 , the vertical projections $p(C_1), \dots, p(C_n)$ tile S . In view of Theorem 1, their total area (which is 2π) has the upper bound $\sum_1^n 2r_i\pi$, from which $\sum_1^n r_i \geq 1$ follows. $\square$

Remark 1. The inequality of Theorem 1 remains true if we replace “triangle T ” by “any convex region C ”.

Proof. Let K be the incircle of $C \cap B^2$. It is easy to see that among the contact points of K and $bd(C \cap B^2)$ one can find three points forming an acute triangle or two antipodal points. In both cases, the tangent lines of K at these points enclose a region which has larger perpendicular projection than the convex region C . $\square$

Corollary 1 is stronger than Tarski’s plank problem. To see this, cover each cell by the narrowest strip and notice that the widths of these strips are greater than the diameters of the incircles.

A joint paper of the author and K. Bezdek [2] on the solution of Conway’s fried potato problem (Conway starts with a convex “potato” of width 1, makes $n - 1$ plane

cuts, cutting only one piece at a time, and claims that at least one of the pieces contains a ball of diameter $1/n$) solves the above convex partition problem in case of Conway-type decompositions. Further references on “plank problems” can be traced through the book of Brass et al. [3].

A complete proof of Theorem 1 was found by the author and independently by G. Kertész (unpublished). The main purpose of this paper is to prove Theorem 1.

Outline of the proof of Theorem 1. In order to simplify the description of our proof, we assign to each triangle T the ratio

$$\lambda(T) = \frac{\text{the surface area of the vertical projection of } T \cap B^2 \text{ to } S}{\text{the inradius of } T \cap B^2}.$$

Note that Theorem 1 is about maximizing ratio $\lambda(T)$. In the rest of the paper, $p(\cdot)$ denotes the vertical projection from B^2 to S , $r(\cdot)$ denotes the inradius of a given convex region, while $\text{area}(\cdot)$ denotes the area (surface area resp.) of a region (of a spherical region resp.). Using this notation

$$\lambda(T) = \frac{\text{area}(p(T \cap B^2))}{r(T \cap B^2)}.$$

Theorem 1 claims that the maximum of $\lambda(T)$ is 2π , with equality only if T is a strip crossing the disk B^2 . We prove this theorem without any direct surface area computation. In Section 2 we show six elementary ways to change triangle T so that ratio $\lambda(T)$ increases. In Section 3 we distinguish four types of triangles (according to how many vertices of triangle T lie outside of the closed disk B^2) and explain how to combine the methods of Section 2 with some additional arguments to prove that strips maximize $\lambda(T)$.

2. Techniques to Increase the Ratio of the Area of Vertical Projection and the Inradius of T

First we state (without proofs) two simple elementary geometric facts:

Remark 2 (A Chord Length Inequality). Suppose a circle S contains another nonconcentric circle K . Among all chords of S tangent to K both the longest and the shortest ones are perpendicular to the line connecting the centers of the circles. Furthermore, rolling the longest chord around circle K gradually decreases its length.

Remark 3 (An Inradii Inequality). Let ABC and A^*B^*C be two homothetic isosceles triangles so that $A^*C = B^*C \leq BC = AC$. Let us shift vertex C parallel to AB away from the symmetry axis of AB and change the positions of vertices A and B on the line AB so that the sides AC and BC keep passing through the fixed points A^* and B^* . Then the inradius of the triangle ABC decreases. In particular, this is true when $A = A^*$, i.e. when only vertex C is moving.

The following six lemmas are about different types of triangles. In each lemma we describe a continuous change of the given triangle T and argue that $\lambda(\cdot)$ is increasing.

Fig. 2. Lemmas 1–3.

In the rest of this section, A , B and C denote the vertices of triangle T , O denotes the center of disk B^2 and S^1 denotes the boundary circle of disk B^2 .

Lemma 1. *If triangle T is contained in the interior of disk B^2 (Fig. 2), then the **homothetic enlargement** of T from the center O (assuming that the image of T stays in the disk B^2) increases $\lambda(T)$.*

Proof. Assume that a very fine grid is laid over triangle T . Let us enlarge T together with the grid by a factor $\mu > 1$ so that the enlarged triangle T' is still contained in B^2 . Notice that each grid cell moves farther away from O and its area increases by a factor of μ^2 , thus $\mu^2 \text{area}(p(T)) < \text{area}(p(T'))$. Since $\mu r(T) = r(T')$, we have that $\mu(\text{area}(p(T))/r(T)) < \text{area}(p(T'))/r(T')$, which means that $\lambda(T)$ increases. $\square$

Lemma 2. *Assume that triangle T is contained in disk B^2 so that exactly one vertex, say B , belongs to the boundary circle S^1 (Fig. 2). Then the **perpendicular affinity** whose axis is OB and which changes triangle T to a triangle T' , which lies in disk B^2 and has a second vertex on S^1 , **increases** the assigned ratio $\lambda(T)$.*

Proof. Assume that a very fine grid is laid over triangle T . Let us apply a perpendicular affinity of a factor $\mu > 1$ to T and to the grid so that the enlarged triangle T' is still contained in B^2 . Each grid cell moves away from O and its area increases by a factor of μ , thus $\mu \text{area}(p(T)) < \text{area}(p(T'))$. Consider now the inverse of this affinity (i.e. the one with factor $1/\mu$). The image of the incircle of T' is an ellipse of half diameters $r(T')$ and $(1/\mu)r(T')$, thus $r(T) > (1/\mu)r(T')$. Dividing the previous inequality with the latter one, we have that $\text{area}(p(T))/r(T) < \text{area}(p(T'))/r(T')$, which means that $\lambda(T)$ increases. $\square$

Lemma 3. *Assume that triangle T is contained in disk B^2 with exactly two vertices, say with A and B , belonging to the boundary circle S^1 (Fig. 2). Then as **one moves the third vertex C of T parallel to AB , away from the center O , while keeping the other two vertices fixed, the ratio $\lambda(T)$ increases.***

Proof. This time, instead of using a fine grid, let us subdivide T using lines parallel to AB into very thin trapezoids. As we move C parallel to AB , we essentially apply

Fig. 3. Lemmas 4-6.

an affine transformation to each trapezoid preserving both their length and width. Since C moves away from the center O , the area of the projection of each trapezoid to the hemisphere increases. Thus, $\text{area}(p(T)) < \text{area}(p(T'))$. According to Remark 3, the inradius decreases, thus $r(T) > r(T')$. Dividing these two inequalities with each other, we get that $\text{area}(p(T))/r(T) < \text{area}(p(T'))/r(T')$, which means that $\lambda(T)$ increases. $\square$

Lemma 4. Assume that only one of the vertices of triangle T , say C , belongs to the interior of disk B^2 and assume that side AB is disjoint from B^2 (Fig. 3). Let A^* (B^* resp.) be the point where the side AC (BC resp.) intersects the boundary circle S^1 . Also assume that side AB is disjoint from B^2 and is parallel to the segment A^*B^* . If triangle T is changed so that (i) **the vertex C moves** parallel to A^*B^* away from the center O , (ii) the side AC (BC resp.) keeps passing through A^* (B^* resp.) and (iii) the line of AB remains unchanged, then $\lambda(T)$ **increases**.

Proof. Let us denote a new position of triangle T by T' . The area argument used in the previous lemma immediately gives $\text{area}(p(T \cap B^2)) < \text{area}(p(T' \cap B^2))$. Now we show that $r(T \cap B^2) > r(T' \cap B^2)$, which, as in the proof of the previous lemma, implies that $\lambda(T)$ increases. Choose triangle EFC so that EF is parallel to AB and EFC and $T \cap B^2$ have the same incircle. Let us move C according to Lemma 4 and find out how the inradius of $T \cap B^2$ changes. It is easy to see that the incircle rolls on S^1 to the right and thus cannot intersect the line EF and therefore it is smaller than the incircle of triangle T . However, the latter, in view of Remark 3, is smaller than the incircle of T . $\square$

Lemma 5. Assume that exactly two of the vertices of triangle T , say B and C , belong to the interior of the disk B^2 and assume $BC \geq AC$ (Fig. 3). Let A^* be the common point of the side AC and the boundary circle S^1 . Let B^* be a point on the side BC so that AB is parallel to A^*B^* . If triangle T is changed so that (i) **the vertex C moves** parallel to A^*B^* away from the center O , (ii) the side AC (BC resp.) keeps passing through A^* (B^* resp.) and (iii) sideline AB remains the same, then $\lambda(T)$ **increases**.

Proof. Let us denote a new position of triangle T by T' . The area argument used in the proof of Lemma 3 gives $\text{area}(p(T \cap B^2)) < \text{area}(p(T' \cap B^2))$. In view of Remark 3 we have $r(T \cap B^2) > r(T' \cap B^2)$. Dividing the area inequality with the inradius

inequality we get that $\text{area}(p(T))/r(T) < \text{area}(p(T'))/r(T')$, which means that $\lambda(T)$ increases. $\square$

Many extreme value problems are solved by local improvements. For example, when finding the smallest area triangle circumscribed to a given circle, one tries to roll one of the sides along the circle, keeping the other two fixed. It is easy to see that the area of the triangle can be decreased this way, unless the contact points are midpoints of the edges of the triangle. The following lemma shows that such improvements, in case of our problem, work only under very special circumstances.

Lemma 6. *Assume that vertex B of triangle T belongs to the interior of disk B^2 . Also assume that side AB touches the incircle K of $T \cap B^2$. Let m be the midpoint of the segment $AB \cap B^2$, let t be the tangency point of the side AB and the incircle of $T \cap B^2$ and let h be the midpoint of the chord of B^2 containing the side BC (Fig. 3). If the order of these three points is different from $t - m - h$ (here t and h might coincide with m), then **rolling the sideline AB around the incircle K in one of the two directions increases the ratio $\lambda(T)$.***

Proof. Without loss of generality, assume that AB is horizontal and A is the left end-point. We prove Lemma 6 indirectly. If A belongs to B^2 , then the indirect assumption (without loss of generality) means that among the points m , h and t , point m is the most left and neither t nor h coincides with it. If A does not belong to B^2 , then m has to be left from h and thus the indirect assumption means again that among the points m , h and t , point m is the most left and neither t nor h coincides with it. We distinguish two cases (Fig. 4):

- (A) In the triplet $\{m, h, t\}$, t is the second point from the left and it is different from h .
- (B) In the triplet $\{m, h, t\}$, h is the second point from the left and it might coincide with t .

In both cases rolling the sideline AB counterclockwise along the incircle of T increases $\lambda(T)$. As Fig. 4 illustrates, the reasons are different in cases (A) and (B). It turns out that the small triangle T' (which represents the area of loss) can be placed in the tri-

Fig. 4. Lemma 6.

angle representing the area of gain so that the corresponding points of these triangles get farther away from O and thus the area of the vertical projection of T to the hemisphere S increases. $\square$

3. Proof of Theorem 1

A simple compactness argument shows that there exists a triangle T or a strip T for which $\lambda(T)$ is maximal. We show that the maximum is not achieved in the case of triangles. We distinguish four types of triangles according to how many of their vertices lie outside of B^2 (see Fig. 5).

Proposition 1. *If triangle T is a subset of B^2 (see Case 0 in Fig. 5), then it cannot maximize $\lambda(T)$.*

Proof. Using Lemmas 1–3 we conclude that when maximizing $\lambda(T)$ we may assume that all three vertices A , B and C of T lie on the boundary circle of B^2 . Let P , Q and R be the points where the incircle of T is touching the sides AC , CB and BA (Fig. 6). Let O^1 be the center of the incircle K of T . Without loss of generality, we may assume that ray OO^1 intersects the shorter circular arc PQ . Let e be the other support line of K which is parallel to AB . Line e has to touch K also on the shorter circular arc PQ , say, at R^* . Without loss of generality we may assume that P and R^* are not separated by line OO^1 . In view of Remark 2, we know that line e cuts off less from disk B^2 than chord AC and thus a strip of width $2r$ has larger $\lambda(\cdot)$. $\square$

Proposition 2. *If triangle T has exactly two of its vertices of T outside of B^2 (see Case 1 in Fig. 5), then it cannot maximize $\lambda(T)$.*

Proof. First we consider a special case:

Lemma 7. *Triangle T with vertices A , B and C cannot maximize $\lambda(T)$ if vertex C lies on the boundary of B^2 and side AB is disjoint from B^2 .*

Proof. Let K be the incircle of $T \cap B^2$ with center O (Fig. 7). It is enough to prove Lemma 7 if circle K is tangent to sides CA , CB and the boundary circle of B^2 . De-

Fig. 5. Four types of triangles.

note the three contact points (in the above order) by P , Q and R . Ray RO intersects K between P and Q , say, at R' . Similarly, ray QO intersects K between P and R , say, at Q' . According to Remark 2, among all chords of B^2 which are tangent to circle K , the one through R' is the longest. The order of the points R' , P and Q' along circle K implies that the chord through P is longer than the one through Q' . The monotonicity of the area cut off from K by chords of decreasing length implies our inequality. $\square$

Now we consider the general case of Proposition 2. Let K be the incircle of $T \cap B^2$ with center O (Fig. 7). We may assume that K is tangent to the sides CA , CB and AB , otherwise Lemma 4 reduces this case to the special case we just discussed. Denote the three contact points (in the above order) by P , Q and R . Ray RO intersects K between P and Q , say at R' . Let x be the chord of B^2 which is parallel to AB and touches K at R' . Let X be the circular segment (by definition a *circular segment* is a region bounded by an arc and a chord) which is cut off from B^2 by the chord x and contains C . Denote by T' a triangle which can be obtained from triangle T by shifting the sideline AB away from C until AB becomes disjoint from B^2 . Let K' be the incircle of $T' \cap B^2$. Let x' be the chord of B^2 which is parallel to AB and touches K' and separates K' from line AB . Finally let X' be the circular segment which is cut off from B^2 by the chord x' and which

Fig. 7. Proposition 2.

Fig. 8. Proposition 3.

separates C from K' . We have that

$$\text{area}(p(X)) < \text{area}(p(X')) < \text{area}(p(B^2 \setminus T')),$$

where the first inequality follows from $X \subset X'$ and where the second inequality follows from Lemma 7 applied for triangle T' . Let Y be that circular segment which is cut off from B^2 by line AB and which does not contain C . Adding $\text{area}(p(Y))$ to the first and to the last terms of the above inequality chain we get that $2\pi - 2r\pi < \text{area}(p(B^2 \setminus T))$, which is what we had to prove. $\square$

Proposition 3. *If triangle T has exactly one of its vertices outside of B^2 (see Case 2 in Fig. 5), then it cannot maximize $\lambda(T)$.*

Proof. Let A be the vertex of T which lies outside of B^2 (Fig. 8). Indirectly assume triangle T maximizes $\lambda(T)$. In view of Lemma 4 we have that BC is the shortest side of T . Without loss of generality assume $AB \geq AC$. Denote by e the line which is parallel to AB and is tangent to incircle K . Let H be the strip bounded by lines AB and e .

We show that the $\lambda(T) < \lambda(H)$, contradicting the maximal choice of T . Equivalently, we show that $\text{area}(p(T \setminus H)) < \text{area}(p(H \setminus T))$. How can we prove this without surface area computation and without changing triangle T ? Let triangle $T \setminus H$ be denoted by T_1 . The basic idea is to show that there is an isometry f from T_1 to $H \setminus T$ (i.e. a copy of triangle T_1 can be placed in $H \setminus T$) so that for each point X of T_1 the tangent plane of S at $p(f(X))$ is steeper than the one at $p(X)$. It is just a technical complication that f will be given as a composition of several isometries, where only the image of the last isometry belongs to the strip $H \setminus T$. It turns out that we achieve our goal by studying two subcases:

Subcase (a): Line e Separates the Center of B^2 from Vertex C . Let EF be the chord $e \cap B^2$. Denote by P the intersection point of EF and BC . We define several congruent triangles:

Definition 1. *Triangle T_1 was defined as $T \setminus H$. Let triangle T_2 be the image of T_1 under the reflection around axis O^1P , where O^1 is the incenter of T .*

Lemma 8. *Triangle T_2 is a subset of strip H .*

Let us return to subcase (a) depicted in Fig. 8. Let PO^1 be the bisector of the angular sector EPB . If the center O of B^2 belongs to the angular sector $EP P'$, then it is obvious that $\text{area}(p(T_1)) < \text{area}(p(T_2))$, which implies Proposition 3. In the rest of the argument we assume that center O of B^2 belongs to the angular sector $P'PF$.

If the contact point of T_3 and S^1 belongs to the horizontal side of T_3 , then a rotation which takes each point of the triangle away from O is used until a second vertex belongs to S^1 . Denote by T_4 this new triangle (T_4 is not shown in Fig. 8). It is easy to see that $\text{area}(p(T_3)) < \text{area}(p(T_4))$. If the contact point of T_3 and S^1 does not belong to the horizontal side of T_3 , then a counterclockwise rotation is used until a second vertex belongs to S^1 . We also denote by T_4 this new triangle. Let us remind ourselves that T_3 is

Fig. 10. Subcase (b) of Proposition 3.

similar to ABC , where $AC \leq CB$. A simple analysis of the intersection of T_3 and thin annuli centered at the center of rotation reveals that again $\text{area}(p(T_3)) < \text{area}(p(T_4))$.

Definition 3. Consider that translate of T_2 which lies in the region $H \setminus T$ and touches the boundary of $H \setminus T$ either at its right-upper or right-lower corner (it is easy to see that at least one corner can be contacted). If it touches the right-upper corner we call it T_5 . If it touches the right-lower corner we call it T_6 .

It is easy to see that both T_5 and T_6 can be rotated around their contact points so that they remain contained in H and have another vertex on the boundary of B^2 . Since the new triangles have the same relative positions with respect to B^2 as triangles T_4 , we can conclude that $\text{area}(p(T_1)) \leq \text{area}(p(T_3)) = \text{area}(p(T_5)) = \text{area}(p(T_6))$, which is what we wanted. $\square$

Subcase (b): Line e Does Not Separate the Center of B^2 from Vertex C . In the rest of the proof we use Fig. 10, where we continue using the notation of Figs. 8 and 9. Let H again be the strip which is parallel to AB and contains the incircle of T . Recall that T_1 denotes the triangle $T \setminus H$. We show that $\text{area}(p(T_1)) < \text{area}(p(H \setminus T))$.

We need to introduce a couple of new points and regions. Let L be the intersection of lines e and AC . Denote by u the region enclosed by AC , e and S^1 . Let u^* be the common part of T_1 and the image of u under the reflection around L . Let v^* be the region $T_1 \setminus u^*$. Let v^{**} be the image of v^* under the reflection around E . Next we show that $v^* \subset H$.

To see this, it is enough to verify that $C^{**} \in H$ and $G^{**} \in H$. The first inclusion follows from the assumption that BC is the shortest side of T . Indeed, this implies that $\angle ACB \geq \pi/3$ and thus $PC^{**} = CP < \sqrt{3}r < 2r$, which means that $C^{**} \in H$. It is harder to see the second inclusion. The above angle inequality implies again that $EE^{**} = 2LP < 2\sqrt{3}$. The composition of the reflections around L and then around P take the circle S^1 to S^{1**} . Notice that the second circle is obtained from S^1 by a horizontal translation of length $< 2\sqrt{3}r$. Simple calculus shows that $MN \leq 2\sqrt{3}r$, thus MN and S^{1**} intersect each other in the interior of B^2 . $\angle ACB \geq \pi/3$ implies that distance between G^* and e is at most r and then the same applies for G^{**} , i.e. we have the desired inclusion $G^{**} \in H$.

Finally change the region v^{**} into region v using the following transformation, called “shift to right”: Treat region v^{**} as a union of horizontal segments and translate each segment horizontally to the right until the right endpoint belongs to S^1 . Every point of u , when reflected around L gets closer to the center O of B^2 . Similarly, every point of v^* , when reflected around E and then “shifted to right”, moves away from the center O of B^2 . The last two observations imply that $\text{area}(p(T_1)) < \text{area}(p(u \cup v)) < \text{area}(p(H \setminus T))$, which is what we wanted to prove.

Proposition 4. *A triangle T , with all three vertices outside of disk B^2 (see Case 3 in Fig. 5), cannot maximize $\lambda(T)$.*

Proof. It is easy to see that rolling one of the sides improves the area of the projection and thus T cannot be optimal. $\square$

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