The On-Line Heilbronn's Triangle Problem in d Dimensions*

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Abstract. In this paper we show a lower bound for the on-line version of Heilbronn's triangle problem in d dimensions. Specifically, we provide an incremental construction for positioning n points in the d -dimensional unit cube, for which every simplex defined by $d + 1$ of these points has volume $\Omega(1/n^{(d+1) \ln(d-2) - 0.265d + 2.269})$ (for $d \geq 5$).

1. Introduction

The *off-line* version of the now famous *triangle problem* was posed by Heilbronn [R] more than 50 years ago. It is formulated as follows:

Given n points in the unit square, what is $\mathcal{H}_2^{\text{off-line}}(n)$, the maximum possible area of the *smallest* triangle defined by some three of these points?

There is a large gap between the best currently known lower and upper bounds on $\mathcal{H}_2^{\text{off-line}}(n)$, $\Omega(\log n/n^2)$ [KPS2] and $O(1/n^{8/7-\varepsilon})$ (for any $\varepsilon > 0$) [KPS1]. Jiang et al. [JLV] showed that the expected area of the smallest triangle, when the n points are put uniformly at random in the unit square, is $\Theta(1/n^3)$. Barequet [B1] generalized the off-line problem to d dimensions:

Given n points in the d -dimensional unit cube, what is $\mathcal{H}_d^{\text{off-line}}(n)$, the maximum possible volume of the *smallest* simplex defined by some $d + 1$ of these points?

The best currently known lower bound on $\mathcal{H}_d^{\text{off-line}}(n)$ is $\Omega(\log n/n^d)$ [L1]. Other versions, in which the dimension of the optimized simplex is lower than that of the cube, were investigated in [L2], [L3], and [BN].

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The *on-line* version of the triangle problem is harder than the off-line version because the value of n is not specified in advance. In other words, the points are positioned one after the other in a d -dimensional unit cube, while n is incremented by one after every point-positioning step. The procedure can be stopped at any time, and the already-positioned points must have the property that every subset of $d + 1$ points defines a polytope whose volume is at least some quantity $\mathcal{H}_d^{\text{on-line}}(n)$, where the goal is to maximize this quantity. Schmidt [S] showed that $\mathcal{H}_2^{\text{on-line}}(n) = \Omega(1/n^2)$. Barequet [B2] used nested packing arguments to demonstrate that $\mathcal{H}_3^{\text{on-line}}(n) = \Omega(1/n^{10/3}) = \Omega(1/n^{3.333\dots})$ and $\mathcal{H}_4^{\text{on-line}}(n) = \Omega(1/n^{127/24}) = \Omega(1/n^{5.292\dots})$.

In this paper we present a nontrivial generalization of the latter method to d dimensions, showing that for a fixed value of $d \geq 5$ we have $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(1/n^{(d+1)\ln(d-2)-0.265d+2.269})$. We provide an incremental procedure for positioning n points (one by one) in a d -dimensional unit cube so that no subset of up to $d + 1$ points is “too dense.” Specifically, the distance between any two points is at least $a_1/n^{1/d}$ (for some constant $a_1 > 0$), no three points define a triangle whose area is less than $a_2/n^{2/(d-1)}$ (for some constant $a_2 > 0$), and so on. The values of the constants are tuned at the end of the construction. It is then proven that all the d -dimensional simplices defined by $(d + 1)$ -tuples of the points have volume $\Omega(1/n^{(d+1)\ln(d-2)-0.265d+2.269})$.

2. The Construction

2.1. Notation and Plan

We use the following notation. Let $p_{i_1}, p_{i_2}, \dots, p_{i_q}$ be any q points in $\mathbb{R}^d$. Then $|p_{i_1}p_{i_2}|$ denotes the distance between two points p_{i_1}, p_{i_2} ; $|p_{i_1}p_{i_2}p_{i_3}|$ denotes the area of the triangle $p_{i_1}p_{i_2}p_{i_3}$; $|p_{i_1}p_{i_2}p_{i_3}p_{i_4}|$ denotes the three-dimensional volume of the tetrahedron $p_{i_1}p_{i_2}p_{i_3}p_{i_4}$; and, in general, $|p_{i_1}p_{i_2} \cdots p_{i_q}|$ denotes the volume of the $(q - 1)$ -dimensional simplex $p_{i_1}p_{i_2} \cdots p_{i_q}$. We denote by C^d the d -dimensional unit cube, and by B_r^d a d -dimensional ball of radius r . The line defined by the pair of points p_{i_1}, p_{i_2} is denoted by $\ell_{i_1i_2}$.

Throughout the construction we refer to d as a fixed constant. Therefore, we omit factors that depend solely on d , except when they appear in powers of n .

We want to construct a set S of n points in C^d such that:

- [1] $|p_{i_1}p_{i_2}| \geq V_1 = a_1/n^{1/d}$, for any pair of distinct points $p_{i_1}, p_{i_2} \in S$ and for some constant $a_1 > 0$.
- [2] $|p_{i_1}p_{i_2}p_{i_3}| \geq V_2 = a_2/n^{2/(d-1)}$, for any triple of distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in S$ and for some constant $a_2 > 0$.
- [3] $|p_{i_1}p_{i_2}p_{i_3}p_{i_4}| \geq V_3 = a_3/n^{(4d^2-5d-1)/(d(d-1)(d-2))}$, for any quadruple of distinct points $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4} \in S$ and for some constant $a_3 > 0$.
- $\vdots$
- [$q - 1$] $|p_{i_1}p_{i_2} \cdots p_{i_q}| \geq V_{q-1} = a_{q-1}V_{q-2}/(a_{q-2}n^{(d(q-2)+q-3)/(d(d-q+2))})$, for any q -tuple ($4 \leq q \leq d + 1$) of distinct points $p_{i_1}, p_{i_2}, \dots, p_{i_q} \in S$ and for some constant $a_{q-1} > 0$.

Note that condition [$q - 1$] holds for $q = 4$, in which case it coincides with [3].

The goal is to construct S incrementally. That is, assume that we have already constructed a subset S_v of v points, for $v < n$, which satisfies conditions [1]–[$q - 1$] above. We want to show that there exists a new point $p \in C^d$ that satisfies:

- [1'] $|pp_{i_1}| \geq V_1 = a_1/n^{1/d}$, for each point $p_{i_1} \in S$.
- [2'] $|pp_{i_1}p_{i_2}| \geq V_2 = a_2/n^{2/(d-1)}$, for any pair of distinct points $p_{i_1}, p_{i_2} \in S$.
- [3'] $|pp_{i_1}p_{i_2}p_{i_3}| \geq V_3 = a_3/n^{(4d^2-5d-1)/(d(d-1)(d-2))}$, for any triple of distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in S$.
- $\vdots$
- [($q - 1$)'] $|pp_{i_1}p_{i_2} \cdots p_{i_{q-1}}| \geq V_{q-1} = a_{q-1}V_{q-2}/(a_{q-2}n^{(d(q-2)+q-3)/(d(d-q+2))})$, for any ($q - 1$)-tuple ($4 \leq q \leq d + 1$) of distinct points $p_{i_1}, p_{i_2}, \dots, p_{i_{q-1}} \in S$.

We will show this by summing up the volumes of the “forbidden” portions of C^d where one of the inequalities [1']–[($q - 1$)'] is violated, and by showing that the sum of these volumes is less than 1. This implies the existence of the desired point p , which we then add to S_v to form S_{v+1} . We continue in this manner until the entire set S is constructed.

2.2. Forbidden Balls

The forbidden regions where one of the inequalities [1'] is violated are v d -dimensional balls of radius $r = a_1/n^{1/d}$.¹ Their total volume is at most

$$v|B_r^d| = O\left(\frac{v}{n}\right) = O(1).$$

2.3. Forbidden Cylinders

The forbidden regions where one of the inequalities [2'] is violated are $\binom{v}{2}$ d -dimensional “cylinders” G_{ij} , for $1 \leq i < j \leq v$. The cylinder G_{ij} is centered at ℓ_{ij} , its length is at most $\sqrt{d}$, and its cross-section perpendicular to ℓ_{ij} is a $(d - 1)$ -dimensional sphere of radius

$$r_{i,j} = \frac{2a_2}{n^{2/(d-1)}|p_i p_j|} = \Theta\left(\frac{1}{n^{2/(d-1)}|p_i p_j|}\right)$$

(see Fig. 1).

The overall volume of the “cylinders” (within C^d) is at most

$$\sum_{1 \leq i < j \leq v} (|B_{r_{i,j}}^{d-1}|\sqrt{d}) = \sum_{1 \leq i < j \leq v} O\left(\frac{1}{n^2|p_i p_j|^{d-1}}\right). \quad (1)$$

To bound this sum, we fix p_i and sum over p_j . We use a d -dimensional spherical

¹ Recall that $|B_r^d| = \pi^{d/2}r^d/\Gamma(d/2 + 1) = \Theta(r^d)$, where $\Gamma(\cdot)$ is the continuous generalization of the factorial function.

Fig. 1. A cylinder in $\mathbb{R}^d$.

packing argument that exploits the fact that S_v satisfies [1]. Specifically, we have

$$\sum_{j \neq i} \frac{1}{|p_i p_j|^{d-1}} \leq \sum_{t=1}^{O(n^{1/d})} \frac{M_t n^{(d-1)/d}}{a_1^{d-1} t^{d-1}}, \quad (2)$$

where M_t is the number of points of S_v that lie in the d -dimensional spherical shell centered at p_i with inner radius $a_1 t/n^{1/d}$ and outer radius $a_1(t+1)/n^{1/d}$; see Fig. 2. There are $O(n^{1/d})$ such spherical shells (within C^d). Because of [1], the number of such

Fig. 2. A spherical packing of balls in $\mathbb{R}^d$.

points is $M_t = O(t^{d-1})$. This follows by an argument of packing spheres of volume $\Theta(1/n)$ within a shell whose volume is $\Theta(t^{d-1}/n)$.

Hence, the sum in (2) is $O(n)$. Summing this over all p_i , we obtain a final bound of $O(vn)$. Substituting this in (1), we see that the total volume of the forbidden cylinders is $O(v/n) = O(1)$.

2.4. Forbidden Prisms

The forbidden regions where one of the inequalities [3'] is violated are $\binom{v}{3}$ d -dimensional “prisms” φ_{ijk} , for $1 \leq i < j < k \leq v$. The base area (a portion of a two-dimensional flat) of φ_{ijk} is at most d , and its “height” is a $(d-2)$ -dimensional sphere of radius

$$r_{i,j,k} = \Theta\left(\frac{1}{n^{(4d^2-5d-1)/(d(d-1)(d-2))}|p_i p_j p_k|}\right).$$

The overall volume of the prisms (within C^d) is at most

$$\sum_{1 \leq i < j < k \leq v} (|B_{r_{i,j,k}}^{d-2}|d) = \sum_{1 \leq i < j < k \leq v} O\left(\frac{1}{n^{(4d^2-5d-1)/(d(d-1))}|p_i p_j p_k|^{d-2}}\right). \quad (3)$$

To bound this sum, we fix p_i, p_j and sum over p_k . We use a d -dimensional cylindrical packing argument that exploits the fact that S_v satisfies [1] and [2]. The cylinders are centered at ℓ_{ij} ; see Fig. 3, where the line ℓ_{ij} emanates from p_i toward p_j through the

Fig. 3. A d -dimensional cylindrical packing (an extruded $(d-1)$ -dimensional spherical packing) of balls in $\mathbb{R}^d$.

d th dimension. Specifically, we have

$$\sum_{k \neq i,j} \frac{1}{|p_i p_j p_k|^{d-2}} \leq \frac{N_0 n^{2(d-2)/(d-1)}}{a_2^{d-2}} + \sum_{t=1}^{O(n^{1/d})} \frac{2^{d-2} N_t n^{(d-2)/d}}{a_1^{d-2} t^{d-2} |p_i p_j|^{d-2}}, \quad (4)$$

where N_0 is the number of points of S_v that lie in the innermost d -dimensional cylinder of the packing (centered at ℓ_{ij} and of radius $a_1/n^{1/d}$), and N_t is the number of points of S_v that lie in the cylindrical shell centered at ℓ_{ij} with inner radius $a_1 t/n^{1/d}$ and outer radius $a_1(t+1)/n^{1/d}$.

Obviously, $N_0 = O(n^{1/d})$, since the volume of the $(d-1)$ -dimensional cross-sectional sphere of the innermost cylinder is $\Theta(1/n^{(d-1)/d})$ and because of [1]. Also, we have $N_t = O(t^{d-2} n^{1/d})$. This follows by an argument of packing spheres of volume $\Theta(1/n)$ within a shell whose volume is $\Theta(t^{d-2}/n^{(d-1)/d})$.

Hence, the quantity in (4) is $O(n^{(2d^2-3d-1)/(d(d-1))} + n/|p_i p_j|^{d-2})$. Substituting this in (3), we obtain the upper bound on the total volume of the forbidden prisms:

$$\begin{aligned} & O \left(\sum_{1 \leq i < j \leq v} \left(\frac{1}{n^2} + \frac{1}{n^{(3d^2-4d-1)/(d(d-1))} |p_i p_j|^{d-2}} \right) \right) \\ &= O \left(\frac{v^2}{n^2} + \frac{1}{n^{(3d^2-4d-1)/(d(d-1))}} \sum_{1 \leq i < j \leq v} \frac{1}{|p_i p_j|^{d-2}} \right). \end{aligned} \quad (5)$$

We bound the sum in the second summand similarly to our bounding of the term in (2) (in Section 2.3). We fix p_i and use a d -dimensional spherical packing argument within spherical shells centered at p_i . Arguing as above, we obtain

$$\sum_{j \neq i} \frac{1}{|p_i p_j|^{d-2}} \leq \sum_{t=1}^{O(n^{1/d})} \frac{M_t n^{(d-2)/d}}{a_1^{d-2} t^{d-2}} = \sum_{t=1}^{O(n^{1/d})} \frac{O(t^{d-1}) n^{(d-2)/d}}{a_1^{d-2} t^{d-2}} = O(n).$$

Summing this over all p_i , we obtain a final bound of $O(vn)$. Substituting this in (5), we see that the total volume of the forbidden prisms is

$$O \left(\frac{v^2}{n^2} + \frac{v}{n^{(2d^2-3d-1)/(d(d-1))}} \right) = O(1).$$

2.5. General Forbidden Zones

In Sections 2.2–2.4 we computed the total volume of the forbidden zones in which the respective inequalities [1']–[3'] are violated. These zones correspond to $q = 2, 3, 4$, respectively. In this section we analyze the general case $4 < q \leq d+1$. (The smallest value of q for which this analysis holds is 5, since for $q = 5$ we need conditions [4] and [3], which are precisely the bottom of the range in which the recursive relation $[q-1]$ holds.)

The forbidden regions where one of the inequalities $[(q-1)']$ is violated are $\binom{v}{q-1}$ d -dimensional zones $\psi_{i_1 i_2 \dots i_{q-1}}$ (for $1 \leq i_1 < i_2 < \dots < i_{q-1} \leq v$ and $4 < q \leq$

$d + 1$), whose “bases” are portions of $(q - 2)$ -dimensional flats with volume at most $d^{(q-2)/2}$. The “height” of the zone $\psi_{i_1 i_2 \dots i_{q-1}}$ is a $(d - q + 2)$ -dimensional sphere of radius $r_{i_1, \dots, i_{q-1}} = \Theta(V_{q-1}/|p_{i_1} p_{i_2} \dots p_{i_{q-1}}|)$. The total volume of the zones (within C^d) is at most

$$\sum_{1 \leq i_1 < \dots < i_{q-1} \leq v} O(|B_{r_{i_1, \dots, i_{q-1}}}^{d-q+2}| d^{(q-2)/2}) = \sum_{1 \leq i_1 < \dots < i_{q-1} \leq v} O\left(\frac{V_{q-1}^{d-q+2}}{|p_{i_1} p_{i_2} \dots p_{i_{q-1}}|^{d-q+2}}\right). \quad (6)$$

To bound this sum, we fix $p_{i_1}, p_{i_2}, \dots, p_{i_{q-2}}$ and sum over $p_{i_{q-1}}$. We use a packing argument that exploits the fact that S_v satisfies [1]–[$q - 2$]. The packing consists of the Cartesian product of the $(q - 3)$ -dimensional flat $\pi = \pi_{i_1 i_2 \dots i_{q-2}}$ that passes through $p_{i_1}, p_{i_2}, \dots, p_{i_{q-2}}$, and spheres whose centers belong to π and extend to the $(d - q + 3)$ -dimensional space orthogonal to π . Specifically, we have

$$\begin{aligned} & \sum_{i_{q-1} \neq i_1, \dots, i_{q-2}} \frac{1}{|p_{i_1} p_{i_2} \dots p_{i_{q-1}}|^{d-q+2}} \\ & \leq \frac{Z_0}{V_{q-2}^{d-q+2}} + \sum_{t=1}^{O(n^{1/d})} \left(Z_t \cdot O\left(\frac{n^{1/d}}{a_1 t |p_{i_1} p_{i_2} \dots p_{i_{q-2}}|}\right)^{d-q+2} \right), \end{aligned} \quad (7)$$

where Z_0 is the number of points of S_v that lie in the innermost shape of the packing (centered at the flat π and of radius $a_1/n^{1/d}$), and Z_t is the number of points of S_v that lie in the shell centered at π with inner radius $a_1 t/n^{1/d}$ and outer radius $a_1(t+1)/n^{1/d}$.

Obviously, $Z_0 = O(n^{(q-3)/d})$, since the volume of the innermost shape is $\Theta(1/n^{(d-q+3)/d})$ and because of [1]. Also, we have $Z_t = O(t^{d-q+2} n^{(q-3)/d})$. This follows by an argument of packing spheres of volume $\Theta(1/n)$ within a shell whose volume is $\Theta(t^{d-q+2}/n^{(d-q+3)/d})$.

Hence, the sum in (7) is

$$O\left(\frac{n^{(q-3)/d}}{V_{q-2}^{d-q+2}} + \frac{n}{|p_{i_1} p_{i_2} \dots p_{i_{q-2}}|^{d-q+2}}\right). \quad (8)$$

Substituting this in (6), we obtain the upper bound on the total volume of the forbidden zones:

$$\begin{aligned} & O\left(V_{q-1}^{d-q+2} \sum_{1 \leq i_1 < \dots < i_{q-2} \leq v} \left(\frac{n^{(q-3)/d}}{V_{q-2}^{d-q+2}} + \frac{n}{|p_{i_1} p_{i_2} \dots p_{i_{q-2}}|^{d-q+2}}\right)\right) \\ & = O\left(n^{(q-3)/d} v^{q-2} \left(\frac{V_{q-1}}{V_{q-2}}\right)^{d-q+2} + \sum_{1 \leq i_1 < \dots < i_{q-2} \leq v} n \left(\frac{V_{q-1}}{V_{q-3}}\right)^{d-q+2}\right). \end{aligned}$$

Combining this with the equality

$$V_{q-1} = \frac{a_{q-1} V_{q-2}}{a_{q-2} n^{(d(q-2)+q-3)/(d(d-q+2))}},$$

we see that the total forbidden volume is

$$O \left(1 + \sum_{1 \leq i_1 < \dots < i_{q-2} \leq v} n \left(\frac{V_{q-1}}{V_{q-3}} \right)^{d-q+2} \right). \quad (9)$$

In order to show that the bound in (9) is $O(1)$, it remains to prove that the second summand in it is $O(1)$. From $[q-1]$ and $[q-2]$ we know that

$$V_{q-1} = \frac{a_{q-1} V_{q-2}}{a_{q-2} n^{(d(q-2)+q-3)/(d(d-q+2))}}$$

and

$$V_{q-2} = \frac{a_{q-2} V_{q-3}}{a_{q-3} n^{(d(q-3)+q-4)/(d(d-q+3))}},$$

respectively. By substituting this in the second summand of (9), we obtain

$$\begin{aligned} & \sum_{1 \leq i_1 < \dots < i_{q-2} \leq v} n \left(\frac{V_{q-1}}{V_{q-3}} \right)^{d-q+2} \\ &= \left(\frac{a_{q-1}}{a_{q-3}} \right)^{d-q+2} \sum_{1 \leq i_1 < \dots < i_{q-2} \leq v} \frac{1}{n^{(2(q-3)d^2 + (-2q^2 + 13q - 22)d - 2q^2 + 12q - 17)/(d(d-q+3))}} \\ &< \left(\frac{a_{q-1}}{a_{q-3}} \right)^{d-q+2} \frac{v^{q-2}}{n^{(2(q-3)d^2 + (-2q^2 + 13q - 22)d - 2q^2 + 12q - 17)/(d(d-q+3))}}. \end{aligned} \quad (10)$$

Thus, it suffices to prove that

$$n^{(2(q-3)d^2 + (-2q^2 + 13q - 22)d - 2q^2 + 12q - 17)/(d(d-q+3))} > v^{q-2},$$

i.e. (knowing that $v \leq n$), that

$$\frac{2(q-3)d^2 + (-2q^2 + 13q - 22)d - 2q^2 + 12q - 17}{d(d-q+3)} > q-2,$$

which, after simple manipulations, is

$$(q-4)d^2 - (q-4)^2d - 2q^2 + 12q - 17 > 0.$$

However, it is easily verified that

$$(q-4)d^2 - (q-4)^2d - 2q^2 + 12q - 17 = (q-4)(d-q+2)(d+2) - 1 \geq 5 > 0,$$

using the inequalities $q \geq 5$, $d-q \geq -1$, and $d \geq 4$.

2.6. Summation

We are now ready to bound $\mathcal{H}_d^{\text{on-line}}(n)$, the maximum possible volume of the smallest simplex defined by some $d + 1$ of the n points put in the d -dimensional unit cube. In other words, we want to bound V_d from below. For this purpose we use its recursive definition and write

$$V_d = \prod_{q=4}^{d+1} \left(\frac{a_{q-1}}{a_{q-2} n^{(d(q-2)+q-3)/(d(d-q+2))}} \right) \cdot V_2 = \frac{a_d}{n^{(\sum_{q=4}^{d+1} ((d(q-2)+q-3)/(d(d-q+2)))) + 2/(d-1)}}. \quad (11)$$

Let us bound from above the power of n in (11):

$$\begin{aligned} \sum_{q=4}^{d+1} \frac{d(q-2)+q-3}{d(d-q+2)} + \frac{2}{d-1} &= \sum_{q=4}^{d+1} \left(\frac{q-1-1/d}{d-q+2} \right) - \frac{1}{d} \sum_{q=4}^{d+1} \left(\frac{d-q+2}{d-q+2} \right) + \frac{2}{d-1} \\ &= \sum_{t=1}^{d-2} \frac{d+1-1/d-t}{t} - (1-2/d) + \frac{2}{d-1} \\ &<_{(1)} (d+1-1/d)(\ln(d-2)+0.735) - (d-2) - (1-2/d) + 2/(d-1) \\ &= (d+1) \ln(d-2) - 0.265d + 1.735 - (\ln(d-2) - 1.265)/d + 2/(d-1) \\ &<_{(2)} (d+1) \ln(d-2) - 0.265d + 2.269, \end{aligned}$$

where we use the facts (which can be verified by elementary calculus) that

1. $\sum_{t=1}^k 1/t < \ln k + 0.735$ for $k \geq 3$ (that is, $d \geq 5$), and
2. $2/(d-1) - (\ln(d-2) - 1.265)/d < 0.534$ for $d \geq 5$.

We see that $V_d > a_d / n^{(d+1) \ln(d-2) - 0.265d + 2.269}$. (Note that when d tends to infinity, some terms vanish and $\sum_{t=1}^{d-2} 1/t - \ln(d-2)$ approaches the Euler–Mascheroni constant, $\gamma = 0.577 \dots$. Thus, for large-enough values of d , we have $V_d > a_d / n^{(d+1) \ln(d-2) - 0.422d + 1.578}$.)

It remains to show that the constants $a_1, a_2, \dots, a_d$ can be fixed so that the total volume of the forbidden zones is strictly less than 1. To this aim note that among these constants, the total volume of the forbidden balls depends solely on a_1 , the total volume of the forbidden cylinders depends only on a_1, a_2 , and so on. This allows us to fix the values of the constants sequentially so that the total volume of any type of forbidden zone is strictly less than $1/d$. Specifically, we fix a_1 to make the total volume of the forbidden balls less than $1/d$. Then, having a_1 already fixed, we fix a_2 to make the total volume of the forbidden cylinders less than $1/d$, and so on. The grand total of the volume of the forbidden zones is, thus, less than $d(1/d) = 1$. (See [B2] for the implementation of this technique for $d = 3, 4$.) Note that the values of $a_1, a_2, \dots, a_d$ depend only on d and not on n . This allows the iterative procedure to continue ad infinitum, while the forbidden zones of each iteration properly contain the respective forbidden zones of the preceding iteration.

This completes the proof of the main theorem:

Theorem 1. $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(1/n^{(d+1)\ln(d-2)-0.265d+2.269})$ for $d \geq 5$.

3. Conclusion

In this paper we show, by using nested packing arguments, that $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(1/n^{(d+1)\ln(d-2)-0.265d+2.269})$ for all $d \geq 5$. (For large-enough values of d we have $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(1/n^{(d+1)\ln(d-2)-0.422d+1.578})$.) This compares favorably with the best-known lower bound [L1] in the off-line case $\mathcal{H}_d^{\text{off-line}}(n) = \Omega(\log n/n^d)$.

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